Upper/Lower Bounds of Generalized H_2 Norms in Sampled-Data Systems with Convergence Rate Analysis and Discretization Viewpoint

Jung Hoon Kim[a,*], **Tomomichi Hagiwara**[b]

[a] *Center for Robotics Research, Korea Institute of Science and Technology (KIST), Seongbuk-gu, Seoul 02792, Republic of Korea*
[b] *Department of Electrical Engineering, Kyoto University, Nishikyo-ku, Kyoto 615-8510, Japan*

Abstract

This paper considers linear time-invariant (LTI) sampled-data systems and studies their generalized H_2 norms. They are defined as the induced norms from L_2 to L_∞, in which two types of the L_∞ norm of the output are considered as the temporal supremum magnitude under the spatial ∞-norm and 2-norm. The input/output relation of sampled-data systems is first formulated under their lifting-based treatment. We then develop a method for computing the generalized H_2 norms with operator-theoretic gridding approximation. This method leads to readily computable upper bounds as well as lower bounds of the generalized H_2 norms, whose gaps tend to 0 at the rate of $1/\sqrt{N}$ with the gridding approximation parameter N. An approximately equivalent discretization method of the generalized plant is further provided as a fundamental step to addressing the controller synthesis problem of minimizing the generalized H_2 norms of sampled-data systems. Finally, a numerical example is given to show the effectiveness of the computation method.

Keywords: Sampled-data systems, L_∞/L_2-induced norm, Discretization, Gridding, operator-theoretic approach

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*Corresponding author

Email addresses: j.h.kim@kist.re.kr (Jung Hoon Kim), hagiwara@kuee.kyoto-u.ac.jp (Tomomichi Hagiwara)

1. Introduction

The H_2 norm has been widely used as a performance measure for the disturbance rejection problem. There are two time-domain ideas to define the H_2 norm for linear time-invariant (LTI) sampled-data systems. By this term, we mean that the continuous-time plant is LTI and controlled by an LTI discrete-time controller. Furthermore, our particular interest lies in the intersample behavior of the continuous-time signals about the plant. The first idea [1] considers the L_2 norm of the output for the impulse input occurring at a sampling instant. However, it does not take account of the periodically time-varying nature of LTI sampled-data systems viewed in continuous time. With this nature taken into consideration, the second idea (which admits an equivalent frequency-domain interpretation) [2, 3, 4] deals with the root mean square (RMS) of the L_2 norms of all the responses for the impulse inputs occurring at τ in the sampling interval $[0, h)$.

If we confine ourselves to single-output LTI continuous-time and discrete-time systems, on the other hand, we could adopt an alternative time-domain definition of the H_2 norm without considering the impulse input. More precisely, it is well known that the H_2 norm coincides with the induced norm from L_2 to L_∞ [5, 6, 7] in such LTI continuous-time systems and that from l_2 to l_∞ [8, 9] in such LTI discrete-time systems. Even though this is not the case for the multi-output LTI continuous-time and discrete-time systems, the induced norms have been regarded as their generalized H_2 norms. In such treatment, two different spatial norms (i.e., the vector ∞ and 2 norms) are often dealt with in defining the L_∞ and l_∞ norms of the output for the multi-output cases.

It is well known that the L_2 input considered in the generalized H_2 norms is also relevant to the H_∞ norm of LTI sampled-data systems because the latter is nothing but the induced norm from L_2 to L_2. Even with the same class of inputs, which is practical for dealing with typical disturbances with finite energy, however, taking these two different norms on the output can be quite meaningful depending on different purposes. Use of the L_∞ norm for the output leads to the generalized H_2 norms and is particularly suitable for such situations where having a large instantaneous peak value of the output (in terms of the L_∞ norm) is much more problematic than having a large 'average' value of the output (in terms of the L_2 norm). Thus, considering the generalized H_2 norms properly matches practical control applications such as avoiding robot manipulators from colliding with their surrounding objects and suppressing chemical plants from being overly pressured caused by unknown disturbances with finite energy; the H_∞ and L_1 norms do not match these problems effectively. In this sense, it is worthwhile to study the generalized H_2 norms of LTI sampled-data systems. They were analytically formulated in [10] for the first time by employing the idea of the lifting technique [10, 11, 12, 13], but their computation was not discussed there. An approximate but asymptotically exact computation method was then provided in [14] by using an idea of fast-sampling rather than lifting.

Recently, the generalized H_2 norms of LTI sampled-data systems were revisited in [15] again with the lifting arguments in such a way that a comparative study of the generalized H_2 norms with the two existing definitions for the H_2 norm of sampled-data systems can also be carried out. The arguments therein revealed for the first time that the generalized H_2 norms coincide with neither of the two existing definitions of the H_2 norm for LTI sampled-data systems. This clearly means that the generalized H_2 norms (i.e., induced norms from L_2 to L_∞ with two alternative underlying spatial norms about the latter) could be interpreted as yet another definition of the H_2 norm of single-output LTI sampled-data systems. However, to compute the generalized H_2 norms with the arguments in [15], one should take the supremum of a suitably constructed function over the sampling interval $[0, h)$. Thus, only an approximate computation approach based on gridding is given in [15] and the arguments in [14] are also subject to the same kind of situation (as well as some more additional assumption on the system). In other words, no upper bounds of the generalized H_2 norms have been derived and their lower bounds are obtained only through gridding in the existing studies.

This paper aims at resolving this issue through an operator-theoretic interpretation of the gridding method [15], and derives computable upper bounds as well as lower bounds of the generalized H_2 norms together with the associated convergence rate. More precisely, such an interpretation leads to a method for computing upper and lower bounds of each of the two generalized H_2 norms for an LTI sampled-data system. Furthermore, it is shown that the gaps between the upper and lower bounds converge to 0 at the rates of $1/\sqrt{N}$, where N is the gridding approximation parameter. This paper also gives an alternative interpretation of the lower bound computation through an approximate discretization process of the generalized plant. To this end, we relate the computation with that of the l_∞/l_2-induced norm of an approximately equivalent LTI discrete-time system constructed with the discretized generalized plant, where the latter computation is readily possible [8, 9]. This interpretation through the discretization of the generalized plant gives a crucial basis for the optimal controller synthesis for minimizing the generalized H_2 norms of LTI sampled-data systems [16].

We remark that the gridding approximation has been extended to another approximation approach in [17] but for the case of SISO systems and only for the spatial ∞ norm. The present paper nevertheless opts to confine itself to the simpler gridding approximation at the sacrifice of slight deterioration in the accuracy of approximation while dealing also with the multi-input/multi-output (MIMO) case and the spatial 2 norm; this allows us to circumvent the introduction of the more involved fast-lifting treatment [18] adopted (even for the gridding approach) in [17] and helps us to provide a less complicated perspective for this simpler approach.

The organization of this paper is as follows. Section 2 gives some mathematical notations. In Section 3, the lifting-based results in [15] relevant to the generalized H_2 norms of sampled-data systems are reviewed. The main results of this paper are given in Section 4. A method for computing upper and lower bounds of the generalized H_2 norms together with the associated convergence rate is provided through the operator-theoretic gridding approximation idea. Furthermore, the lower bound computation is related to discretization of the continuous-time generalized plant to open a further direction for the associated optimal controller synthesis. Finally, a numerical example is given in Section 5 to demonstrate the effectiveness of the developed computation method.

2. Mathematical Notations

This section gives the mathematical notations used in this paper. The notations $\mathbb{N}$, $\mathbb{R}^\nu_\infty$ and $\mathbb{R}^\nu_2$ denote the set of positive integers, the Banach space of ν-dimensional real vectors equipped with vector ∞-norm (denoted by $|\cdot|_\infty$) and the Hilbert space of ν-dimensional real vectors equipped with the usual inner product and the associated Euclidean norm (denoted by $|\cdot|_2$), respectively. The notation $\mathbb{N}_0$ is further used to imply $\mathbb{N} \cup \{0\}$.

The notations $|\cdot|_{p/2}$ ($p = \infty, 2$) are used to imply the induced norms of a matrix as a mapping from $\mathbb{R}_2^{\nu_1}$ to $\mathbb{R}_p^{\nu_2}$, i.e.,

$$|T|_{p/2} := \sup_{w \in \mathbb{R}_2^{\nu_1}} \frac{|Tw|_p}{|w|_2} \quad (p = \infty, 2)$$

The notations $\|\cdot\|_{(\infty,p)}$ $(p = \infty, 2)$ are used to mean the $L_\infty[0,h)$ norms under the spatial ∞-norm and 2-norm, respectively, i.e.,

$$\|z(\cdot)\|_{(\infty,\infty)} := \operatorname*{ess\,sup}_{0 \le t < h} |z(t)|_\infty = \operatorname*{ess\,sup}_{0 \le t < h} \max_{1 \le i < \nu} |z_i(t)|,$$

$$\|z(\cdot)\|_{(\infty,2)} := \operatorname*{ess\,sup}_{0 \le t < h} |z(t)|_2 = \operatorname*{ess\,sup}_{0 \le t < h} \left(z^T(t) z(t)\right)^{1/2},$$

or those with h replaced by an integer fraction $h' = h/N$ or ∞, whose distinction will be clear from the context (the same comment applies to the following norm notations used in common for slightly different types of quantities). The notation $\|\cdot\|_{(2,2)}$ is used to mean either the $L_2[0,h)$ norm of a real-vector-valued function, i.e.,

$$\|w(\cdot)\|_{(2,2)} := \left(\int_0^h |w(t)|_2^2 dt\right)^{1/2},$$

or that with h replaced by $h' = h/N$ or ∞.

On the other hand, for an operator $\mathbf{T}$ from $(L_2[0,h))^{\nu_1}$ to $(L_\infty[0,h))^{\nu_2}$, the notations $\|\cdot\|_{(\infty,p)/(2,2)}$ $(p = \infty, 2)$ are used to denote either the induced norms

$$\|\mathbf{T}\|_{(\infty,p)/(2,2)} := \sup_{w \in (L_2[0,h))^{\nu_1}} \frac{\|\mathbf{T}w\|_{(\infty,p)}}{\|w\|_{(2,2)}} \quad (p = \infty, 2),$$

or that with h replaced by ∞. This notation is also applied to the discrete-time case, i.e., the induced norms from l_2 to l_∞ equipped with two spatial (∞ and 2) norms for the l_∞ norm.

The notations $\|\cdot\|_{(\infty,p)/2}$ $(p = \infty, 2)$ are used to imply the induced norms from $\mathbb{R}_2^{\nu_1}$ to $(L_\infty[0,h))^{\nu_2}$, i.e.,

$$\|\mathbf{T}\|_{(\infty,p)/2} := \sup_{x \in \mathbb{R}_2^{\nu_1}} \frac{\|\mathbf{T}x\|_{(\infty,p)}}{|x|_2} \quad (p = \infty, 2)$$

or that with h replaced by h'.

Furthermore, the notation $\|\cdot\|_{2/2}$ is used to imply the induced norm from l_2 or $(L_2[0,h))^{\nu_1}$ to $\mathbb{R}_2^{\nu_2}$, or that with h replaced by h'. Finally, we use the notations $d_{\max}(\cdot)$ and $\lambda_{\max}(\cdot)$ to denote the maximum diagonal entry and maximum eigenvalue of a real symmetric matrix, respectively.

3. Characterization of the Generalized H_2 Norms in Sampled-Data Systems

This section reviews the results in [15] and gives an explicit characterization for the generalized H_2 norms of LTI sampled-data systems through the lifting treatment [10, 11, 12, 13]. A similar result was first derived in [14] without applying the lifting technique.

Let us consider the stable sampled-data system Σ_{SD} shown in Fig. 1, where P represents the continuous-time LTI generalized plant, while Ψ, $\mathcal{H}$ and $\mathcal{S}$ represent the discrete-time LTI controller, the zero-order hold and the ideal sampler, respectively, operating with sampling period h in a synchronous fashion. Solid lines and dashed lines in Fig. 1 are used for continuous-time signals and discrete-time signals, respectively. Suppose that P and Ψ are given respectively by

$$P : \begin{cases} \dot{x} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{12} u \\ y = C_2 x, \end{cases} \qquad \Psi : \begin{cases} \psi_{k+1} = A_\Psi \psi_k + B_\Psi y_k \\ u_k = C_\Psi \psi_k + D_\Psi y_k \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}_2^n$, $w(t) \in \mathbb{R}_2^{n_w}$, $u(t) \in \mathbb{R}_2^{n_u}$, $y(t) \in \mathbb{R}_2^{n_y}$, $\psi_k \in \mathbb{R}_2^{n_\Psi}$, $y_k = y(kh)$ and $u(t) = u_k$ $(kh \le t < (k+1)h)$. Furthermore, we regard $z(t)$ to belong to $\mathbb{R}_p^{n_z}$, where p is either ∞ or 2, depending on the context. Note that we have assumed '$D_{11} = 0$' and '$D_{21} = 0$' for the continuous-time generalized plant P in (1). These assumptions are necessary (and sufficient by the stability of Σ_{SD}) for the generalized H_2 norms

$$\|\Sigma_{\mathrm{SD}}\|_{(\infty,p)/(2,2)} := \sup_{\|w\|_{(2,2)} \le 1} \|z\|_{(\infty,p)} \quad (p = \infty, 2) \tag{2}$$

to be well-defined/bounded.

To alleviate the difficulty in the treatment of Σ_{SD} caused by its periodically time-varying nature, we apply the lifting technique [10, 11, 12, 13] to the

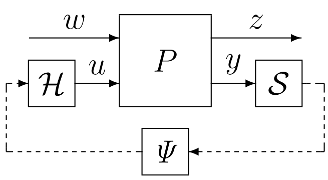

Figure 1: Sampled-data system Σ_{SD}.

sampled-data system Σ_{SD}; given $f \in (L_\infty)^\nu$ or $f \in (L_2)^\nu$, its lifting $\{\widehat{f}_k\}_{k=0}^{\infty}$ (with sampling period h) with $\widehat{f}_k \in (L_\infty[0,h))^\nu$ or $(L_2[0,h))^\nu$ ($k \in \mathbb{N}_0$) is defined by

$$\widehat{f}_k(\theta) = f(kh+\theta) \quad (0 \le \theta < h) \tag{3}$$

Here, for $\widehat{f} := [\widehat{f}_0^T,\ \widehat{f}_1^T,\ \cdots]^T$, we define $\|\widehat{f}\|_{(\infty,p)}$ ($p=\infty,2$) and $\|\widehat{f}\|_{(2,2)}$ by $\sup_k \|\widehat{f}_k\|_{(\infty,p)}$ ($p=\infty,2$) and $(\sum_{k=0}^{\infty}\|\widehat{f}_k\|_{(2,2)}^2)^{1/2}$, respectively, and thus lifting is norm-preserving in both L_∞ and L_2. By applying lifting to $w \in (L_2)^{n_w}$ and $z \in (L_\infty)^{n_z}$, the lifted representation of Σ_{SD} is given by

$$\begin{cases} \xi_{k+1} &= \mathcal{A}\xi_k + \mathcal{B}\widehat{w}_k \\ \widehat{z}_k &= \mathcal{C}\xi_k + \mathcal{D}\widehat{w}_k \end{cases} \tag{4}$$

with $\xi_k := [x_k^T\ \psi_k^T]^T$ ($x_k := x(kh)$), the stable matrix

$$\mathcal{A} = \begin{bmatrix} A_d + B_{2d}D_\Psi C_{2d} & B_{2d}C_\Psi \\ B_\Psi C_{2d} & A_\Psi \end{bmatrix} : \mathbb{R}_2^{n+n_\Psi} \to \mathbb{R}_2^{n+n_\Psi} \tag{5}$$

and the operators

$$\mathcal{B} = J_\Sigma \mathbf{B}_1 : (L_2[0,h))^{n_w} \to \mathbb{R}_2^{n+n_\Psi} \tag{6}$$

$$\mathcal{C} = \mathbf{M}_1 C_\Sigma : \mathbb{R}_2^{n+n_\Psi} \to (L_\infty[0,h))^{n_z} \tag{7}$$

$$\mathcal{D} = \mathbf{D}_{11} : (L_2[0,h))^{n_w} \to (L_\infty[0,h))^{n_z} \tag{8}$$

where

$$A_d := \exp(Ah),\ \ B_{2d} := \int_0^h \exp(A\theta)B_2 d\theta,\ \ C_{2d} := C_2 \tag{9}$$

$$\mathbf{B}_1 w = \int_0^h \exp(A(h-\theta))B_1 w(\theta)d\theta \tag{10}$$

$$\left(\mathbf{M}_1 \begin{bmatrix} x \\ u \end{bmatrix}\right)(\theta) = M_1 \exp(A_2\theta)\begin{bmatrix} x \\ u \end{bmatrix},$$

$$M_1 := \begin{bmatrix} C_1 & D_{12} \end{bmatrix} : \mathbb{R}_2^{n+n_u} \to \mathbb{R}_p^{n_z},\ \ A_2 := \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix} : \mathbb{R}_2^{n+n_u} \to \mathbb{R}_2^{n+n_u} \tag{11}$$

$$(\mathbf{D}_{11}w)(\theta) = \int_0^\theta C_1 \exp(A(\theta-\tau))B_1 w(\tau)d\tau \tag{12}$$

$$J_\Sigma := \begin{bmatrix} I \\ 0 \end{bmatrix} : \mathbb{R}_2^n \to \mathbb{R}_2^{n+n_\Psi},\ \ C_\Sigma := \begin{bmatrix} I & 0 \\ D_\Psi C_{2d} & C_\Psi \end{bmatrix} : \mathbb{R}_2^{n+n_\Psi} \to \mathbb{R}_2^{n+n_u} \tag{13}$$

From the stability assumption of $\Sigma_{\rm SD}$, $\mathcal{A}$ has all its eigenvalues in the open unit disc.

We first note (4) and describe the closed-loop relation between $\widehat{w}_k$ and $\widehat{z}_k$ ($k = 0, \cdots, \infty$) as follows:

$$\begin{bmatrix} \widehat{z}_0 \\ \widehat{z}_1 \\ \widehat{z}_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \mathcal{D} & 0 & \cdots & & \\ \mathcal{CB} & \mathcal{D} & 0 & \cdots & \\ \mathcal{CAB} & \mathcal{CB} & \mathcal{D} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \widehat{w}_0 \\ \widehat{w}_1 \\ \widehat{w}_2 \\ \vdots \end{bmatrix} \tag{14}$$

Because lifting is norm-preserving in both L_2 and L_∞, the generalized H_2 norms of the sampled-data system $\Sigma_{\rm SD}$ coincides with the associated induced norm of the above operator in the right hand side. Since this operator has a Toeplitz structure (and thus every row is an extension of the previous row), it readily follows from $\|z\|_{(\infty,p)} = \sup_k \|\widehat{z}_k\|_{(\infty,p)}$ ($p = \infty, 2$) that the generalized H_2 norms of $\Sigma_{\rm SD}$ coincide with the generalized H_2 norms of its "last" block row, i.e., (after reordering without affecting the generalized H_2 norms)

$$\mathcal{F} := \begin{bmatrix} \mathcal{D} & \mathcal{CB} & \mathcal{CAB} & \mathcal{CA}^2\mathcal{B} & \cdots \end{bmatrix} \tag{15}$$

Here, with a slight abuse of the terminology, the generalized H_2 norms of $\mathcal{F}$ for $p = \infty, 2$ refer to

$$\begin{aligned} \|\mathcal{F}\|_{(\infty,p)/(2,2)} &:= \sup_{\|\widehat{w}\|_{(2,2)} \le 1} \|(\mathcal{F}\widehat{w})(\cdot)\|_{(\infty,p)} = \sup_{\|\widehat{w}\|_{(2,2)} \le 1} \sup_{0 \le \theta < h} |(\mathcal{F}\widehat{w})(\theta)|_p \\ &= \sup_{0 \le \theta < h} \sup_{\|\widehat{w}\|_{(2,2)} \le 1} |(\mathcal{F}\widehat{w})(\theta)|_p \end{aligned} \tag{16}$$

To explicitly characterize the generalized H_2 norms $\|\mathcal{F}\|_{(\infty,p)/(2,2)}$ ($p = \infty, 2$), we briefly sketch the results in [15] as follows. For each $\theta \in [0, h]$, we first introduce the matrices

$$W_\theta := \int_0^\theta \exp(A(\theta - \tau)) B_1 B_1^T \exp(A^T(\theta - \tau)) d\tau \tag{17}$$

$$C_\theta := M_1 \exp(A_2 \theta) C_\Sigma \tag{18}$$

Then, with W_h, consider the solution X_h to the discrete-time Lyapunov equation

$$\mathcal{A} X_h \mathcal{A}^T - X_h + \begin{bmatrix} W_h & 0 \\ 0 & 0 \end{bmatrix} = 0 \tag{19}$$

Let us further introduce the matrix function $F(\theta)$ defined as

$$F(\theta) := C_1 W_\theta C_1^T + C_\theta X_h C_\theta^T \quad (\theta \in [0, h]) \tag{20}$$

Then, it was shown in [15] that

$$\sup_{\|\widehat{w}\|_{(2,2)} \leq 1} |(\mathcal{F}\widehat{w})(\theta)|_\infty = d_{\max}^{1/2}(F(\theta)) \tag{21}$$

$$\sup_{\|\widehat{w}\|_{(2,2)} \leq 1} |(\mathcal{F}\widehat{w})(\theta)|_2 = \lambda_{\max}^{1/2}(F(\theta)) \tag{22}$$

This together with (16) leads to the following result.

Theorem 1 ([15]). *The generalized H_2 norms associated with the MIMO LTI sampled-data system $\Sigma_{\rm SD}$ are given by*

$$\|\mathcal{F}\|_{(\infty,\infty)/(2,2)} = \sup_{0 \leq \theta < h} d_{\max}^{1/2}(F(\theta)) \tag{23}$$

$$\|\mathcal{F}\|_{(\infty,2)/(2,2)} = \sup_{0 \leq \theta < h} \lambda_{\max}^{1/2}(F(\theta)) \tag{24}$$

Theorem 1 almost gives a direct computation method for the generalized H_2 norms $\|\mathcal{F}\|_{(\infty,p)/(2,2)}$ $(p = \infty, 2)$. However, when a simple gridding idea is taken and thus lower bounds are obtained (i.e., by computing $F(\theta)$ for N equally spaced points in $[0, h)$), it is not clear how much the lower bounds could deviate from the exact norms in the worst case. This is closely related with the fact that this theorem cannot lead to easily obtainable upper bounds of the generalized H_2 norms $\|\mathcal{F}\|_{(\infty,p)/(2,2)}$ $(p = \infty, 2)$. In this regard, the following section is devoted to giving readily computable upper bounds as well as lower bounds of the generalized H_2 norms $\|\mathcal{F}\|_{(\infty,p)/(2,2)}$. More precisely, we develop an alternative operator-theoretic method for approximately computing $\|\mathcal{F}\|_{(\infty,p)/(2,2)}$ $(p = \infty, 2)$, which is free from the gridding idea in its usual sense. The key idea in that direction is very close to a similar one that has proved very useful in the L_∞-induced norm problem of sampled-data systems [19, 20], but we exploit in a slightly simplified way to facilitate the overall arguments. We call the method (operator-theoretic) gridding approximation, and analyze the convergence rate in N with which the gaps between the upper and lower bounds converge to 0, where N is the gridding approximation parameter.

4. Main Results

In this section, we first introduce an operator-theoretic interpretation of the gridding approximation. Then, we provide a method for computing upper and lower bounds of the generalized H_2 norms $\|\mathcal{F}\|_{(\infty,p)/(2,2)}$ ($p=\infty,2$) together with the associated convergence rate in their gridding approximation treatment. Finally, an approximately equivalent discretization method of the continuous-time generalized plant P is discussed.

4.1. Gridding Approximation

We first introduce the operator $\mathbf{H}_{N0}$ with $N \in \mathbb{N}$ and $h' := h/N$ given by

$$(\mathbf{H}_{N0}z)(\theta) = z\left(\left\lfloor \frac{\theta}{h'} \right\rfloor h'\right) \quad (0 \leq \theta < h) \tag{25}$$

where z is a function on $[0,h)$ and $\lfloor\cdot\rfloor$ denotes the largest integer that does not exceed $(\cdot)$. For z continuous on $[0,h)$, the function $(\mathbf{H}_{N0}z)(\theta)$ is piecewise constant on $[0,h)$, and $\mathbf{H}_{N0}$ is used in defining the gridding approximation of $\mathcal{F}$. More precisely, the gridding approximation is defined as

$$\mathcal{F}_{N\mathrm{G}} := \begin{bmatrix}\mathbf{H}_{N0}\mathcal{D} & \mathbf{H}_{N0}\mathcal{C}\mathcal{B} & \mathbf{H}_{N0}\mathcal{C}\mathcal{A}\mathcal{B} & \mathbf{H}_{N0}\mathcal{C}\mathcal{A}^2\mathcal{B} & \cdots\end{bmatrix} \tag{26}$$

Here, it is obvious that $(\mathcal{F}_{N\mathrm{G}}\widehat{w})(\theta) = (\mathcal{F}\widehat{w})(\theta)$ for every $\widehat{w}$ such that $\|\widehat{w}\|_{(2,2)} \leq 1$ and $\theta \in \mathcal{K}_N := \{0, h', 2h', \cdots, (N-1)h'\}$. Since

$$\|\mathcal{F}_{N\mathrm{G}}\|_{(\infty,p)/(2,2)} = \sup_{0\leq\theta<h} \sup_{\|\widehat{w}\|_{(2,2)}\leq 1} |(\mathcal{F}_{N\mathrm{G}}\widehat{w})(\theta)|_p \tag{27}$$

as in (16), it follows from (21) and (22) that

$$\|\mathcal{F}_{N\mathrm{G}}\|_{(\infty,\infty)/(2,2)} = \max_{\theta\in\mathcal{K}_N} d_{\max}^{1/2}(F(\theta)) \tag{28}$$

$$\|\mathcal{F}_{N\mathrm{G}}\|_{(\infty,2)/(2,2)} = \max_{\theta\in\mathcal{K}_N} \lambda_{\max}^{1/2}(F(\theta)) \tag{29}$$

To put it another way, computing the operator norms $\|\mathcal{F}_{N\mathrm{G}}\|_{(\infty,p)/(2,2)}$ ($p = \infty, 2$) corresponds to very simple gridding treatment of the matrix function $F(\theta)$ with N equally spaced points in (23) and (24). This is why the operator $\mathcal{F}_{N\mathrm{G}}$ is called the gridding approximation of $\mathcal{F}$.

Because of the equivalence to the gridding treatment of $F(\theta)$, the norm $\|\mathcal{F}_{N\mathrm{G}}\|_{(\infty,p)/(2,2)}$ obviously corresponds to a lower bound of $\|\mathcal{F}\|_{(\infty,p)/(2,2)}$ for each $p = \infty, 2$. Furthermore, an exact computation of these lower bounds $\|\mathcal{F}_{NG}\|_{(\infty,p)/(2,2)}$ ($p = \infty, 2$) is straightforward if we recall the explicit definition of $F(\theta)$ in (20). Actually, this gridding treatment merely in terms of $F(\theta)$ (i.e., without introducing $\mathcal{F}_{N\mathrm{G}}$) is nothing but the approximate computation method for $\|\mathcal{F}\|_{(\infty,p)/(2,2)}$ ($p = \infty, 2$) given in [14, 15], but this method is not necessarily satisfactory in the following two respects.

First, it does not provide a method for giving upper bounds of $\|\mathcal{F}\|_{(\infty,p)/(2,2)}$ ($p = \infty, 2$). Secondly, the computation formula has no clear relationship with an associated (discrete-time) dynamical system, so that it is not clear how we could extend the computation method to arguments for designing optimal controllers minimizing the generalized H_2 norms. The reason why $\mathcal{F}_{N\mathrm{G}}$ has been introduced in the present paper is that it is quite helpful in resolving these issues, in spite of the aforementioned equivalence in dealing with $F(\theta)$ and $\mathcal{F}_{N\mathrm{G}}$ in some restricted aspect (i.e., lower bound computation). Indeed, the remainder of this section is devoted to providing upper bounds for $\|\mathcal{F}\|_{(\infty,p)/(2,2)}$ ($p = \infty, 2$) and the associated convergence rate analysis in N about the upper and lower bounds through the use of $\mathcal{F}_{N\mathrm{G}}$. Furthermore, we derive alternative exact computation methods of $\|\mathcal{F}_{N\mathrm{G}}\|_{(\infty,p)/(2,2)}$ ($p = \infty, 2$) through appropriate discretization of the generalized plant P.

4.2. Upper/Lower Bounds and Convergence Rate Analysis

The following two lemmas play important roles in deriving upper and lower bounds of the generalized H_2 norms and the associated convergence rate for the gridding approximation approach.

Lemma 1. *For each $p = \infty, 2$, the inequality*

$$\|(I - \mathbf{H}_{N0})\mathbf{D}_{11}\|_{(\infty,p)/(2,2)} \leq \frac{K_{ND}^{[p]}}{\sqrt{N}} + \frac{K_{ND0}^{[p]}}{N} \tag{30}$$

holds, where

$$K_{ND}^{[\infty]}/\sqrt{N} := d_{\max}^{1/2}(C_1 W_{h'} C_1^T) \tag{31}$$

$$K_{ND}^{[2]}/\sqrt{N} := \lambda_{\max}^{1/2}(C_1 W_{h'} C_1^T) \tag{32}$$

$$K_{ND0}^{[p]} := h \cdot |C_1 A|_{p/2} \cdot e^{h'|A|_{2/2}} \cdot |V_{(N-1)h'}|_{2/2} \quad (p = \infty, 2) \tag{33}$$

with the matrix V_θ defined by

$$V_\theta V_\theta^T = W_\theta \quad (\theta \in [0, h]) \tag{34}$$

for W_θ given in (17). Furthermore, $K_{ND}^{[p]}$ and $K_{ND0}^{[p]}$ have uniform upper bounds with respect to N given respectively by $K_D^{[p]} := \sqrt{h} \cdot |C_1|_{p/2} \cdot e^{h|A|_{2/2}} \cdot |B_1|_{2/2}$ and $K_{D0}^{[p]} := h \cdot |C_1 A|_{p/2} \cdot e^{h|A|_{2/2}} \cdot |V_h|_{2/2}$.

Lemma 2. *For each $p = \infty, 2$, the inequality*

$$\|(I - \mathbf{H}_{N0})\mathbf{M}_1\|_{(\infty,p)/2} \le \frac{K_{NC0}^{[p]}}{N} \tag{35}$$

holds, where

$$K_{NC0}^{[p]} := h \cdot |M_1 A_2|_{p/2} \cdot e^{h'|A_2|_{2/2}} \cdot \max_{0 \le i \le N-1} |(A'_{2d})^i|_{2/2} \tag{36}$$

with $A'_{2d} := \exp(A_2 h')$. Furthermore, $K_{NC0}^{[p]}$ has a uniform upper bound with respect to N given by $K_{C0}^{[p]} := h \cdot |M_1 A_2|_{p/2} \cdot e^{h|A_2|_{2/2}}$.

The proofs of these lemmas are given in the appendix. From Lemmas 1 and 2, we readily have the following result.

Theorem 2. *For each $p = \infty, 2$, the inequality*

$$\|\mathcal{F} - \mathcal{F}_{N\mathrm{G}}\|_{(\infty,p)/(2,2)} \le \frac{K_{ND}^{[p]}}{\sqrt{N}} + \frac{K_{N0}^{[p]}}{N} \tag{37}$$

holds with

$$K_{N0}^{[p]} := K_{ND0}^{[p]} + K_{NC0}^{[p]} \cdot (|C_\Sigma X_h C_\Sigma^T|_{2/2})^{1/2} \tag{38}$$

where C_Σ and X_h are described by (13) and (19), respectively. Furthermore, $K_{N0}^{[p]}$ has a uniform upper bound with respect to N given by $K_0^{[p]} := K_{D0}^{[p]} + K_{C0}^{[p]} \cdot (|C_\Sigma X_h C_\Sigma^T|_{2/2})^{1/2}$, while $K_{ND}^{[p]}$ has a uniform upper bound with respect to N given by $K_D^{[p]}$.

Proof. We first recall that $\mathcal{D} = \mathbf{D}_{11}$, $\mathcal{C} = \mathbf{M}_1 C_\Sigma$ and $\mathcal{B} = J_\Sigma \mathbf{B}_1$. Then, we have for each $p = \infty, 2$ that

$$\begin{aligned}
&\|\mathcal{F} - \mathcal{F}_{N\mathrm{G}}\|_{(\infty,p)/(2,2)} \\
&= \| \begin{bmatrix}(I - \mathbf{H}_{N0})\mathcal{D} & (I - \mathbf{H}_{N0})\mathcal{C}\mathcal{B} & (I - \mathbf{H}_{N0})\mathcal{C}\mathcal{A}\mathcal{B} & \cdots\end{bmatrix} \|_{(\infty,p)/(2,2)} \\
&\le \|(I - \mathbf{H}_{N0})\mathbf{D}_{11}\|_{(\infty,p)/(2,2)} \\
&\qquad + \|(I - \mathbf{H}_{N0})\mathbf{M}_1\|_{(\infty,p)/2} \cdot \| \begin{bmatrix} C_\Sigma J_\Sigma \mathbf{B}_1 & C_\Sigma \mathcal{A} J_\Sigma \mathbf{B}_1 & \cdots \end{bmatrix} \|_{2/2}
\end{aligned} \tag{39}$$

where the last factor denotes the induced norm from the lifting image of $(L_2[0,\infty))^{n_w}$ to $\mathbb{R}_2^{n+n_u}$. Since the unit-ball image of $\mathbf{B}_1 : (L_2[0,h))^{n_w} \to \mathbb{R}_2^n$ coincides with that of $V_h : \mathbb{R}_2^{n_V} \to \mathbb{R}_2^n$ (where n_V is the number of columns of V_h) and, equivalently, $\{\mathbf{B}_1 w : \|w\|_{(2,2)} \leq \alpha\} = \{V_h w_d : |w_d|_2 \leq \alpha\}$ for every $\alpha > 0$, we readily see that

$$\|\begin{bmatrix} C_\Sigma J_\Sigma \mathbf{B}_1 & C_\Sigma \mathcal{A}\mathbf{B}_1 & \cdots \end{bmatrix}\|_{2/2} = \|\begin{bmatrix} C_\Sigma J_\Sigma V_h & C_\Sigma \mathcal{A} J_\Sigma V_h & \cdots \end{bmatrix}\|_{2/2} =: \|\Pi_{\mathrm{D}}\|_{2/2} \tag{40}$$

where the latter two $\|\cdot\|_{2/2}$ denotes the induced norm from l_2 to $\mathbb{R}_2^{n+n_u}$. Here, Π_{D} can be regarded as a finite-rank (and thus compact) operator acting on Hilbert spaces, and thus $\|\Pi_{\mathrm{D}}\|_{2/2}$ can be computed with the adjoint operator $\Pi_{\mathrm{D}}^* : \mathbb{R}_2^{n+n_u} \to l_2$ of Π_{D} defined as

$$\Pi_{\mathrm{D}}^* = \begin{bmatrix} C_\Sigma J_\Sigma V_h & C_\Sigma \mathcal{A} J_\Sigma V_h & \cdots \end{bmatrix}^T \tag{41}$$

through the relation $\|\Pi_{\mathrm{D}}\|_{2/2}^2 = \|\Pi_{\mathrm{D}}\Pi_{\mathrm{D}}^*\|_{2/2} = |C_\Sigma X_h C_\Sigma^T|_{2/2}$. Combining the above arguments with (30) and (35) readily completes the proof. Q.E.D.

Since $\|\mathcal{F}_{N\mathrm{G}}\|_{(\infty,p)/(2,2)}$ $(p = \infty, 2)$ can be explicitly computed (recall (28) and (29)), it is quite meaningful to apply the triangle inequality to (37) to obtain the following result, which gives upper and lower bounds of the generalized H_2 norms together with the associated convergence rate of $1/\sqrt{N}$.

Corollary 3. *The following inequalities hold:*

$$\|\mathcal{F}_{N\mathrm{G}}\|_{(\infty,\infty)/(2,2)} \leq \|\mathcal{F}\|_{(\infty,\infty)/(2,2)} \leq \|\mathcal{F}_{N\mathrm{G}}\|_{(\infty,\infty)/(2,2)} + \frac{K_{ND}^{[\infty]}}{\sqrt{N}} + \frac{K_{N0}^{[\infty]}}{N} \tag{42}$$

$$\|\mathcal{F}_{N\mathrm{G}}\|_{(\infty,2)/(2,2)} \leq \|\mathcal{F}\|_{(\infty,\infty)/(2,2)} \leq \|\mathcal{F}_{N\mathrm{G}}\|_{(\infty,2)/(2,2)} + \frac{K_{ND}^{[2]}}{\sqrt{N}} + \frac{K_{N0}^{[2]}}{N} \tag{43}$$

4.3. Discretization of Continuous-Time Generalized Plant

In this subsection, we give alternative methods for exactly computing $\|\mathcal{F}_{N\mathrm{G}}\|_{(\infty,p)/(2,2)}$ $(p = \infty, 2)$ through discretization treatment of the generalized plant P. For $i = 0, 1, \cdots, N-1$, we first note by (25) that

$$\left(\mathbf{H}_{N0}\mathbf{M}_1 \begin{bmatrix} x \\ u \end{bmatrix}\right)(\theta) = M_1 (A'_{2d})^i \begin{bmatrix} x \\ u \end{bmatrix} \quad (ih' \le \theta < (i+1)h') \tag{44}$$

$$\begin{aligned}(\mathbf{H}_{N0}\mathbf{D}_{11}w)(\theta) &= \int_0^{ih'} C_1 \exp(A(ih' - \tau))B_1 w(\tau)d\tau \\ &= C_1 \int_0^{ih'} \exp(A(ih' - \tau))B_1 w(\tau)d\tau =: C_1 \mathbf{B}'_{1i} w \quad (ih' \le \theta < (i+1)h')\end{aligned} \tag{45}$$

Note that $\mathbf{B}'_{1i}$ corresponds to $\mathbf{B}_1$ with the underlying horizon replaced by $[0, ih')$. This, together with essentially the same arguments as the derivation of (40), immediately implies that for $p = \infty$, $\|\mathcal{F}_{N\mathrm{G}}\|_{(\infty,\infty)/(2,2)}$ coincides with the induced norm of the infinite-dimensional matrix

$$F_{N\mathrm{G}} := \left[\overline{C_1} V'_{dN} \quad \overline{M_1} A'_{2dN} C_\Sigma J_\Sigma V_h \quad \overline{M_1} A'_{2dN} C_\Sigma \mathcal{A} J_\Sigma V_h \quad \cdots \right] \tag{46}$$

from l_2 to $\mathbb{R}^{Mn_z}_\infty$, where $\overline{(\cdot)}$ denotes $\mathrm{diag}[(\cdot), \cdots, (\cdot)]$ consisting of N copies of $(\cdot)$ and

$$V'_{dN} := \begin{bmatrix} 0 \\ V_{h'} \\ \vdots \\ V_{(N-1)h'} \end{bmatrix}, \quad A'_{2dN} = \begin{bmatrix} I \\ A'_{2d} \\ \vdots \\ (A'_{2d})^{N-1} \end{bmatrix} \tag{47}$$

Note that we can define $V_{ih'}$ $(i = 1, \cdots, N-1)$ from (34) so that they have the same number of columns as V_h. For $p = 2$ on the other hand, $\|\mathcal{F}_{N\mathrm{G}}\|_{(\infty,2)/(2,2)}$ coincides with

$$\max_{1 \le j \le N} \|F_{N\mathrm{G}j}\|_{2/2} \tag{48}$$

where $F_{N\mathrm{G}j}$ denotes the matrix with n_z rows constituting the jth block row of $F_{N\mathrm{G}}$ and $\|\cdot\|_{2/2}$ denotes the induced norm from l_2 to $\mathbb{R}^{n_z}_2$. Here, let us further define $D_{N\mathrm{G}} := \overline{C_1} V'_{dN}$ and $M_{N0} := \overline{M_1} A'_{2dN} C_\Sigma$, for simplicity. Then, we readily see that the matrix $F_{N\mathrm{G}}$ corresponds to the (reversed) 'last' block row of the infinite-dimensional Toeplitz matrix representation of the input/output relation of the discrete-time system

$$\Sigma_{N\mathrm{G}} : \begin{cases} \xi_{k+1} &= \mathcal{A}\xi_k + J_\Sigma V_h w_k \\ z_k &= M_{N0}\xi_k + D_{N\mathrm{G}} w_k \end{cases} \tag{49}$$

Furthermore, the discrete-time system $\varSigma_{N\mathrm{G}}$ of (49) coincides with the closed-loop system obtained by connecting $\varPsi$ to the discrete-time generalized plant

$$P_{N\mathrm{G}}: \begin{cases} x_{k+1} &= A_d x_k + V_h w_k + B_{2d} u_k \\ z_k &= C_{Nd} x_k + D_{N\mathrm{G}} w_k + D_{Nd} u_k \\ y_k &= C_{2d} x_k \end{cases} \tag{50}$$

where the matrices $C_{Nd}: \mathbb{R}_2^n \to \mathbb{R}_p^{Nn_z}$ and $D_{Nd}: \mathbb{R}_2^{n_u} \to \mathbb{R}_p^{Nn_z}$ are given by

$$\begin{bmatrix} C_{Nd} & D_{Nd} \end{bmatrix} := \overline{M_1} A'_{2dN} \tag{51}$$

Next, for $p = 2$, we define N subsystems $P_{N\mathrm{G}j}$ $(j = 1, \cdots, N)$ of $P_{N\mathrm{G}}$ constructed by replacing $C_{Nd}, D_{N\mathrm{G}}$ and D_{Nd} in $P_{N\mathrm{G}}$ with the matrices with n_z rows constituting the jth block rows of C_{Nd}, $D_{N\mathrm{G}}$ and D_{Nd}, respectively. Furthermore, let $\varSigma_{N\mathrm{G}j}$ be the discrete-time system obtained by connecting $\varPsi$ to the discrete-time generalized plant $P_{N\mathrm{G}j}$. We then readily have the following result giving alternative characterizations of the gridding approximation $\|\mathcal{F}_{N\mathrm{G}}\|_{(\infty,p)/(2,2)}$ $(p = \infty, 2)$ through the computations of the induced norms from l_2 to l_∞ equipped with two spatial (∞ and 2) norms for the l_∞ norm discussed in [8].

Theorem 4. *The following relations hold:*

$$\|\mathcal{F}_{N\mathrm{G}}\|_{(\infty,\infty)/(2,2)} = \|\varSigma_{N\mathrm{G}}\|_{(\infty,\infty)/(2,2)} \tag{52}$$

$$\|\mathcal{F}_{N\mathrm{G}}\|_{(\infty,2)/(2,2)} = \max_{1 \le j \le N} \|\varSigma_{N\mathrm{G}j}\|_{(\infty,2)/(2,2)} \tag{53}$$

The relevant studies associated with the generalized H_2 norms of sampled-data systems in [14, 15] were confined to the analysis problem and extending those arguments to allow controller synthesis (nor deriving the upper bounds in Corollary 3) has not been discussed. In this connection, the above theorem can be interpreted as providing a fundamental step to addressing the controller synthesis problem of minimizing the generalized H_2 norms of sampled-data systems by working on $\varSigma_{N\mathrm{G}}$. Indeed, one would expect that such a controller synthesis problem could be approximately reducible to that of minimizing the discrete-time generalized H_2 norms through the discretized generalized plant $P_{N\mathrm{G}}$ in (50) if N is relatively large (because the last two terms in (42) and (43) tend to 0 at the rate of $1/\sqrt{N}$). This expectation can actually be fully justified and thus the arguments in the present paper eventually have a further crucial impact on the associated controller synthesis problem. However, it is only after the present arguments have been adequately extended as briefly discussed for the SISO case with only $p = \infty$ in [16]. The reason why the present arguments alone cannot immediately justify the above expectation is that $K_{N0}^{[p]}$ ($p = \infty, 2$) in Corollary 3 depend on the controller Ψ itself as seen from (38). The arguments in [16] and their extension to the MIMO case and $p = 2$ require more involved treatment and space, and thus it is not suitable to discuss the full details in this paper. Instead, we provide a relevant controller synthesis procedure only as a sort of conjecture so that the meaning and an ultimate research goal of the present study will be clearly understood. That is, once we generalize the arguments in [16] and formally establish the claim that

> the discrete-time controller Ψ minimizing $\|\Sigma_{N\mathrm{G}}\|_{(\infty,p)/(2,2)}$ ($p = \infty, 2$) is a suboptimal controller for the sampled-data system Σ_{SD} under the generalized H_2 norm $\|\Sigma_{\mathrm{SD}}\|_{(\infty,p)/(2,2)}$ ($p = \infty, 2$) in the sense that the gap between the perfomance by the designed controller and that by the optimal controller for Σ_{SD} is ensured to be bounded by an order of $1/\sqrt{N}$

(the details of this claim will be presented elsewhere due to limited space), then we are led to the following consequence:

> the optimal controller synthesis for Σ_{SD} virtually reduces to that for $\Sigma_{N\mathrm{G}}$ (with the approximately discretized generalized plant $P_{N\mathrm{G}}$ in (50)) with a sufficiently large N.

The latter problem is simply that for discrete-time systems and can be tackled by only sightly modifying the arguments in [21, 22] for the LMI-based synthesis method for the optimal controller minimizing the H_2 norm of the closed-loop system; for example, if we consider the extension of the arguments in [22], we are immediately led to the minimization problem of γ under the LMI constraints

$$\begin{bmatrix} P & J & A_d X + B_{2d} L & A_d + B_{2d} R C_{2d} & V_h \\ * & H & Q & Y A_d + F C_{2d} & Y V_h \\ * & * & X + X^T - P & I + S^T - J & 0 \\ * & * & * & Y + Y^T - H & 0 \\ * & * & * & * & I \end{bmatrix} > 0 \tag{54}$$

$$\begin{bmatrix} W & C_{Nd}X + D_{Nd}L & C_{Nd} + D_{Nd}RC_{2d} & D_{N\mathrm{G}} \\ * & X + X^T - P & I + S^T - J & 0 \\ * & * & Y + Y^T - H & 0 \\ * & * & * & I \end{bmatrix} > 0 \tag{55}$$

$$\mu_p(W) < \gamma^2 \tag{56}$$

where the decision variables are the matrices Q, F, L, R, X, Y, $P > 0$, J, $H > 0$, S and $W > 0$ together with the scalar $\gamma > 0$. Furthermore, $\mu_\infty(W)$ denotes $\max_{i=1,\ldots,N} d_{\max}(W_{ii})$, where $W_{ii} \in \mathbb{R}^{n_z \times n_z}$ denotes the ith diagonal submatrix of $W =: (W_{ij})_{i,j=1,\ldots,N} \in \mathbb{R}^{Nn_z \times Nn_z}$ (hence $\mu_\infty(W)$ can actually be also represented simply as $d_{\max}(W)$), while $\mu_2(W)$ denotes $\max_{i=1,\ldots,N} \lambda_{\max}(W_{ii})$. The LMI condition for the optimal H_2 controller synthesis in [22] involves $\mathrm{tr}(W) < \gamma^2$ instead of (56), and this will be the only essential difference arising in the synthesis of the optimal controller minimizing the corresponding generalized H_2 norm.

5. Numerical Example

This section examines the effectiveness of the computation methods developed in this paper through a numerical example.

We consider the 5-mass-spring-damper system shown in Figure 2, where $m_1, \ldots, m_5$ denote the masses, $k_1, \ldots, k_5$ denote the spring constants, $c_1, \ldots, c_5$ denote the damper constants, $l_1, \ldots, l_5$ denote the displacements of masses from their equilibrium positions, $d_1, \ldots, d_5$ denote the unknown disturbances in L_2 affecting the masses $m_1, \ldots, m_5$, respectively, and u denotes the control input force applied on the mass m_5. Then, the motion of this system is given

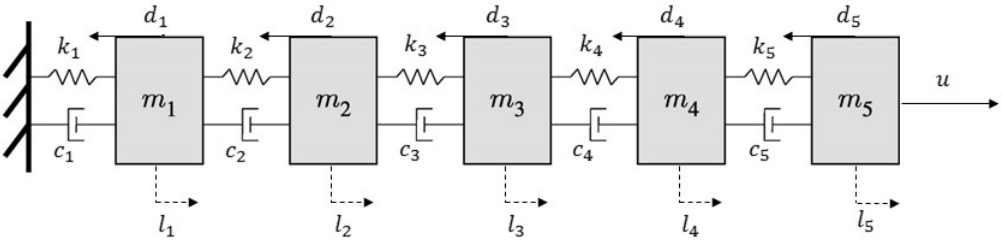

Figure 2: 5-mass-spring-damper system.

by the 10th-order state equation

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0_{5\times5} & I_5 \\ A_{mk} & A_{mc} \end{bmatrix} x + \begin{bmatrix} 0_{5\times5} \\ \mathrm{diag}[-\frac{1}{m_1},\dots,-\frac{1}{m_5}] \end{bmatrix} w + \begin{bmatrix} 0_{9\times1} \\ \frac{1}{m_5} \end{bmatrix} u \\ &=: Ax + B_1 w + B_2 u \end{aligned} \tag{57}$$

where $x := \begin{bmatrix} l_1 & \cdots & l_5 & \dot{l}_1 & \cdots & \dot{l}_5 \end{bmatrix}^T$, $w := [d_1 \ \cdots \ d_5]^T$ and

$$A_{mk} := \begin{bmatrix} -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_2+k_3}{m_2} & \frac{k_3}{m_2} & 0 & 0 \\ 0 & \frac{k_3}{m_3} & -\frac{k_3+k_4}{m_3} & \frac{k_4}{m_3} & 0 \\ 0 & 0 & \frac{k_4}{m_4} & -\frac{k_4+k_5}{m_4} & \frac{k_5}{m_4} \\ 0 & 0 & 0 & \frac{k_5}{m_5} & -\frac{k_5}{m_5} \end{bmatrix} \tag{58}$$

$$A_{mc} := \begin{bmatrix} -\frac{c_1+c_2}{m_1} & \frac{c_2}{m_1} & 0 & 0 & 0 \\ \frac{c_2}{m_2} & -\frac{c_2+c_3}{m_2} & \frac{c_3}{m_2} & 0 & 0 \\ 0 & \frac{c_3}{m_3} & -\frac{c_3+c_4}{m_3} & \frac{c_4}{m_3} & 0 \\ 0 & 0 & \frac{c_4}{m_4} & -\frac{c_4+c_5}{m_4} & \frac{c_5}{m_4} \\ 0 & 0 & 0 & \frac{c_5}{m_5} & -\frac{c_5}{m_5} \end{bmatrix} \tag{59}$$

Suppose that we only have the measurements of the mass positions, i.e., $y = [l_1 \ \cdots l_5]^T$, so that $C_2 = [I_5 \ 0_{5\times5}]$. We further suppose that the controlled output is given by

$$z := \begin{bmatrix} \sqrt{\frac{m_1}{2}}\dot{l}_1 & \cdots & \sqrt{\frac{m_5}{2}}\dot{l}_5 \end{bmatrix}^T \tag{60}$$

which implies that each entry of z corresponds to (the square root of) the kinetic energy of each mass. Hence, $\|z\|_p^2$ represents the largest energy of the masses when $p = \infty$ while it represents the total energy of the masses when $p = 2$. Thus, both $\|z\|_\infty$ and $\|z\|_2$ are suitable as a possible measure for the control performance of the controller Ψ with respect to the unknown disturbance in L_2 when the objective is to suppress the vibration of the masses in some way or another. This problem leads to

$$C_1 := \begin{bmatrix} 0_{5\times5} & \mathrm{diag}[\sqrt{\frac{m_1}{2}},\dots,\sqrt{\frac{m_5}{2}}] \end{bmatrix} \tag{61}$$

$$D_{12} := 0_{5\times1} \tag{62}$$

and computing the generalized H_2 norms $\|\mathcal{F}\|_{(\infty,p)/(2,2)}$ $(p = \infty, 2)$ between w and z. For simplicity, let us assume in the following that $m_1 = \cdots =$

$m_5 = 1,\ k_1 = \cdots = k_5 = 0.5$ and $c_1 = \cdots = c_5 = 0.2$ and $h = 0.5$. We next consider the case when the discrete-time controller Ψ is described by the static output feedback gain

$$-\begin{bmatrix}0.5 & 0.5 & 0.5 & 0.5 & 1\end{bmatrix} \tag{63}$$

We compute estimates of the generalized H_2 norms $\|\mathcal{F}\|_{(\infty,p)/(2,2)}$ $(p = \infty, 2)$ by taking the gridding approximation parameter N ranging from 200 to 4000. The results of the estimate $\|\Sigma_{N\mathrm{G}}\|_{(\infty,\infty)/(2,2)}$ and the error bound $K_{ND}^{[\infty]}/\sqrt{N} + K_{N0}^{[\infty]}/N$ associated with (42) are shown in Table 1, while the relevant results associated with (43) are shown in Table 2.

Upper bounds as well as lower bounds of $\|\mathcal{F}\|_{(\infty,p)/(2,2)}$ $(p = \infty, 2)$ can be obtained immediately from Corollary 3 by using the results given in Tables 1 and 2 and their gaps tend to 0 with the rate no slower than $1/\sqrt{N}$. This observation demonstrates the effectiveness of the theoretical results derived in this paper for the numerical analysis of $\|\mathcal{F}\|_{(\infty,p)/(2,2)}$ $(p = \infty, 2)$.

Table 1: Computation results for $\|\Sigma_{N\mathrm{G}}\|_{(\infty,\infty)/(2,2)}$ and error bound.

N	200	500	1000	2000	4000
$\|\Sigma_{N\mathrm{G}}\|_{(\infty,\infty)/(2,2)}$	4.1043	4.1043	4.1043	4.1043	4.1043
$K_{ND}^{[\infty]}/\sqrt{N} + K_{N0}^{[\infty]}/N$	0.0753	0.0383	0.0238	0.0152	0.0099

Table 2: Computation results for $\max_{1\le j\le N} \|\Sigma_{N\mathrm{G}j}\|_{(\infty,2)/(2,2)}$ and error bound.

N	200	500	1000	2000	4000
$\max_{1\le j\le N} \|\Sigma_{N\mathrm{G}j}\|_{(\infty,2)/(2,2)}$	5.6696	5.6696	5.6696	5.6696	5.6696
$K_{ND}^{[2]}/\sqrt{N} + K_{N0}^{[2]}/N$	0.0957	0.0465	0.0279	0.0172	0.0109

6. Conclusion

This paper provided a method for computing the upper bounds as well as the lower bounds of the generalized H_2 norms (i.e., the induced norms from L_2 to L_∞) in MIMO LTI sampled-data systems through an operator-theoretic gridding approximation approach. As a fundamental step to tackling the optimal controller synthesis problem of minimizing the generalized H_2 norms of sampled-data systems, an approximately equivalent discretization method of the generalized plant was discussed in this paper; the corresponding l_∞/l_2-induced norms give the lower bounds and are readily computable [8, 9]. It was further remarked that the discretized plant can eventually be justified as what should be used in reducing the associated optimal controller synthesis problem for sampled-data systems to the discrete-time controller synthesis problem of minimizing the discrete-time generalized H_2 norms (i.e., l_∞/l_2 induced norms); due to limited space, the basic idea was stated only as a sort of conjecture to stress the meaning and an ultimate research goal of the present study but further details will be presented elsewhere. It was also shown that the gaps between the bounds converge to 0 at the rate of $1/\sqrt{N}$, where N is the gridding approximation parameter with which the sampling interval is fractioned in the analysis. We then examined effectiveness of the developed method through a numerical example.

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References

[1] T. Chen, B. A. Francis, H_2-optimal sampled-data control, *IEEE Trans. Automat. Contr.* 36(4)(1991), 387–397.

[2] B. Bamieh, J. B. Pearson, The H^2 problem for sampled-data systems, *Syst. Control Lett.* 19(1)(1992) 1–12.

[3] T. Hagiwara, M. Araki, FR-operator approach to the H_2 analysis and synthesis of sampled-data systems, *IEEE Trans. Automat. Contr.* 40(8)(1995) 1411–1421.

[4] P. P. Khargonekar, N. Sivashankar, H_2 optimal control for sampled-data systems, *Syst. Control Lett.* 17(6)(1991) 425–436.

[5] V. Chellabonia, W. M. Haddad, D. S. Bernstein, D. A. Wilson, Induced convolution operator norms of linear dynamical systems, *Math. Control Signals Systems* 13(3)(2000) 216–239.

[6] M. A. Rotea, The generalized H_2 control problem, *Automatica* 29(2)(1993) 373–385.

[7] D. A. Wilson, Convolution and Hankel operator norms for linear systems, *IEEE Trans. Automat. Contr.* 34(1)(1989) 94–97.

[8] V. Chellabonia, W. M. Haddad, D. S. Bernstein, D. A. Wilson, Induced convolution operator norms for discrete-time linear systems, *Proc. 38th IEEE Conference on Decision and Control*, (1999), 487–492.

[9] D. A. Wilson, M. A. Nekoui, G. D. Halikias, An LQR weight selection approach to the discrete generalized H_2 control problem, *Int. J. Control* 71(1)(1998) 93–101.

[10] B. Bamieh, J. B. Pearson, B. A. Francis, A. Tannenbaum, A lifting technique for linear periodic systems with applications to sampled-data systems, *Syst. Control Lett.* 17(2)(1991) 79–88.

[11] B. Bamieh, J. B. Pearson, A general framework for linear periodic systems with application to H^∞ sampled-data systems, *IEEE Trans. Automat. Contr.* 37(4)(1992) 418–435.

[12] H. T. Toivonen, Sampled-data control of continuous-time systems with an H_∞ optimality criterion, *Automatica* 28(1)(1992) 45–54.

[13] Y. Yamamoto, A function space approach to sampled-data control systems and their tracking problems, *IEEE Trans. Automat. Contr.* 39(4)(1994) 703–713.

[14] G. G. Zhu, R. E. Skelton, L_2 to L_∞ gains for sampled-data systems, *Int. J. Control* 61(1)(1995) 19–32.

[15] J. H. Kim, T. Hagiwara, Extensive theoretical/numerical comparative studies on H_2 and generalized H_2 norms in sampled-data systems, *Int. J. Control* accepted for publication (dx.doi.org/10.1080/00207179.2016.1257158).

[16] J. H. Kim, T. Hagiwara, A study on discretization approach to the L_∞/L_2 optimal controller synthesis problem in sampled-data systems, *Proc. 55th IEEE Conference on Decision and Control* (2016) 4021–4026.

[17] J. H. Kim, T. Hagiwara, Computation of the induced norm from L_2 to L_∞ in SISO sampled-data systems: discretization approach with convergence rate analysis, *Proc. 54th IEEE Conference on Decision and Control* (2015) 1750–1755.

[18] T. Hagiwara, H. Umeda, Modified fast-sample/fast-hold approximation for sampled-data system analysis, *Eur. J. Control* 14(4)(2008) 286–296.

[19] J. H. Kim, T. Hagiwara, L_∞-induced norm analysis of sampled-data systems via piecewise constant and linear approximations, *Automatica* 51(1)(2015) 223–232.

[20] J. H. Kim, T. Hagiwara, Further results on the L_1 analysis of sampled-data systems via kernel approximation approach, *Int. J. Control* 89(8)(2016) 1684–1697.

[21] C. W. Scherer, P. Gahinet, M. Chilali, Multi-objective output-feedback control via LMI optimization, *IEEE Trans. Automat. Contr.* 42(7)(2002) 896–911.

[22] M. C. de Oliveira, J. C. Geromel, J. Bernussou, Extended H_2 and H_∞ norm characterizations and controller parametrizations for discrete-time systems, *Int. J. Control* 75(9)(2002) 666-679.

Appendix A. Proof of Lemmas

This appendix is concerned with the proofs of Lemmas given in this paper.

Proof of Lemma 1:

It readily follows that

$$\begin{aligned}
&\|(I-\mathbf{H}_{N0})\mathbf{D}_{11}\|_{(\infty,p)/(2,2)} = \max_{0\le i\le N-1}\sup_{0\le\theta'<h'}\sup_{\|w\|_{(2,2)}\le 1}\Big|\int_0^{ih'+\theta'} C_1 \\
&\exp(A(ih'+\theta'-\tau)B_1 w(\tau)d\tau - \int_0^{ih'} C_1\exp(A(ih'-\tau)B_1 w(\tau)d\tau\Big|_p \\
&\le \max_{0\le i\le N-1}\sup_{0\le\theta'<h'}\sup_{\|w\|_{(2,2)}\le 1}\Big|\int_{ih'}^{ih'+\theta'} C_1\exp(A(ih'+\theta'-\tau)B_1 w(\tau)d\tau\Big|_p \\
&+ \max_{0\le i\le N-1}\sup_{0\le\theta'<h'}\sup_{\|w\|_{(2,2)}\le 1}\Big|C_1(\exp(A\theta')-I)\int_0^{ih'}\exp(A(ih'-\tau))B_1 w(\tau)d\tau\Big|_p \\
&\le \sup_{\|w\|_{(2,2)}\le 1}\Big|\int_0^{h'} C_1\exp(A(h'-\tau)B_1 w(\tau)d\tau\Big|_p + \sup_{0\le\theta'<h'}\Big|C_1(\exp(A\theta')-I)\Big|_{p/2} \\
&\cdot \sup_{\|w\|_{(2,2)}\le 1}\Big|\int_0^{(N-1)h'}\exp(A((N-1)h'-\tau)B_1 w(\tau)d\tau\Big|_2 \qquad \text{(A.1)}
\end{aligned}$$

Here, we first consider the first term of (A.1). For $p = \infty$, applying the continuous-time Cauchy-Schwarz inequality leads to

$$\begin{aligned}
&\sup_{\|w\|_{(2,2)}\le 1} \left| \int_0^{h'} C_1 \exp(A(h'-\tau)B_1 w(\tau)d\tau \right|_\infty^2 \\
&= \max_{1\le j\le n_z} \int_0^{h'} C_{1j} \exp(A(h'-\theta'))B_1 B_1^T \exp(A^T(h'-\theta'))C_{1j}^T d\theta' \\
&= \max_{1\le j\le n_z} C_{1j} W_{h'} C_{1j}^T = d_{\max}^{1/2}(C_1 W_{h'} C_1^T) \qquad \text{(A.2)}
\end{aligned}$$

where C_{1j} denotes the jth row of C_1. For $p = 2$, on the other hand, it readily follows that

$$\sup_{\|w\|_{(2,2)}\le 1} \left| \int_0^{h'} C_1 \exp(A(h'-\tau)B_1 w(\tau)d\tau \right|_2 = \|C_1 \mathbf{B}'_{11}\|_{2/2} \qquad \text{(A.3)}$$

(with $\mathbf{B}'_{1i}$ defined in (45)) and this can be exactly computed with the adjoint operator $(\mathbf{B}'_{11})^* : \mathbb{R}_2^n \to (L_2[0,h'))^{n_w}$ of $\mathbf{B}'_{11}$ through the relation

$$\|C_1 \mathbf{B}'_{11}\|_{2/2}^2 = |C_1 \mathbf{B}'_{11}(\mathbf{B}'_{11})^* C_1^T|_{2/2} = \lambda_{\max}(C_1 W_{h'} C_1^T) \qquad \text{(A.4)}$$

Next, let us deal with the two factors in the second term of (A.1). It readily follows from the Taylor expansion of $\exp(A\theta')$ that the first factor satisfies

$$\begin{aligned}
&\sup_{0\le\theta'<h'} |C_1(\exp(A\theta') - I)|_{p/2} \le \sup_{0\le\theta'<h'} \left| C_1 \sum_{i=1}^{\infty} \frac{A^i(\theta')^i}{i!} \right|_{p/2} \\
&\le h'|C_1 A|_{p/2} \sum_{i=1}^{\infty} \frac{|A|_{2/2}^{i-1}(h')^{i-1}}{i!} \le h'|C_1 A|_{p/2} e^{h'|A|_{2/2}} \qquad \text{(A.5)}
\end{aligned}$$

and the second factor is nothing but $\|\mathbf{B}'_{1(N-1)}\|_{2/2} = |V_{(N-1)h'}|_{2/2}$. Combining the above arguments lead to the first assertion of Lemma 1.

The second assertion of Lemma 1 follows if we note that applying the continuous-time Cauchy-Schwarz inequality to the first term of (A.1) derives

$$
\begin{aligned}
&\sup_{\|w\|_{(2,2)}\le 1} \left| \int_0^{h'} C_1 \exp(A(h'-\theta')) B_1 w(\theta') d\theta' \right|_p \\
&\le \sup_{\|w\|_{(2,2)}\le 1} \int_0^{h'} |C_1 \exp(A(h'-\theta')) B_1 w(\theta')|_p \, d\theta' \\
&\le \sup_{\|w\|_{(2,2)}\le 1} \int_0^{h'} \big|C_1 \exp(A(h'-\theta')) B_1|_{p/2} \cdot |w(\theta')\big|_2 \, d\theta' \\
&\le \Big(\int_0^{h'} |C_1 \exp(A(h'-\theta')) B_1|_{p/2}^2 d\theta' \Big)^{1/2} \cdot \sup_{\|w\|_{(2,2)}\le 1} \Big(\int_0^{h'} |w(\theta')|_2^2 d\theta' \Big)^{1/2} \\
&\le \sqrt{h'} |C_1|_{p/2} e^{h'|A|_{2/2}} \cdot |B_1|_{2/2} \le \sqrt{h} |C_1|_{p/2} e^{h|A|_{2/2}} \cdot |B_1|_{2/2} \qquad \text{(A.6)}
\end{aligned}
$$

and that $V_{(N-1)h'} V_{(N-1)h'}^T \le V_h V_h^T$.

Proof of Lemma 2:

It immediately follows that

$$
\begin{aligned}
&\|(I - \mathbf{H}_{N0}) \mathbf{M}_1\|_{(\infty,p)/2} \\
&= \max_{0\le i\le N-1} \sup_{0\le\theta'<h'} |M_1 \exp(A_2(ih'+\theta')) - M_1 \exp(A_2(ih'))|_{p/2} \\
&= \max_{0\le i\le N-1} \sup_{0\le\theta'<h'} |M_1 (\exp(A_2\theta') - I)(A'_{2d})^i|_{p/2} \\
&\le \sup_{0\le\theta'<h'} |M_1(\exp(A_2\theta') - I)|_{p/2} \cdot \max_{0\le i\le N-1} |(A'_{2d})^i|_{2/2} \qquad \text{(A.7)}
\end{aligned}
$$

Here, applying the essentially the same arguments in (A.5) to the first term of the right-hand-side of (A.7) readily derives

$$
\sup_{0\le\theta'<h'} |M_1(\exp(A_2\theta') - I)|_{p/2} \le h' |M_1 A_2|_{p/2} e^{h'|A_2|_{2/2}} \qquad \text{(A.8)}
$$

This together with (A.7) clearly leads to the first assertion of Lemma 2. The second assertion of Lemma 2 follows if we note that

$$
e^{h'|A_2|_{2/2}} \cdot \max_{1\le i\le N-1} |(A'_{2d})^i|_{2/2} \le e^{h'|A_2|_{2/2}} \cdot e^{(N-1)h'|A_2|_{2/2}} = e^{h|A_2|_{2/2}} \qquad \text{(A.9)}
$$