# Further Results on the $L_1$ Analysis of Sampled-Data Systems via Kernel Approximation Approach

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(*v5.0 released February 2015*)

This paper gives two methods for the $L_1$ analysis of sampled-data systems, by which we mean computing the $L_\infty$-induced norm of sampled-data systems. This is achieved by developing what we call the kernel approximation approach in the setting of sampled-data systems. We first consider the lifting treatment of sampled-data systems and give an operator theoretic representation of their input/output relation. We further apply the fast-lifting technique by which the sampling interval $[0, h)$ is divided into $M$ subintervals with an equal width, and provide methods for computing the $L_\infty$-induced norm. In contrast to a similar approach developed earlier called the input approximation approach, we use an idea of kernel approximation, in which the kernel function of an input operator and the hold function of an output operator are approximated by piecewise constant or piecewise linear functions. Furthermore, it is shown that the approximation errors in the piecewise constant approximation or piecewise linear approximation scheme converge to 0 at the rate of $1/M$ or $1/M^2$, respectively. In comparison with the existing input approximation approach, in which the input function (rather than the kernel function) of the input operator is approximated by piecewise constant or piecewise linear functions, we show that the kernel approximation approach gives improved computation results. More precisely, even though the convergence rates in the kernel approximation approach remain qualitatively the same as those in the input approximation approach, the newly developed former approach could lead to quantitatively improved approximation errors than the latter approach particularly when the piecewise linear approximation scheme is taken. Finally, a numerical example is given to demonstrate the effectiveness of the kernel approximation approach with this scheme.

**Keywords:** sampled-data systems, $L_\infty$-induced norm, kernel approximation approach, fast-lifting

## 1. Introduction

Sampled-data systems (Araki and Ito, 1996; Bamieh and Dahleh, 1993; Bamieh and Pearson, 1992a,b; Chen and Francis, 1991; Dullerud and Francis, 1992; Hagiwara and Araki, 1995; Hagiwara and Okada, 2009; Hagiwara and Umeda, 2008; Keller and Anderson, 1992; Khargonekar and Sivashankar, 1991; Kim and Hagiwara, 2015a,c, 2016; Mirkin and Rotstein, 1999a,b; Sivashankar and Khargonekar, 1992; Tadmor, 1992; Toivonen, 1992; Yamamoto, 1994; Yamamoto and Madievski, 1999) arise in feedback control when continuous-time plants are controlled by discrete-time controllers, and they occur naturally in feedback control applications, such as process control, attitude control and so on. Thus, there have been a number of studies associated with sampled-data systems taking account of their inter-sample behavior, and the studies can be classified by the type of system norms considered, where the typical studies are the $H_\infty$ problem (Bamieh and Pearson, 1992b; Hagiwara and Okada, 2009; Mirkin and Rotstein, 1999a,b; Tadmor, 1992;

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Toivonen, 1992) and the $H_2$ problem (Bamieh and Pearson, 1992a; Chen and Francis, 1991; Hagiwara and Araki, 1995; Khargonekar and Sivashankar, 1991; Mirkin and Rotstein, 1999a,b). Even though $H_\infty$ and $H_2$ norms play important roles in the analysis and synthesis for sampled-data systems relevant to practical control problems, they cannot be used for dealing with the problems of bounded persistent disturbances.

In connection with this, the $L_1$ problem of sampled-data systems, which deals with the $L_\infty$-induced norm of such systems, has been studied (Bamieh and Dahleh, 1993; Dullerud and Francis, 1992; Kim and Hagiwara, 2015a; Sivashankar and Khargonekar, 1992). More precisely, in Bamieh and Dahleh (1993); Dullerud and Francis (1992); Sivashankar and Khargonekar (1992), computation methods for the $L_\infty$-induced norm of sampled-data systems have been developed by introducing the idea of fast-sample/fast-hold (FSFH) approximation; a sampled-data system is "approximated" by a discrete-time system through the FSFH approximation technique (Keller and Anderson, 1992), and it is shown that the $l_\infty$-induced norm of the approximating discrete-time system converges to the $L_\infty$-induced norm of the original sampled-data systems as the FSFH approximation parameter $M$ tends to infinity. However, these studies do not evaluate how close the $l_\infty$-induced norm for a given $M$ is to the exact value of the $L_\infty$-induced norm. In other words, these studies cannot derive readily obtainable upper and lower bounds of the $L_\infty$-induced norm of sampled-data systems. In contrast, our recent study (Kim and Hagiwara, 2015a) derives readily computable upper and lower bounds of the $L_\infty$-induced norm of sampled-data systems by using the ideas of the fast-lifting technique (Hagiwara and Umeda, 2008) and the input approximation approach (Kim and Hagiwara, 2014). The latter approach was first developed by the motivation to compute the $L_\infty[0, h)$-induced norm of compression operators, which are closely related to the $L_1$ analysis problem of continuous-time systems. This approach takes an idea of approximating the input of a relevant operator with piecewise constant or piecewise linear functions by dividing the sampling interval $[0, h)$ into $M$ subintervals with an equal width (without applying sampling of signals), and fast-lifting plays a key role in such approximations aiming at simplifying the $L_\infty$-induced norm computation of sampled-data systems. For the fast-lifting parameter $M$, it was shown that the gap between the upper and lower bounds of the induced norm converges to 0 at the convergence rates of $1/M$ and $1/M^2$ in the piecewise constant approximation scheme and the piecewise linear approximation scheme, respectively.

As a significant advance over the existing result, this paper further develops alternative methods for computing upper and lower bounds of the $L_\infty$-induced norm of sampled-data systems by developing a framework in which the kernel approximation approach (Kim and Hagiwara, 2015b) can be exploited in the study of sampled-data systems; even though the kernel approximation approach was first introduced in Kim and Hagiwara (2015b) for continuous-time systems as an improved approach and effectively uses the same ideas of piecewise constant approximation and piecewise linear approximation schemes, the $h$-periodic nature of sampled-data systems requires us to construct more involved arguments in the present paper, as was the case with the relationship with the input approximation approach in sampled-data systems (Kim and Hagiwara, 2015a) and that in continuous-time systems (Kim and Hagiwara, 2015b). Nevertheless, it is shown that the advantage of the kernel approximation approach over the input approximation approach is inherited to the $L_1$ analysis of sampled-data systems by providing readily computable upper and lower bounds of the induced norm and showing that their gap becomes smaller.

The organization of this paper is as follows. We first review sampled-data systems and the lifting approach to such systems in Section 2. We next provide computation methods for the $L_\infty$-induced norm of sampled-data systems in Section 3. More precisely, we apply the ideas of fast-lifting (Hagiwara and Umeda, 2008) and kernel approximation (Kim and Hagiwara, 2015b) to the $L_\infty$-induced norm analysis of sampled-data systems. By an adequate introduction of piecewise constant approximation and piecewise linear approximation schemes, we show that the $L_\infty$-induced norm of sampled-data systems is approximated by the $\infty$-norm of a suitably constructed matrix in each scheme, and further give an upper bound and lower bound of the $L_\infty$-induced norm that can be obtained easily. We then show that the approximation errors stemming from the approximation treatment converge to 0 at the rates of $1/M$ or $1/M^2$ in the piecewise constant approximation or piecewise linear approximation scheme, respectively. It is further shown that (even though the above 'qualitative assertions' on the convergence rates remain the same as those in the input approximation approach) the kernel approximation introduced in this paper is quantitatively superior to the existing input approximation approach. We finally demonstrate the effectiveness of the developed computation methods through a numerical example in Section 4; the present paper is an extended and enhanced version of our conference paper (Kim and Hagiwara, 2015c) in studying such an example and providing proofs of the arguments among other amendments in the details of descriptions.

In the following, the notations $\mathbb{N}$, $\mathbb{R}^\nu$ and $\mathcal{K}_\nu$ are used to mean the set of positive integers, the set of $\nu$-dimensional real vectors and the Banach space $(L_\infty[0,h))^\nu$, respectively. The notations $I_\nu \in \mathbb{R}^{\nu\times\nu}$ and $0_{\nu\times\mu} \in \mathbb{R}^{\nu\times\mu}$ are used to denote the identity matrix on $\mathbb{R}^\nu$ and the $\nu \times \mu$ zero matrix, respectively. We further use the notation $\mathbb{N}_0$ to imply $\mathbb{N} \cup \{0\}$. The notation $\|\cdot\|$ is used to mean either the $L_\infty[0,h)$ norm of a vector valued function, i.e.,

$$\|f(\cdot)\| := \max_i \operatorname*{ess\,sup}_{0\le t<h} |f(t)|, \tag{1}$$

the $L_\infty[0,h)$-induced norm of an operator (or these with $h$ replaced by $h/M$), or the $\infty$-norm of a finite-dimensional matrix, whose distinction will be clear from the context.

## 2. Lifted Representation of Sampled-Data Systems

Let us consider the stable sampled-data system $\Sigma_{\rm SD}$ shown in Figure 1, where $P$ denotes the continuous-time linear time-invariant (LTI) generalized plant, while $\Psi$, $\mathcal{H}$ and $\mathcal{S}$ denote the discrete-time LTI controller, the zero-order hold and the ideal sampler, respectively, operating with sampling period $h$ in a synchronous fashion. Solid lines and dashed lines in Fig. 1 are used to represent continuous-time signals and discrete-time signals, respectively. Suppose that $P$ and $\Psi$ are described respectively by

$$P: \begin{cases} \dfrac{dx}{dt} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{11} w + D_{12} u \\ y = C_2 x \end{cases}, \qquad \Psi: \begin{cases} \psi_{k+1} = A_\Psi \psi_k + B_\Psi y_k \\ u_k = C_\Psi \psi_k + D_\Psi y_k \end{cases} \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^{n_w}$, $u(t) \in \mathbb{R}^{n_u}$, $z(t) \in \mathbb{R}^{n_z}$, $y(t) \in \mathbb{R}^{n_y}$, $\psi_k \in \mathbb{R}^{n_\psi}$, $y_k = y(kh)$ and $u(t) = u_k$ $(kh \le t < (k+1)h)$.

Because the sampled-data system $\Sigma_{\rm SD}$ is a hybrid continuous-time/discrete-time system, this system can be viewed as a (periodically) time-varying system in the continuous-time viewpoint. To deal with $\Sigma_{\rm SD}$ as a time-invariant system, we apply the lifting technique (Bamieh and Pearson, 1992b; Toivonen, 1992; Yamamoto, 1994), in which a given $f \in (L_\infty)^\nu$ is dealt with through its

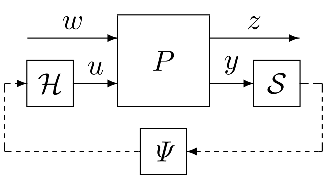


Figure 1. Sampled-data system $\Sigma_{\rm SD}$.

lifting representation $\{\widehat{f}_k\}_{k=0}^{\infty}$ (with sampling period $h$) with $\widehat{f}_k \in \mathcal{K}_\nu$ defined as follows:

$$\widehat{f}_k(\theta) = f(kh+\theta) \quad (0 \le \theta < h) \tag{3}$$

Applying the lifting technique to $w$ and $z$ leads to the lifted representation $\widehat{\Sigma}_{\mathrm{SD}}$ of the sampled-data system $\Sigma_{\mathrm{SD}}$ described by

$$\widehat{\Sigma}_{\mathrm{SD}} : \begin{cases} \xi_{k+1} &= \mathcal{A}\xi_k + \mathcal{B}\widehat{w}_k \\ \widehat{z}_k &= \mathcal{C}\xi_k + \mathcal{D}\widehat{w}_k \end{cases} \tag{4}$$

with $\xi_k := [x_k^T \ \psi_k^T]^T$ ($x_k := x(kh)$), the matrix

$$\mathcal{A} = \begin{bmatrix} A_d + B_{2d}D_\Psi C_{2d} & B_{2d}C_\Psi \\ B_\psi C_{2d} & A_\Psi \end{bmatrix} : \mathbb{R}^{n+n_\Psi} \to \mathbb{R}^{n+n_\Psi} \tag{5}$$

and the operators

$$\mathcal{B} = J_\Sigma \mathbf{B}_1 : \mathcal{K}_{n_w} \to \mathbb{R}^{n+n_\Psi}, \quad \mathcal{C} = \mathbf{M}_1 C_\Sigma : \mathbb{R}^{n+n_\Psi} \to \mathcal{K}_{n_z}, \quad \mathcal{D} = \mathbf{D}_{11} : \mathcal{K}_{n_w} \to \mathcal{K}_{n_z} \tag{6}$$

where

$$A_d := \exp(Ah), \quad B_{2d} := \int_0^h \exp(A\theta)B_2 d\theta, \quad C_{2d} := C_2, \quad C_\Sigma := \begin{bmatrix} I_n & 0_{n\times n_\Psi} \\ D_\Psi C_{2d} & C_\Psi \end{bmatrix} \tag{7}$$

$$\mathbf{B}_1 w = \int_0^h \exp(A(h-\theta))B_1 w(\theta)d\theta, \quad J_\Sigma := \begin{bmatrix} I_n \\ 0_{n_\Psi \times n} \end{bmatrix} \tag{8}$$

$$\left(\mathbf{M}_1 \begin{bmatrix} x \\ u \end{bmatrix}\right)(\theta) = C_0 \exp(A_2\theta) \begin{bmatrix} x \\ u \end{bmatrix}, \quad A_2 := \begin{bmatrix} A & B_2 \\ 0_{n_u\times n} & 0_{n_u\times n_u} \end{bmatrix}, \quad C_0 := \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \tag{9}$$

$$(\mathbf{D}_{11}w)(\theta) = \int_0^\theta C_1 \exp(A(\theta-\tau))B_1 w(\tau)d\tau + D_{11}w(\theta) \tag{10}$$

From the stability assumption of $\Sigma_{\mathrm{SD}}$, $\mathcal{A}$ is stable, i.e., has all its eigenvalues in the open unit disc.

## 3. Computation of the $L_\infty$-Induced Norm of Sampled-Data Systems via Kernel Approximation Approach

This section gives the main results of this paper, i.e., two methods for computing the $L_\infty$-induced norm of the sampled-data system $\Sigma_{\mathrm{SD}}$. More precisely, we apply the fast-lifting technique (Hagiwara and Umeda, 2008) on top of the lifting treatment of the sampled-data system $\Sigma_{\mathrm{SD}}$, and derive upper and lower bounds of its $L_\infty$-induced norm, whose gap converges to zero as the fast-lifting parameter $M$ tends to $\infty$. This is accomplished by introducing the kernel approximation approach (Kim and Hagiwara, 2015b) (as opposed to the input approximation approach (Kim and Hagiwara, 2014)) into the setting of sampled-data systems, through either of the two independent ideas with piecewise constant approximation and piecewise linear approximation schemes. Furthermore, we clarify the associated convergence rates with respect to $M$ for each of the two approximation schemes. The fast-lifting technique mentioned above plays a key role in developing such two piecewise approximation schemes; for $M \in \mathbb{N}$ and $h' := h/M$, fast-lifting (Hagiwara and Umeda, 2008) of a signal $f \in \mathcal{K}_\nu$, denoted by $\check{f} = \mathbf{L}_M f$, is defined by $\check{f} := [(f^{(1)})^T \ \cdots \ (f^{(M)})^T]^T \in (\mathcal{K}'_\nu)^M$ with

$$f^{(i)}(\theta') := f((i-1)h' + \theta') \quad (0 \le \theta' < h'), \tag{11}$$

where $\mathcal{K}'_\nu$ is a shorthand notation for $(L_\infty[0, h'))^\nu$. It is easy to see that $\mathbf{L}_M$ is norm-preserving (i.e., $\|\mathbf{L}_M f\| = \|f\|$), which plays a crucial role in the following arguments.

### 3.1 *Toeplitz Structure of Input/Output Relation and Truncation*

To compute the $L_\infty$-induced norm of the sampled-data system $\Sigma_{\rm SD}$, we first note that the closed-loop input/output relation of the lifted sampled-data system $\widehat{\Sigma}_{\rm SD}$ between $\widehat{w}_k$ and $\widehat{z}_k$ $(k = 0, \cdots, \infty)$ can be described from (4) as follows:

$$\begin{bmatrix} \widehat{z}_0 \\ \widehat{z}_1 \\ \widehat{z}_2 \\ \widehat{z}_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} \mathcal{D} & 0 & \cdots & & & \\ \mathcal{CB} & \mathcal{D} & 0 & \cdots & & \\ \mathcal{CAB} & \mathcal{CB} & \mathcal{D} & 0 & \cdots & \\ \mathcal{CA}^2\mathcal{B} & \mathcal{CAB} & \mathcal{CB} & \mathcal{D} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \widehat{w}_0 \\ \widehat{w}_1 \\ \widehat{w}_2 \\ \widehat{w}_3 \\ \vdots \end{bmatrix} \tag{12}$$

Since the operator on the right hand side has a Toeplitz structure (so that each block row is an extension of the row above it), it follows readily from $\|z\| = \sup_k \|\widehat{z}_k\|$ that the $L_\infty$-induced norm of $\Sigma_{\rm SD}$ (which is the same as the corresponding induced norm of $\widehat{\Sigma}_{\rm SD}$) coincides with the $L_\infty$-induced norm of its "last" block row, i.e., (after reordering columns without affecting the $L_\infty$-induced norm)

$$\mathcal{F} := \begin{bmatrix} \mathcal{D} & \mathcal{CB} & \mathcal{CAB} & \mathcal{CA}^2\mathcal{B} & \cdots \end{bmatrix} \tag{13}$$

It is, however, still difficult to compute $\|\mathcal{F}\|$ since $\mathcal{F}$ consists of an infinite number of columns. To alleviate this difficulty, we first take a sufficiently large $N \in \mathbb{N}$, decompose $\mathcal{F}$ into

$$\mathcal{F} = \mathcal{F}_N^- + \mathcal{F}_N^+ \tag{14}$$

$$\mathcal{F}_N^- := \begin{bmatrix} \mathcal{D} & \cdots & \mathcal{CA}^N\mathcal{B} & 0 & 0 & \cdots \end{bmatrix}, \quad \mathcal{F}_N^+ := \begin{bmatrix} 0 & \cdots & 0 & \mathcal{CA}^{N+1}\mathcal{B} & \mathcal{CA}^{N+2}\mathcal{B} & \cdots \end{bmatrix} \tag{15}$$

and compute the $L_\infty$-induced norm $\|\mathcal{F}_N^-\|$ as accurately as possible while the computation of $\|\mathcal{F}_N^+\|$ is treated in a comparatively simple way (because the latter norm is expected to be small when $N$ is large enough). This basic idea is exactly the same as that in the input approximation approach in the setting of sampled-data systems (Kim and Hagiwara, 2015a), in which the computation of an upper bound of $\|\mathcal{F}_N^+\|$ has also been studied. Hence, the contributions of the present paper lie in developing alternative methods for approximating $\mathcal{F}_N^-$ through the kernel approximation approach and computing upper and lower bounds of its norm. As it turns out, this new direction leads to improved results for the computation of the $L_\infty$-induced norm of sampled-data systems.

### 3.2 *Fast-Lifting Treatment of $\mathcal{F}_N^-$*

In this subsection, we review the fast-lifting treatment of $\mathcal{F}_N^-$, which plays a key role for the introduction of the kernel approximation approach. Because the fast-lifting $\mathbf{L}_M$ has a norm-preserving property, it immediately follows that

$$\|\mathcal{F}_N^-\| = \left\| \begin{bmatrix} \mathbf{L}_M \mathcal{D} \mathbf{L}_M^{-1} & \cdots & \mathbf{L}_M \mathcal{CA}^N \mathcal{B} \mathbf{L}_M^{-1} \end{bmatrix} \right\| \tag{16}$$

To facilitate the treatment of the right-hand side, we introduce $\mathbf{D}'_{11}$, $\mathbf{B}'_1$ and $\mathbf{M}'_1$ defined as $\mathbf{D}_{11}$, $\mathbf{B}_1$ and $\mathbf{M}_1$, respectively, with the interval $[0, h)$ replaced by $[0, h')$ $(= [0, h/M))$, and also introduce the matrices

$$A'_d := \exp(Ah'), \quad A'_{2d} := \exp(A_2 h'), \quad J := \begin{bmatrix} I_n \\ 0_{n_u \times n} \end{bmatrix} \tag{17}$$

Then, (as in the standard arguments employing fast-lifting, e.g., Hagiwara and Umeda (2008)), it readily follows from direct computations that $\mathbf{L}_M \mathcal{D} \mathbf{L}_M^{-1}$ and $\mathbf{L}_M \mathcal{C} \mathcal{A}^j \mathcal{B} \mathbf{L}_M^{-1}$ $(j = 0, \cdots, N)$ in (16) are given respectively by

$$\mathbf{L}_M \mathcal{D} \mathbf{L}_M^{-1} = \overline{\mathbf{M}'_1} \Delta_M^0 \overline{\mathbf{B}'_1} + \overline{\mathbf{D}'_{11}}, \quad \mathbf{L}_M \mathcal{C} \mathcal{A}^j \mathcal{B} \mathbf{L}_M^{-1} = \overline{\mathbf{M}'_1} A'_{2dM} C_\Sigma \mathcal{A}^j J_\Sigma A'_{dM} \overline{\mathbf{B}'_1} \tag{18}$$

where

$$A'_{dM} := \begin{bmatrix} (A'_d)^{M-1} & \cdots I \end{bmatrix}, \quad A'_{2dM} := \begin{bmatrix} I \\ \vdots \\ (A'_{2d})^{M-1} \end{bmatrix}, \quad \Delta_M^0 := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ J & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ (A'_{2d})^{M-2} J & \cdots & J & 0 \end{bmatrix} \tag{19}$$

and $\overline{(\cdot)}$ denotes diag$[(\cdot), \ \cdots, \ (\cdot)]$ consisting of $M$ copies of $(\cdot)$. Hence, the operator matrix on the right hand side of (16) admits the representation

$$\mathcal{F}_{NM}^- = \begin{bmatrix} \overline{\mathbf{M}'_1} \Delta_M^0 \overline{\mathbf{B}'_1} + \overline{\mathbf{D}'_{11}} & \overline{\mathbf{M}'_1} \mathcal{J}_{M0} \overline{\mathbf{B}'_1} & \cdots & \overline{\mathbf{M}'_1} \mathcal{J}_{MN} \overline{\mathbf{B}'_1} \end{bmatrix} \tag{20}$$

where

$$\mathcal{J}_{Mj} := A'_{2dM} C_\Sigma \mathcal{A}^j J_\Sigma A'_{dM} \quad (j = 0, \cdots, N) \tag{21}$$

Applying fast-lifting introduces no approximation treatment but reduces the interval on which the actions of operators are described (i.e., from $[0, h)$ to $[0, h')$). This gives us a better chance to approximate the operators more accurately with tractable ones through the idea of kernel approximation as discussed in the following.

### 3.3 *Kernel Approximation Approach to* $\mathcal{F}_{NM}^-$

This subsection provides main results of the present paper, i.e., a framework for computing $\|\mathcal{F}_{NM}^-\|$ $(= \|\mathcal{F}_N^-\|)$ through the kernel approximation approach (Kim and Hagiwara, 2015b) developed in the setting of sampled-data systems. The kernel approximation approach was first introduced in Kim and Hagiwara (2015b) to compute the $L_\infty$-induced norm of continuous-time LTI systems. More precisely, it was shown therein that the kernel approximation approach leads to more quantitatively effective methods for computing upper and lower bounds of the $L_\infty$-induced norm of continuous-time systems than the existing input approximation approach (Kim and Hagiwara, 2014). Motivated by this achievement, this paper is interested in the extension of the kernel approximation approach to the $L_\infty$-induced norm computation of sampled-data systems and its comparison with the methods through the existing input approximation approach (Kim and Hagiwara, 2015a). More precisely, we consider constant and linear approximations to the 'kernel function' $\exp(A(h' - \theta'))B_1$ of $\mathbf{B}'_1$ together with constant and linear approximations of the 'hold function' $C_0 \exp(A_2 \theta')$ of $\mathbf{M}'_1$, which respectively lead to piecewise constant and piecewise linear approximations of signals if they are viewed on $[0,h)$ rather than $[0,h')$. Furthermore, we aim at deriving the associated convergence rates with respect to the fast-lifting parameter $M$ in these two approximation schemes.

### 3.3.1 Piecewise Constant Approximation Scheme

We introduce the operators $\mathbf{B}'_{\mathrm{k0}} : \mathcal{K}'_{n_w} \to \mathbb{R}^n$, $\mathbf{M}'_{\mathrm{a0}} : \mathbb{R}^{n+n_u} \to \mathcal{K}'_{n_z}$ and $\mathbf{D}'_{\mathrm{a0}} : \mathcal{K}'_{n_w} \to \mathcal{K}'_{n_z}$ defined respectively as

$$\mathbf{B}'_{\mathrm{k0}} w = \int_0^{h'} A'_d B_1 w(\theta') d\theta' \tag{22}$$

$$\left(\mathbf{M}'_{\mathrm{a0}} \begin{bmatrix} x \\ u \end{bmatrix}\right)(\theta') = C_0 \begin{bmatrix} x \\ u \end{bmatrix} \quad (0 \le \theta' < h') \tag{23}$$

$$(\mathbf{D}'_{\mathrm{a0}} w)(\theta') = D_{11} w(\theta') \tag{24}$$

Introducing the operator $\mathbf{B}'_{\mathrm{k0}}$ corresponds to the zero-order approximation of the kernel function $\exp(A(h'-\theta'))B_1 = A'_d \sum_{i=0}^{\infty} \frac{(-A\theta')^i}{i!} B_1$ of the operator $\mathbf{B}'_1$. Such an approximation is essentially the same as that in the continuous-time case in Kim and Hagiwara (2015b), but constitutes the key difference from the input approximation approach in the case of sampled-data systems (Kim and Hagiwara, 2015a). Similarly for $\mathbf{B}'_{\mathrm{k1}}$ relevant to the piecewise linear approximation scheme given later. The operator $\mathbf{M}'_{\mathrm{a0}}$ corresponds to the zero-order approximation of the hold function $C_0 \exp(A_2\theta')$ of the operator $\mathbf{M}'_1$, i.e., the zero-order approximation of the Taylor expansion of the output of $\mathbf{M}'_1$. The operator $\mathbf{D}'_{\mathrm{a0}}$ means the operator of multiplication by the matrix $D_{11}$. The latter two operators are exactly the same as those used for approximating $\mathbf{M}'_1$ and $\mathbf{D}'_{11}$, respectively, in the input approximation approach in the setting of sampled-data systems (Kim and Hagiwara, 2015a). Hence, the subscript 'a' (rather than 'k') is used to indicate approximations.

**Remark 1:** The operators $\mathbf{M}'_1$ and $\mathbf{D}'_{11}$ are comparatively simple to deal with and thus there seem to be little variations for their reasonable approximations. Hence, the main contribution of the present paper over the existing results (Kim and Hagiwara, 2015a) is a new approximation approach to the operator $\mathbf{B}'_1$. Essentially the same comments apply to the following arguments associated with the piecewise linear approximation scheme. As it turns out (see the last paragraph of Subsection 3.5), however, the new method for approximating the operator $\mathbf{B}'_1$ in this paper leads to an improved method for computing upper and lower bounds of the $L_\infty$-induced norm than the existing input approximation approach (Kim and Hagiwara, 2015a).

To proceed with the kernel approximation approach with the piecewise constant approximation scheme, we consider the operator $\mathcal{F}^-_{NM0}$ obtained by replacing $\mathbf{B}'_1$, $\mathbf{M}'_1$ and $\mathbf{D}'_{11}$ with $\mathbf{B}'_{\mathrm{k0}}$, $\mathbf{M}'_{\mathrm{a0}}$ and $\mathbf{D}'_{\mathrm{a0}}$, respectively, in (20):

$$\mathcal{F}^-_{NM\mathrm{k0}} = \begin{bmatrix} \overline{\mathbf{M}'_{\mathrm{a0}}} \Delta^0_M \overline{\mathbf{B}'_{\mathrm{k0}}} + \overline{\mathbf{D}'_{\mathrm{a0}}} & \overline{\mathbf{M}'_{\mathrm{a0}}} \mathcal{J}_{M0} \overline{\mathbf{B}'_{\mathrm{k0}}} & \cdots & \overline{\mathbf{M}'_{\mathrm{a0}}} \mathcal{J}_{MN} \overline{\mathbf{B}'_{\mathrm{k0}}} \end{bmatrix} \tag{25}$$

This paper shows that $\|\mathcal{F}^-_{NM\mathrm{k0}}\|$ can be computed exactly and tends to $\|\mathcal{F}^-_N\|$ as $M$ tends to infinity at the convergence rate of $1/M$. The following two lemmas play important roles in establishing the latter fact.

**Lemma 1:** *The inequality*

$$\|\mathbf{L}_M\mathcal{D}\mathbf{L}_M^{-1} - (\overline{\mathbf{M}'_{\mathrm{a}0}}\Delta_M^0\overline{\mathbf{B}'_{\mathrm{k}0}} + \overline{\mathbf{D}'_{\mathrm{a}0}})\| \leq \frac{K_{M\mathcal{D}\mathrm{k}0}}{M} \tag{26}$$

*holds with* $K_{M\mathcal{D}\mathrm{k}0}$ *defined as*

$$\begin{aligned} K_{M\mathcal{D}\mathrm{k}0} :=& h\|C_1\| \cdot \|B_1\| e^{\|A\|h/M} + \frac{h^2}{M}\|A\| \cdot \|B_1\| e^{\|A\|h/M} \\ &\cdot \sum_{k=0}^{M-2}\left(\frac{1}{2}\|C_1(A'_d)^{k+1}\| + \|C_1(A'_d)^k\| e^{\|A\|h/M}\right) \end{aligned} \tag{27}$$

*Furthermore,* $K_{M\mathcal{D}\mathrm{k}0}$ *has a uniform upper bound with respect to* $M$ *given by*

$$K^U_{\mathcal{D}\mathrm{k}0} := h\|C_1\| \cdot \|B_1\| e^{\|A\|h} + \frac{3h^2}{2}\|C_1\| \cdot \|A\| \cdot \|B_1\| e^{\|A\|h} \tag{28}$$

**Lemma 2:** *The inequality*

$$\|\overline{\mathbf{M}'_1}\mathcal{J}_{Mj}\overline{\mathbf{B}'_1} - \overline{\mathbf{M}'_{\mathrm{a}0}}\mathcal{J}_{Mj}\overline{\mathbf{B}'_{\mathrm{k}0}}\| \leq \frac{K_{Mj\mathrm{k}0}}{M} \tag{29}$$

*holds for* $j = 0, \cdots, N$*, where*

$$K_{Mj\mathrm{k}0} := e^{\|A\|h/M}\|\mathcal{J}_{Mj}\|\frac{h^2}{M} \cdot \left\{\|C_0A_2\| \cdot \|B_1\| e^{\|A_2\|h/M} + \frac{1}{2}\|C_0\| \cdot \|A\| \cdot \|A'_dB_1\|\right\} \tag{30}$$

*Furthermore,* $K_{Mj\mathrm{k}0}$ *has a uniform upper bound with respect to* $M$ *and* $j$ *given by*

$$K^U_{\mathcal{CAB}\mathrm{k}0} := h^2 e^{\|A\|h} \cdot \|B_1\| \cdot K_* \cdot \left\{\|C_0A_2\| e^{\|A_2\|h} + \frac{1}{2}\|C_0\| \cdot \|A\| e^{\|A\|h}\right\} \tag{31}$$

*where* $K_* := \max_{i\in\mathbb{N}_0}\|\mathcal{A}^i\| \cdot e^{(\|A\|+\|A_2\|)h} \cdot \|C_\Sigma\|$.

**Remark 2:** $\max_{i\in\mathbb{N}_0}\|\mathcal{A}^i\|$ exists since $\mathcal{A}^i \to 0$ as $i \to \infty$ by the stability assumption of $\Sigma_{\mathrm{SD}}$.

The proofs of these lemmas are given in Appendix A. These lemmas immediately lead to the following result.

**Proposition 1:** *The inequality*

$$\|\mathcal{F}^-_{NM} - \mathcal{F}^-_{NM\mathrm{k}0}\| \leq \frac{K_{M\mathrm{k}0}}{M} \tag{32}$$

*holds where*

$$K_{M\mathrm{k}0} := K_{M\mathcal{D}\mathrm{k}0} + \sum_{j=0}^{N} K_{Mj\mathrm{k}0} \tag{33}$$

*In addition,* $K_{M\mathrm{k}0}$ *has a uniform upper bound with respect to* $M$ *given by*

$$K^U_{\mathrm{k}0} := K^U_{\mathcal{D}\mathrm{k}0} + (N+1) \cdot K^U_{\mathcal{CAB}\mathrm{k}0} \tag{34}$$

Proposition 1 clearly demonstrates the mathematical validity of the piecewise constant approximation scheme since it ensures that $\|\mathcal{F}_{NM\mathrm{k}0}^-\|$ tends to $\|\mathcal{F}_{NM}^-\|$ ($= \|\mathcal{F}_N^-\|$) with the convergence rate of $1/M$. To exploit the convergence property in practical computations, we next provide a method for (*exactly*) computing $\|\mathcal{F}_{NM\mathrm{k}0}^-\|$. To facilitate the arguments, we first suppose that $D_{11} = 0$ (so that $D'_{\mathrm{a}0} = 0$) for a while, even though we will eventually deal with the case of $D_{11} \neq 0$. It readily follows from (25) and the definition of $\mathbf{M}'_{\mathrm{a}0}$ that the output of $\mathcal{F}_{NM\mathrm{k}0}^-$ is a constant function determined by the matrix $C_0$. Furthermore, the input of $\mathcal{F}_{NM\mathrm{k}0}^-$ may always be assumed to be a constant function when we evaluate $\|\mathcal{F}_{NM\mathrm{k}0}^-\|$. This is because we can see easily from (22) (by considering the constant function $w_0 = (1/h')\int_0^{h'} w(\theta')d\theta'$) that

$$\{\mathbf{B}'_{\mathrm{k}0} w_0 \mid w_0 \text{ is a constant function}, \|w_0\| \leq 1\} = \{\mathbf{B}'_{\mathrm{k}0} w \mid \|w\| \leq 1\} \tag{35}$$

Hence, $\|\mathcal{F}_{NM\mathrm{k}0}^-\|$ coincides with the $\infty$-norm of the matrix obtained by replacing the operators $\mathbf{B}'_{\mathrm{k}0}$ and $\mathbf{M}'_{\mathrm{a}0}$ in (25) with $A'_d B_1 h'$ and $C_0$, respectively. Combining the above arguments leads to the following result.

**Proposition 2:** $\|\mathcal{F}_{NM\mathrm{k}0}^-\|$ *coincides with the $\infty$-norm of the finite-dimensional matrix* $F_{NM\mathrm{k}0}^-$ *given by*

$$F_{NM\mathrm{k}0}^- := \left[\overline{D_{11}} \quad \overline{C_0}\Delta_M^0 \overline{A'_d B_1 h'} \quad \overline{C_0}\mathcal{J}_{M0}\overline{A'_d B_1 h'} \cdots \quad \overline{C_0}\mathcal{J}_{MN}\overline{A'_d B_1 h'}\right] \tag{36}$$

**Remark 3:** The above arguments under the assumption $D_{11} = 0$ immediately lead to (36) without the extra entry $\overline{D_{11}}$, but it is not hard to see that dealing with $D_{11} \neq 0$ and thus the corresponding multiplication operator $\overline{\mathbf{D}'_{\mathrm{a}0}}$ in (25) simply leads to introducing this extra entry by the property of $L_\infty[0, h')$; the treatment of $D_{11}$ is essentially the same as that in Sivashankar and Khargonekar (1992). Similarly for Proposition 4 given later.

Combining the above propositions leads to the following first main result in this paper.

**Theorem 1:** *The inequality*

$$\|F_{NM\mathrm{k}0}^-\| - \frac{K_{M\mathrm{k}0}}{M} \leq \|\mathcal{F}_N^-\| \leq \|F_{NM\mathrm{k}0}^-\| + \frac{K_{M\mathrm{k}0}}{M} \tag{37}$$

*holds with the matrix* $F_{NM\mathrm{k}0}^-$ *given by (36), where* $K_{M\mathrm{k}0}$ *has a uniform upper bound with respect to* $M$ *given by (34).*

This implies that upper and lower bounds of $\|\mathcal{F}_N^-\|$ can be readily obtained through the matrix $\infty$-norm $\|F_{NM\mathrm{k}0}^-\|$ together with $K_{M\mathrm{k}0}/M$. Furthermore, by taking the fast-lifting parameter $M$ larger, the gap between those upper and lower bounds converges to 0 at no slower convergence rate than $1/M$ (because $K_{M\mathrm{k}0}$ has a uniform upper bound $K_{\mathrm{k}0}^U$).

*3.3.2 Piecewise Linear Approximation Scheme*

Aiming at developing an improved method, we next introduce the operators $\mathbf{B}'_{\mathrm{k}1} : \mathcal{K}'_{n_w} \to \mathbb{R}^n$, $\mathbf{M}'_{\mathrm{a}1} : \mathbb{R}^{n+n_u} \to \mathcal{K}'_{n_z}$ and $\mathbf{D}'_{\mathrm{a}1} : \mathcal{K}'_{n_w} \to \mathcal{K}'_{n_z}$ defined respectively as

$$\mathbf{B}'_{\mathrm{k}1} w = \int_0^{h'} A'_d (I - A\theta') B_1 w(\theta') d\theta' \tag{38}$$

$$\left( \mathbf{M}'_{\mathrm{a}1} \begin{bmatrix} x \\ u \end{bmatrix} \right) (\theta') = C_0 (I + A_2 \theta') \begin{bmatrix} x \\ u \end{bmatrix} \quad (0 \leq \theta' < h') \tag{39}$$

$$(\mathbf{D}'_{\mathrm{a}1} w)(\theta') = \int_0^{\theta'} C_1 B_1 w(\theta') d\theta' + D_{11} w(\theta') \tag{40}$$

Introducing the operator $\mathbf{B}'_{\mathrm{k}1}$ corresponds to the first-order approximation of the kernel function of the operator $\mathbf{B}'_1$ and is a key for developing the piecewise linear approximation scheme in the kernel approximation approach. Introducing the operators $\mathbf{M}'_{\mathrm{a}1}$ and $\mathbf{D}'_{\mathrm{a}1}$, as in the input approximation approach, corresponds to the first-order approximation of the hold function of $\mathbf{M}'_1$ and the kernel function of the 'compact part' of $\mathbf{D}'_{11}$, respectively.

To proceed with the approximation arguments, we consider the operator $\mathcal{F}^-_{NM\mathrm{k}1}$ obtained by replacing $\mathbf{B}'_1$, $\mathbf{M}'_1$ and $\mathbf{D}'_{11}$ with $\mathbf{B}'_{\mathrm{k}1}$, $\mathbf{M}'_{\mathrm{a}1}$ and $\mathbf{D}'_{\mathrm{a}1}$, respectively, in (20):

$$\mathcal{F}^-_{NM\mathrm{k}1} = \left[ \overline{\mathbf{M}'_{\mathrm{a}1}} \Delta^0_M \overline{\mathbf{B}'_{\mathrm{k}1}} + \overline{\mathbf{D}'_{\mathrm{a}1}} \quad \overline{\mathbf{M}'_{\mathrm{a}1}} \mathcal{J}_{M0} \overline{\mathbf{B}'_{\mathrm{k}1}} \quad \cdots \quad \overline{\mathbf{M}'_{\mathrm{a}1}} \mathcal{J}_{MN} \overline{\mathbf{B}'_{\mathrm{k}1}} \right] \tag{41}$$

Another main result of this paper is that $\|\mathcal{F}^-_{NM\mathrm{k}1}\|$ can be computed exactly and converges to $\|\mathcal{F}^-_N\|$ at the rate of $1/M^2$. The following two lemmas are important in establishing the latter fact.

**Lemma 3:** *The inequality*

$$\|\mathbf{L}_M \mathcal{D} \mathbf{L}_M^{-1} - (\overline{\mathbf{M}'_{\mathrm{a}1}} \Delta^0_M \overline{\mathbf{B}'_{\mathrm{k}1}} + \overline{\mathbf{D}'_{\mathrm{a}1}})\| \leq \frac{K_{M\mathcal{D}\mathrm{k}1}}{M^2} \tag{42}$$

*holds with $K_{M\mathcal{D}\mathrm{k}1}$ defined as*

$$\begin{aligned} K_{M\mathcal{D}\mathrm{k}1} :=& \frac{1}{2}\|C_1\| \cdot \|A\| \cdot \|B_1\| h^2 e^{\|A\|h/M} + \frac{1}{2}\|A\|^2 \cdot \|B_1\| e^{\|A\|h/M} \frac{h^3}{M} \\ & \cdot \sum_{k=0}^{M-2} \left\{ \left\| C_1 (A'_d)^k \right\| e^{\|A\|h/M} + \frac{1}{3} \sup_{0 \leq \theta' < h'} \left\| C_1 (I + A\theta')(A'_d)^{k+1} \right\| \right\} \end{aligned} \tag{43}$$

*Furthermore, $K_{M\mathcal{D}\mathrm{k}1}$ has a uniform upper bound with respect to $M$ given by*

$$K^U_{\mathcal{D}\mathrm{k}1} := \frac{1}{2}\|C_1\| \cdot \|A\| \cdot \|B_1\| h^2 e^{\|A\|h} + \frac{1}{2}\|C_1\| \cdot \|A\|^2 \cdot \|B_1\| h^3 e^{\|A\|h} \left( \frac{4 + \|A\|h}{3} \right) \tag{44}$$

**Lemma 4:** *The inequality*

$$\|\overline{\mathbf{M}'_1} \mathcal{J}_{Mj} \overline{\mathbf{B}'_1} - \overline{\mathbf{M}'_{\mathrm{a}1}} \mathcal{J}_{Mj} \overline{\mathbf{B}'_{\mathrm{k}1}}\| \leq \frac{K_{Mj\mathrm{k}1}}{M^2} \tag{45}$$

*holds for* $j = 0, \cdots, N$, *where*

$$K_{Mj\mathrm{k}1} = \frac{1}{2} e^{\|A\|h/M} \|\mathcal{J}_{Mj}\| \frac{h^3}{M} \cdot \left\{ \frac{1}{3} \sup_{0 \le \theta' < h'} \|C_0(I + A_2\theta')\| \cdot \|A\|^2 \cdot \|A'_d B_1\| + \left\|C_0 A_2^2\right\| e^{\|A_2\|h/M} \|B_1\| \right\} \tag{46}$$

*Furthermore,* $K_{Mj\mathrm{k}1}$ *has a uniform upper bound with respect to* $M$ *and* $j$ *defined as*

$$K^U_{\mathcal{CAB}\mathrm{k}1} := \frac{1}{2} h^3 e^{\|A\|h} \|B_1\| K_* \cdot \left\{ \frac{1}{3} \left(\|C_0\| + \|C_0 A_2\|\, h\right) \|A\|^2 e^{\|A\|h} + \left\|C_0 A_2^2\right\| e^{\|A_2\|h} \right\} \tag{47}$$

The proofs of these lemmas are also given in Appendix A. We immediately arrive at the following result by combining these lemmas.

**Proposition 3:** *The inequality*

$$\|\mathcal{F}^-_{NM} - \mathcal{F}^-_{NM\mathrm{k}1}\| \le \frac{K_{M\mathrm{k}1}}{M^2} \tag{48}$$

*holds where*

$$K_{M\mathrm{k}1} := K_{M\mathcal{D}\mathrm{k}1} + \sum_{j=0}^{N} K_{Mj\mathrm{k}1} \tag{49}$$

*In addition,* $K_{M\mathrm{k}1}$ *has a uniform upper bound with respect to* $M$ *given by*

$$K^U_{\mathrm{k}1} := K^U_{\mathcal{D}\mathrm{k}1} + (N+1) K^U_{\mathcal{CAB}\mathrm{k}1} \tag{50}$$

Proposition 3 clearly implies that the error in the approximation of $\|\mathcal{F}^-_N\|$ $(= \|\mathcal{F}^-_{NM}\|)$ by $\|\mathcal{F}^-_{NM\mathrm{k}1}\|$ in the piecewise linear approximation scheme converges to 0 at the rate of $1/M^2$. To exploit the convergence result in practical computations, we next provide a method for (*exactly*) computing $\|\mathcal{F}^-_{NM\mathrm{k}1}\|$ as follows.

**Proposition 4:** *Let* $V^{[0]}$ *be the matrix consisting of the* $L_1[0, h')$ *norm of each entry of the matrix linear function* $\overline{C_0}\varDelta^0_M \overline{A'_d(I - A\theta')B_1}$*, while let* $V^{[h']}$ *be the matrix constructed in the same way from* $\overline{C_0(I + A_2 h')}\varDelta^0_M \overline{A'_d(I - A\theta')B_1}$*. Furthermore, let* $T_j^{[0]}$ $(j = 0, \cdots, N)$ *be the matrix consisting of the* $L_1[0, h')$ *norm of each entry of the matrix linear function* $\overline{C_0}\mathcal{J}_{Mj}\overline{A'_d(I - A\theta')B_1}$*, while let* $T_j^{[h']}$ $(j = 0, \cdots, N)$ *be the matrix constructed in the same way from* $\overline{C_0(I + A_2 h')}\mathcal{J}_{Mj}\overline{A'_d(I - A\theta')B_1}$*. Then,* $\|\mathcal{F}^-_{NM\mathrm{k}1}\|$ *coincides with the* $\infty$*-norm of the finite-dimensional matrix* $F^-_{NM\mathrm{k}1}$ *given by*

$$F^-_{NM\mathrm{k}1} := \begin{bmatrix} \overline{D_{11}} & 0 & V^{[0]} & T_0^{[0]} & \cdots & T_N^{[0]} \\ \overline{D_{11}} & C_1 B_1 h' & V^{[h']} & T_0^{[h']} & \cdots & T_N^{[h']} \end{bmatrix} \tag{51}$$

The proof of Proposition 4 is given in Appendix B. Combining the above propositions leads to the following second main result in this paper.

**Theorem 2:** *The inequality*

$$\|F_{NM\mathrm{k1}}^{-}\| - \frac{K_{M\mathrm{k1}}}{M^2} \le \|\mathcal{F}_N^{-}\| \le \|F_{NM\mathrm{k1}}^{-}\| + \frac{K_{M\mathrm{k1}}}{M^2} \tag{52}$$

*holds with the matrix $F_{NM\mathrm{k1}}^{-}$ given by (51), where $K_{M\mathrm{k1}}$ has a uniform upper bound with respect to $M$ given by (50).*

This implies that upper and lower bounds of $\|\mathcal{F}_N^{-}\|$ can be obtained through $\|F_{NM\mathrm{k1}}^{-}\|$ together with $K_{M\mathrm{k1}}/M^2$. In addition, by taking the fast-lifting parameter $M$ larger, the gap between those upper and lower bounds converges to 0 at no slower convergence rate than $1/M^2$ (because $K_{M\mathrm{k1}}$ has a uniform upper bound $K_{\mathrm{k1}}^{U}$).

### 3.4 *Upper Bound of $\|\mathcal{F}_N^{+}\|$ and Computation of $\|\mathcal{F}\|$*

In the preceding subsection, two methods for computing $\|\mathcal{F}_N^{-}\|$ $(= \|\mathcal{F}_{NM}^{-}\|)$ are given through the kernel approximation approach with the piecewise constant approximation and piecewise linear approximation schemes. Hence, the last task is to compute $\|\mathcal{F}_N^{+}\|$, for which an upper bound has been given in Kim and Hagiwara (2015a) as follows; note that the stability assumption of $\Sigma_{\mathrm{SD}}$ ensures the existence of $L$ such that $\|\mathcal{A}^L\| < 1$.

**Proposition 5** (Kim and Hagiwara (2015a), Proposition 3)**:** *If $\left\|\mathcal{A}^L\right\| < 1$, then*

$$\|\mathcal{F}_N^{+}\| \le \frac{\|C_\Sigma \mathcal{A}_{NL}\|}{1 - \|\mathcal{A}^L\|} \|C_0\| e^{\|A_2\|h} h e^{\|A\|h} B_1 =: K_{NL} \tag{53}$$

*where $\mathcal{A}_{NL} := \begin{bmatrix} \mathcal{A}^{N+1} & \mathcal{A}^{N+2} & \cdots & \mathcal{A}^{N+L} \end{bmatrix}$, and $K_{NL}$ converges to 0 regardless of $L$ as $N \to \infty$ with the exponential convergence rate of $\rho^N$ $(0 < \rho < 1)$.*

Here, we note that the definitions of $\mathcal{F}_N^{-}$ and $\mathcal{F}_N^{+}$ together with $\mathcal{F}$ in (14) and (15) immediately lead to the following inequality:

$$\|\mathcal{F}_N^{-}\| \le \|\mathcal{F}\| \le \|\mathcal{F}_N^{-}\| + \|\mathcal{F}_N^{+}\| \tag{54}$$

This is an improved result compared with our previous result in Kim and Hagiwara (2015a,c), in which $\|\mathcal{F}_N^{-}\| - \|\mathcal{F}_N^{+}\|$ was used as a lower bound of $\|\mathcal{F}\|$; such a lower bound was derived as a direct consequence of the triangle inequality, but if we note that $\mathcal{F}_N^{-}$ is a restriction of $\mathcal{F}$ (because the entries of $\mathcal{F}_N^{-}$ are zero except first $N+2$ entries), the above improved lower bound follows immediately. By combining Theorems 1 and 2 and Proposition 5 together with (54), we can readily have the following result, giving upper and lower bounds of $\|\mathcal{F}\|$ such that their gap converge to 0 as the parameters $M$ and $N$ tends to $\infty$.

**Theorem 3:** *If $\|\mathcal{A}^L\| < 1$, then*

$$\|F_{NM\mathrm{k0}}^{-}\| - \frac{K_{M\mathrm{k0}}}{M} \le \|\mathcal{F}\| \le \|F_{NM\mathrm{k0}}^{-}\| + \frac{K_{M\mathrm{k0}}}{M} + K_{NL} \tag{55}$$

$$\|F_{NM\mathrm{k1}}^{-}\| - \frac{K_{M\mathrm{k1}}}{M^2} \le \|\mathcal{F}\| \le \|F_{NM\mathrm{k1}}^{-}\| + \frac{K_{M\mathrm{k1}}}{M^2} + K_{NL} \tag{56}$$

*Furthermore, $K_{M\mathrm{k0}}/M$ and $K_{M\mathrm{k1}}/M^2$ converge to 0 as $M \to \infty$ at no slower rate than $1/M$ and $1/M^2$, respectively, while $K_{NL}$ converges to 0 regardless of $L$ as $N \to \infty$.*

### 3.5 *Comparison with Input Approximation Approach*

In this section, we are in a position to discuss the effectiveness of the developed computation method for $\|\mathcal{F}\|$ with the kernel approximation approach, compared with the method in Kim and Hagiwara (2015a) through the input approximation approach. To make the comparison fair, however, we state the modified versions of the results in (Kim and Hagiwara, 2015a, Theorem 3) that follow immediately by using the improved estimation of $\|\mathcal{F}\|$ in (54); we readily have the inequalities

$$\|F_{NM\mathrm{i}0}^{-}\| \leq \|\mathcal{F}\| \leq \|F_{NM\mathrm{i}0}^{-}\| + \frac{K_{M\mathrm{i}0}}{M} + K_{NL} \tag{57}$$

$$\|F_{NM\mathrm{i}1}^{-}\| - \frac{K_{M\mathrm{i}1}}{M^2} \leq \|\mathcal{F}\| \leq \|F_{NM\mathrm{i}1}^{-}\| + \frac{K_{M\mathrm{i}1}}{M^2} + K_{NL} \tag{58}$$

for the piecewise constant approximation and piecewise linear approximation schemes, respectively, with appropriately defined finite-dimensional matrices $F_{NM\mathrm{i}0}^{-}$ and $F_{NM\mathrm{i}1}^{-}$ and constants $K_{M\mathrm{i}0}$ and $K_{M\mathrm{i}1}$ (see Kim and Hagiwara (2015a) for the definitions of the matrices $F_{NM\mathrm{i}0}^{-}$ and $F_{NM\mathrm{i}1}^{-}$). The lower bounds for $\|\mathcal{F}\|$ in these inequalities are less conservative than those in (Kim and Hagiwara, 2015a, Theorem 3) given by $\|F_{NM\mathrm{i}0}^{-}\| - K_{NL}$ and $\|F_{NM\mathrm{i}1}^{-}\| - \frac{K_{M\mathrm{i}1}}{M^2} - K_{NL}$, respectively. We further recall that the constants $K_{M\mathrm{i}0}$ and $K_{M\mathrm{i}1}$ are given respectively by

$$K_{M\mathrm{i}0} = K_{M\mathcal{D}\mathrm{i}0} + \sum_{j=0}^{N} K_{Mj\mathrm{i}0} \tag{59}$$

$$K_{M\mathrm{i}1} = K_{M\mathcal{D}\mathrm{i}1} + \sum_{j=0}^{N} K_{Mj\mathrm{i}1} \tag{60}$$

where

$$K_{M\mathcal{D}\mathrm{i}0} := h\|C_1\| \cdot \|B_1\| + \frac{h^2}{M}\|A\| \cdot \|B_1\| e^{\|A\|h/M} \cdot \sum_{k=0}^{M-2} \{\|C_1(A_d')^{k+1}\| + \|C_1(A_d')^k\| e^{\|A\|h/M}\} \tag{61}$$

$$K_{Mj\mathrm{i}0} := e^{\|A\|h/M}\|\mathcal{J}_{Mj}\|\frac{h^2}{M} \cdot \Big\{\|C_0A_2\| \cdot \|B_1\| e^{\|A_2\|h/M} + \|C_0\| \cdot \|A\| \cdot \|A_d'B_1\|\Big\} \tag{62}$$

$$K_{M\mathcal{D}\mathrm{i}1} := \frac{1}{2}\|C_1\| \cdot \|A\| \cdot \|B_1\| h^2 e^{\|A\|h/M} + \frac{1}{2}\|A\|^2 \cdot \|B_1\| e^{\|A\|h/M}\frac{h^3}{M} \cdot \sum_{k=0}^{M-2} \left\{\|C_1(A_d')^k\| e^{\|A\|h/M} + \sup_{0\leq\theta'<h} \|C_1(I + A\theta')(A_d')^{k+1}\|\right\} \tag{63}$$

$$K_{Mj\mathrm{i}1} := \frac{1}{2}e^{\|A\|h/M}\|\mathcal{J}_{Mj}\|\frac{h^3}{M} \cdot \left\{\sup_{0\leq\theta'<h'} \|C_0(I + A_2\theta')\| \cdot \|A\|^2 \cdot \|A_d'B_1\| + \|C_0A_2^2\| e^{\|A_2\|h/M}\|B_1\|\right\} \tag{64}$$

Thus, we can see from (27), (30), (33), (43), (46), (49) and (59)–(64) that the constants $K_{M\mathrm{k}0}$ and $K_{M\mathrm{k}1}$ we have derived in this paper are smaller than $K_{M\mathrm{i}0}$ and $K_{M\mathrm{i}1}$, respectively, in the above inequalities derived through parallel arguments. This situation is closely related to the $L_\infty$-induced norm analysis of continuous-time systems (Kim and Hagiwara, 2014, 2015b), in which the operator $\mathbf{B}'$, which plays a key role in the analysis of continuous-time systems and has a form similar to $\mathbf{B}_1'$, was approximated by using either the input approximation or kernel approximation approach and it was shown that the kernel approximation approach works more effectively than the input approximation approach. In particular, in the inequalities corresponding to (55)–(58) for continuous-time systems, it was shown that $K_{M\mathrm{k}0} = (1/2)K_{M\mathrm{i}0}$ and $K_{M\mathrm{k}1} = (1/3)K_{M\mathrm{i}1}$. However, this kind of equalities do not follow in the case of sampled-data systems because the existence of an $h$-periodic nature of sampled-data systems complicates the representations of these constants. We can only show that $(1/2)K_{M\mathrm{i}0} < K_{M\mathrm{k}0} < K_{M\mathrm{i}0}$ for the piecewise constant approximation scheme and $K_{M\mathrm{k}1} < K_{M\mathrm{i}1}$ for the piecewise linear approximation scheme. It is, however, still important to note that the latter inequality means that the gap between the upper and lower bounds in (56) through the kernel approximation approach is smaller than that in (58) through the input approximation approach when we select the piecewise linear approximation scheme. For the case when the piecewise constant approximation scheme is used, on the other hand, the former inequality implies that the input approximation approach is superior to the kernel approximation approach in the case of sampled-data systems. Interestingly enough, this consequence is not consistent with what has been clarified in our study (Kim and Hagiwara, 2015b) for continuous-time systems. This kind of discrepancy between the continuous-time systems and sampled-data systems under the piecewise constant approximation scheme is interpreted as stemming from the following reason. The $h$-periodic nature of the input/output relation of the sampled-data system $\Sigma_{\mathrm{SD}}$ requires us to deal with not only the operator $\mathbf{B}_1$ but also the operators $\mathbf{M}_1$ and $\mathbf{D}_{11}$ (more precisely, not only the operator $\mathbf{B}_1'$ but also the operators $\mathbf{M}_1'$ and $\mathbf{D}_{11}'$) to compute the $L_\infty$-induced norm, and the constants $K_{M\mathrm{k}0}$ and $K_{M\mathrm{k}1}$ are dependent on all the approximations of the operators $\mathbf{B}_1$, $\mathbf{M}_1$ and $\mathbf{D}_{11}$, while in continuous-time systems we only need to approximate the operator $\mathbf{B}$.

To summarize, the gap between the upper and lower bounds in (56) for the kernel approximation approach is smaller than that in (58) for the input approximation approach when we choose the piecewise linear approximation scheme. However, for the piecewise constant approximation scheme, the gap in (57) for the input approximation approach is smaller than in (55). Meanwhile, for the input approximation approach, it will be numerically verified in the following section that the piecewise linear approximation scheme is superior to the piecewise constant approximation scheme in the $L_\infty$-induced norm computation of $\mathcal{F}$. Combining these arguments clearly (implies that the piecewise linear approximation scheme is superior to the piecewise constant approximation scheme also in the kernel approximation approach as can be verified in the numerical example too, and) indicates an advantage of the computation method with a combined use of the piecewise linear approximation scheme and the kernel approximation approach over the other three methods.

## 4. Numerical Example

In this section, we examine effectiveness of the kernel approximation approach (especially the method with a combined use of the piecewise linear approximation scheme and the kernel approximation approach) through a numerical example.

Let us consider the mass-spring-damper system shown in Figure 2 consisting of 5 mass-spring-damper units, where $m_1, \ldots, m_5$ denote the masses, $k_1, \ldots, k_5$ denote the spring constants, $c_1, \ldots, c_5$ denote the damper constants, $l_1, \ldots, l_5$ denote the displacements of masses from their equilibrium positions, $w$ denotes the bounded persistent disturbance on the mass $m_3$ and $u$ denotes the control input force applied on the mass $m_5$. Then, the motion of this system is described by the 10th-order state equation

$$\frac{dx}{dt} = \begin{bmatrix} 0_{5\times 5} & I_5 \\ A_{mk} & A_{mc} \end{bmatrix} x + \begin{bmatrix} 0_{7\times 1} \\ -\frac{1}{m_3} \\ 0_{2\times 1} \end{bmatrix} w + \begin{bmatrix} 0_{9\times 1} \\ \frac{1}{m_5} \end{bmatrix} u =: Ax + B_1 w + B_2 u \tag{65}$$

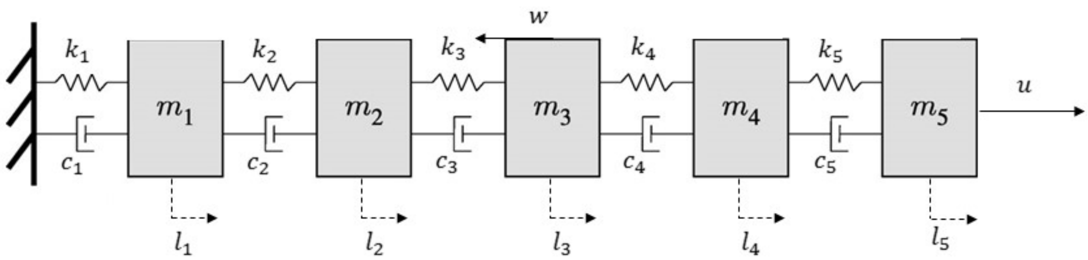


Figure 2. 5-mass-spring-damper system.

where $x := \begin{bmatrix} l_1 & \cdots & l_5 & \frac{dl_1}{dt} & \cdots & \frac{dl_5}{dt} \end{bmatrix}^T$ and

$$A_{mk} := \begin{bmatrix} -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_2+k_3}{m_2} & \frac{k_3}{m_2} & 0 & 0 \\ 0 & \frac{k_3}{m_3} & -\frac{k_3+k_4}{m_3} & \frac{k_4}{m_3} & 0 \\ 0 & 0 & \frac{k_4}{m_4} & -\frac{k_4+k_5}{m_4} & \frac{k_5}{m_4} \\ 0 & 0 & 0 & \frac{k_5}{m_5} & -\frac{k_5}{m_5} \end{bmatrix} \tag{66}$$

$$A_{mc} := \begin{bmatrix} -\frac{c_1+c_2}{m_1} & \frac{c_2}{m_1} & 0 & 0 & 0 \\ \frac{c_2}{m_2} & -\frac{c_2+c_3}{m_2} & \frac{c_3}{m_2} & 0 & 0 \\ 0 & \frac{c_3}{m_3} & -\frac{c_3+c_4}{m_3} & \frac{c_4}{m_3} & 0 \\ 0 & 0 & \frac{c_4}{m_4} & -\frac{c_4+c_5}{m_4} & \frac{c_5}{m_4} \\ 0 & 0 & 0 & \frac{c_5}{m_5} & -\frac{c_5}{m_5} \end{bmatrix} \tag{67}$$

Suppose that a stabilizing discrete-time full-state feedback controller $\Psi$ is given (assuming that $y = x$) and that we are interested in evaluating the performance of the controller in suppressing the worst displacement $l_3$, or more precisely, computing the $L_\infty$-induced norm $\|\mathcal{F}\|$ of the system between $w$ and $z := l_3$. This corresponds to $C_1 := [0_{2\times 1} \;\; 1 \;\; 0_{7\times 1}]$, $C_2 = I_{10}$, $D_{11} = 0$ and $D_{12} = 0$. Here, for simplicity, we assume that $m_1 = \cdots = m_5 = 2$, $k_1 = \cdots = k_5 = 1$ and $c_1 = \cdots = c_5 = 1.6$ and $h = 2$. We further consider the case when the state feedback gain is given by

$$-\begin{bmatrix} 0.3620 & 0.5946 & 0.6702 & 0.6410 & 0.6024 & 1.2290 & 2.0098 & 2.1962 & 1.9878 & 1.7530 \end{bmatrix} \tag{68}$$

We compute estimates of the $L_\infty$-induced norm $\|\mathcal{F}\|$ by taking the fast-lifting parameter $M$ ranging from 400 to 1000 on the condition that $L = 5$ and $N = 20$, which lead to $K_{NL} = 2.90 \times 10^{-5}$. The results for the upper and lower bounds of $\|\mathcal{F}\|$ obtained by (55)–(58) under the piecewise constant approximation and piecewise linear approximation schemes are shown in Tables 1 and 2, respectively. For each of these two approximation schemes, we are interested in the comparison between the kernel approximation approach developed in this paper and the existing input approximation approach. Hence, these tables consist of Case (a) for the existing approach and Case (b) for the new developed approach.

We can see from Tables 1 and 2 that the error bounds for the computation of $\|\mathcal{F}\|$ (i.e., the gaps between the upper and lower bounds) decrease by taking the fast-lifting parameter $M$ larger for all estimates. Thus, we can confirm validity of all the four approximation methods provided in this paper and our earlier study (Kim and Hagiwara, 2015a) for computing the $L_\infty$-induced norm $\|\mathcal{F}\|$ of sampled-data systems. However, a more important concern in this paper lies in the effectiveness comparison between (a) the input approximation approach and (b) the kernel approximation ap-

Table 1. Results with piecewise constant approximation scheme ($L = 5$ and $N = 20$).

Case (a): Input approximation approach

| $M$ | 400 | 600 | 800 | 1000 |
|---|---|---|---|---|
| Upper Bound | 3.0741 | 2.4717 | 2.1740 | 1.9966 |
| Lower Bound | 1.2952 | 1.2952 | 1.2952 | 1.2952 |
| Time (sec) | 5.5206 | 16.3935 | 36.4463 | 334.7224 |

Case (b): Kernel approximation approach

| $M$ | 400 | 600 | 800 | 1000 |
|---|---|---|---|---|
| Upper Bound | 2.3523 | 1.9933 | 1.8164 | 1.7110 |
| Lower Bound | 0.2403 | 0.5986 | 0.7752 | 0.8804 |
| Time (sec) | 5.3859 | 16.3646 | 36.3375 | 327.3898 |

Table 2. Results with piecewise linear approximation scheme ($L = 5$ and $N = 20$).

Case (a): Input approximation approach

| $M$ | 400 | 600 | 800 | 1000 |
|---|---|---|---|---|
| Upper Bound | 1.3185 | 1.3055 | 1.3010 | 1.2989 |
| Lower Bound | 1.2720 | 1.2850 | 1.2895 | 1.2916 |
| Time (sec) | 56.5413 | 129.1668 | 235.0635 | 559.3790 |

Case (b): Kernel approximation approach

| $M$ | 400 | 600 | 800 | 1000 |
|---|---|---|---|---|
| Upper Bound | 1.3059 | 1.3000 | 1.2979 | 1.2969 |
| Lower Bound | 1.2846 | 1.2906 | 1.2926 | 1.2936 |
| Time (sec) | 54.6002 | 125.0941 | 229.1151 | 507.6987 |

proach. In this regard, we had an earlier argument in Section 3.5, which implies that, under the piecewise *constant* approximation scheme, the (new) kernel approximation approach can provide no advantage over the input approximation approach in reducing the gap between the computed upper and lower bounds. As seen from Table 1 for the piecewise constant approximation scheme, the input approximation approach indeed leads to smaller approximation errors (i.e., smaller gaps between the computed upper and lower bounds) than the kernel approximation approach, under the same fast-lifting parameter $M$. However, this table also shows that the convergence of the approximation errors is not fast with respect to $M$ even for the input approximation approach. In connection with this, we next observe the corresponding results through the piecewise *linear* approximation scheme in Table 2 exhibiting much faster convergence. This demonstrates that the piecewise linear approximation scheme works much more effectively than the piecewise constant approximation scheme. In spite of the advantage of the former scheme in this respect, however, it should be observed that the former scheme requires much larger computation times than the latter scheme under the same parameter $M$. By inspection of these two tables with these trade-off aspects in mind, we can further see that the gaps between the upper and lower bounds in the piecewise linear approximation scheme with $M = 400$ under both the input approximation and kernel approximation approaches are much smaller than those in the piecewise constant approximation scheme with $M = 1000$ under both the input approximation and kernel approximation approaches, while the computation times for the former are smaller than those for the latter. These observations imply that the piecewise linear approximation scheme drastically outperforms the piecewise constant approximation scheme, under both the input approximation and kernel approximation approaches.

We can further see from these tables that once we switch to the piecewise linear approximation scheme (i.e., in Table 2), an advantage of the kernel approximation approach over the input approximation approach is prominent in the following sense: the range between the upper and lower bounds obtained by the kernel approximation approach is always contained in (and thus less conservative than) that by the input approximation approach for the same parameter $M$. Furthermore, the computation times in the kernel approximation approach are slightly smaller than those in the input approximation approach under the same parameter $M$. As an overall evaluation, the kernel approximation approach with the piecewise linear approximation scheme exhibits the smallest range for the $L_\infty$-induced norm estimates with relatively short computation times among the four computation methods.

## 5. Conclusion

Stimulated by the success of the kernel approximation approach in computing the $L_\infty$-induced norm of continuous-time systems, we developed a new framework for exploiting the idea of the kernel approximation approach in computing the $L_\infty$-induced norm of sampled-data systems. Piecewise constant approximation and piecewise linear approximation schemes were applied via the fast-lifting treatment of sampled-data systems, so that the kernel function of the input operator and the hold function of the output operator associated with sampled-data systems are approximated by piecewise constant or piecewise linear functions. We next showed that upper and lower bounds of the $L_\infty$-induced norm can be readily computed through such approximations and that the gap between the upper bound and lower bound in the piecewise constant approximation scheme or piecewise linear approximation scheme is ensured to converge to 0 at the rate of $1/M$ or $1/M^2$, respectively, where $M$ is the fast-lifting parameter. We also showed that even though these convergence rates are qualitatively the same as those in the existing input approximation approach, the approximation errors through the kernel approximation approach introduced in this paper are smaller than those through the existing input approximation approach (Kim and Hagiwara, 2015a) when the piecewise linear approximation scheme is used. We then compared the effectiveness of the two approximation approaches through a numerical study and confirmed that the kennel approximation approach with the piecewise linear approximation scheme gave the smallest range for the $L_\infty$-induced norm estimates with relatively short computation times among the four computation methods including the two methods in the existing input approximation approach.

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## Appendix A. Proofs of Lemmas 1, 2, 3 and 4:

This appendix is concerned with the proofs of the lemmas in Section 3. They are based on the Taylor expansion of the matrix exponential of $A\theta'$ (or $A_2\theta'$), and the proofs of Lemmas 1–3 proceed in essentially the same way as that of Lemma 4. Hence, only the proof of Lemma 4 is given.

It readily follows that

$$\begin{aligned}\|(\mathbf{B}_1' - \mathbf{B}_{\mathrm{k}1}')w\| &= \left\| \int_0^{h'} A_d'\{\exp(-A\theta') - (I - A\theta')\} B_1 w(\theta') d\theta' \right\| \\ &\le \int_0^{h'} \sum_{j=2}^{\infty} \frac{\|A\|^j (\theta')^j}{j!} d\theta' \cdot \|A_d' B_1\| \cdot \|w\| \le \frac{1}{6}(h')^3 \|A\|^2 e^{\|A\| h'} \|A_d' B_1\| \cdot \|w\| \end{aligned} \quad \text{(A1)}$$

On the other hand, we have the following inequalities (Kim and Hagiwara, 2015a).

$$\|\mathbf{M}_1' - \mathbf{M}_{\mathrm{a}1}'\| = \left\| C_0 \sum_{i=2}^{\infty} \frac{(A_2\theta')^i}{i!} \right\| \leq \frac{(h')^2}{2} \|C_0 A_2^2\| \, e^{\|A_2\|h'} \tag{A2}$$

$$\|\mathbf{B}_1'\| \leq h' e^{\|A\|h'} \|B_1\|, \quad \|\mathbf{M}_{\mathrm{a}1}'\| \leq \sup_{0\leq\theta'<h'} \|C_0(I + A_2\theta')\| \tag{A3}$$

Combining the inequalities (A1)–(A3) leads to the following result.

$$\begin{aligned} \|\overline{\mathbf{M}_1'}\mathcal{J}_{Mj}\overline{\mathbf{B}_1'} - \overline{\mathbf{M}_{\mathrm{a}1}'}\mathcal{J}_{Mj}\overline{\mathbf{B}_{\mathrm{k}1}'}\| \leq & \|\overline{\mathbf{M}_{\mathrm{a}1}'}\mathcal{J}_{Mj}\left(\overline{\mathbf{B}_1'} - \overline{\mathbf{B}_{\mathrm{k}1}'}\right)\| + \|(\overline{\mathbf{M}_1'} - \overline{\mathbf{M}_{\mathrm{a}1}'})\mathcal{J}_{Mj}\overline{\mathbf{B}_1'}\| \\ & \leq \sup_{0\leq\theta'<h'} \|C_0(I + A_2\theta')\| \cdot \|\mathcal{J}_{Mj}\| \cdot \frac{(h')^3}{6}\|A\|^2\|B_1\|e^{\|A\|h'} \\ & + \frac{(h')^2}{2}\|C_0A_2^2\|e^{\|A_2\|h'} \cdot \|\mathcal{J}_{Mj}\| \cdot h'e^{\|A\|h'}\|B_1\| \end{aligned} \tag{A4}$$

This is nothing but the first assertion of this lemma. The second assertion follows readily if we note that

$$e^{\|A\|h/M} \leq e^{\|A\|h}, \quad e^{\|A_2\|h/M} \leq e^{\|A_2\|h}, \quad \frac{\|\mathcal{J}_{Mj}\|}{M} \leq K_*, \quad \|A_d' B_1\| \leq e^{\|A\|h}\|B_1\| \tag{A5}$$

This completes the proof.

## Appendix B. Proof of Proposition 4:

This appendix is concerned with the proof of Proposition 4, which is pertinent to the computation of $\|\mathcal{F}_{NM\mathrm{k}1}^-\|$.

We first focus on the first entry of $\mathcal{F}_{NM\mathrm{k}1}^-$ in (41) and consider the matrix function $((\overline{\mathbf{M}_{\mathrm{a}1}'}\varDelta_M^0\overline{\mathbf{B}_{\mathrm{k}1}'} + \overline{\mathbf{D}_{\mathrm{a}1}'})w)(\theta')$ (assuming that $D_{11} = 0$). First note that $\varDelta_M^0$ has a strictly block lower triangular structure while $\overline{\mathbf{D}_{\mathrm{a}1}'}$ has a block diagonal structure. Hence, by the property of the $L_\infty$-induced norm, it follows that when we are to compute the induced norm $\|\overline{\mathbf{M}_{\mathrm{a}1}'}\varDelta_M^0\overline{\mathbf{B}_{\mathrm{k}1}'} + \overline{\mathbf{D}_{\mathrm{a}1}'}\|$ by considering $((\overline{\mathbf{M}_{\mathrm{a}1}'}\varDelta_M^0\overline{\mathbf{B}_{\mathrm{k}1}'} + \overline{\mathbf{D}_{\mathrm{a}1}'})w)(\theta') = \overline{\mathbf{D}_{\mathrm{a}1}'}w + \overline{\mathbf{M}_{\mathrm{a}1}'}\varDelta_M^0\overline{\mathbf{B}_{\mathrm{k}1}'}w$, the input $w$ in the first term may be handled independently of that in the second term (i.e., they may be regarded to be different functions). This is equivalent to saying that $\mathcal{F}_{NM\mathrm{k}1}^-$ may redefined as

$$\mathcal{F}_{NM\mathrm{k}1}^- = \begin{bmatrix} \overline{\mathbf{D}_{\mathrm{a}1}'} & \overline{\mathbf{M}_{\mathrm{a}1}'}\varDelta_M^0\overline{\mathbf{B}_{\mathrm{k}1}'} & \overline{\mathbf{M}_{\mathrm{a}1}'}\mathcal{J}_{M0}\overline{\mathbf{B}_{\mathrm{k}1}'} & \cdots & \overline{\mathbf{M}_{\mathrm{a}1}'}\mathcal{J}_{MN}\overline{\mathbf{B}_{\mathrm{k}1}'} \end{bmatrix} \tag{B1}$$

without changing $\|\mathcal{F}_{NM\mathrm{k}1}^-\|$. Let us further introduce the partitioned notation $w =: [w_0^T, \cdots, w_{N+2}^T]^T$ for the input of $\mathcal{F}_{NM\mathrm{k}1}^-$. Then, we have

$$(\mathcal{F}_{NM\mathrm{k}1}^- w)(\theta') = \sum_{j=0}^{N} (\overline{\mathbf{M}_{\mathrm{a}1}'}\mathcal{J}_{Mj}\overline{\mathbf{B}_{\mathrm{k}1}'}w_{j+2})(\theta') + (\overline{\mathbf{M}_{\mathrm{a}1}'}\varDelta_M^0\overline{\mathbf{B}_{\mathrm{k}1}'}w_1)(\theta') + (\overline{\mathbf{D}_{\mathrm{a}1}'}w_0)(\theta') \tag{B2}$$

Since all the terms in the right hand side of (B2) are linear functions except the last and since $\overline{\mathbf{D}_{\mathrm{a}1}'}$ is simply an integral operator, it follows readily that the input of $\overline{\mathbf{D}_{\mathrm{a}1}'}$ may be restricted to a constant function in its treatment for evaluating $\|\mathcal{F}_{NM\mathrm{k}1}^{-}\|$. An immediate consequence of this restriction is that $(\mathcal{F}_{NM\mathrm{k}1}^{-}w)(\theta')$ becomes a linear vector function, and this suggests that

$$\|\mathcal{F}_{NM\mathrm{k}1}^{-}\| = \sup_{\|w\|\le 1} \left\| \begin{bmatrix} (\mathcal{F}_{NM\mathrm{k}1}^{-}w)(0) \\ (\mathcal{F}_{NM\mathrm{k}1}^{-}w)(h') \end{bmatrix} \right\| \tag{B3}$$

where $\sup_{\|w\|\le 1}$ in fact is taken under the additional assumption that $w_0$ is a constant function. Let us consider the following representations relevant to the right hand side of (B3).

$$(\mathcal{F}_{NM\mathrm{k}1}^{-}w)(0) = (\overline{\mathbf{D}'_{\mathrm{a}1}}w_0)(0) + \overline{C_0}\varDelta_M^0\overline{\mathbf{B}'_{\mathrm{k}1}}w_1 + \sum_{j=0}^{N}\overline{C_0}\mathcal{J}_{Mj}\overline{\mathbf{B}'_{\mathrm{k}1}}w_{j+2} \tag{B4}$$

$$(\mathcal{F}_{NM\mathrm{k}1}^{-}w)(h') = (\overline{\mathbf{D}'_{\mathrm{a}1}}w_0)(h') + \overline{C_0(I+A_2h')}\varDelta_M^0\overline{\mathbf{B}'_{\mathrm{k}1}}w_1 + \sum_{j=0}^{N}\overline{C_0(I+A_2h')}\mathcal{J}_{Mj}\overline{\mathbf{B}'_{\mathrm{k}1}}w_{j+2} \tag{B5}$$

By the definitions of $\mathbf{B}'_{\mathrm{k}1}$ and $\mathbf{D}'_{\mathrm{a}1}$, it readily follows that $(\mathcal{F}_{NM\mathrm{k}1}^{-}w)(0)$ is determined by the mappings

$$w_0 \mapsto 0 \tag{B6}$$
$$w_1 \mapsto \overline{C_0}\varDelta_M^0\mathrm{diag}[\mathbf{B}'_{\mathrm{k}1}w_1^{(1)},\ \cdots,\ \mathbf{B}'_{\mathrm{k}1}w_1^{(M)}] \tag{B7}$$
$$w_{j+2} \mapsto \overline{C_0}\mathcal{J}_{Mj}\mathrm{diag}[\mathbf{B}'_{\mathrm{k}1}w_{j+2}^{(1)},\ \cdots,\ \mathbf{B}'_{\mathrm{k}1}w_{j+2}^{(M)}] \quad (j=0,\cdots,N) \tag{B8}$$

where $\mathbf{L}_M w_j =: [(w_j^{(1)})^T\ \cdots\ (w_j^{(M)})^T]^T$ $(j=0,\cdots,N+2)$. Similarly, $(\mathcal{F}_{NM\mathrm{k}1}^{-}w)(h')$ is determined by the mappings

$$w_0 \mapsto \mathrm{diag}[(\mathbf{D}'_{\mathrm{a}1}w_0^{(1)})(h'),\ \cdots,\ (\mathbf{D}'_{\mathrm{a}1}w_0^{(M)})(h')] \tag{B9}$$
$$w_1 \mapsto \overline{C_0(I+A_2h')}\varDelta_M^0\mathrm{diag}[\mathbf{B}'_{\mathrm{k}1}w_1^{(1)},\ \cdots,\ \mathbf{B}'_{\mathrm{k}1}w_1^{(M)}] \tag{B10}$$
$$w_{j+2} \mapsto \overline{C_0(I+A_2h')}\mathcal{J}_{Mj}\mathrm{diag}[\mathbf{B}'_{\mathrm{k}1}w_{j+2}^{(1)},\ \cdots,\ \mathbf{B}'_{\mathrm{k}1}w_{j+2}^{(M)}] \quad (j=0,\cdots,N) \tag{B11}$$

The above mappings immediately lead to a procedure for the computation of $\|\mathcal{F}_{NM\mathrm{k}1}^{-}\|$ through the right hand side of (B3). For example, it would require us to compute the $L_1[0,h')$ norm of each entry of $\overline{C_0}\mathcal{J}_{Mj}\overline{A'_d(I-A\theta')B_1}$ when we compute the induced norm of the operator representing the action (B8) because $\mathbf{B}'_{\mathrm{k}1}w_1^{(k)}$ $(k=1,\cdots,M)$ is given by $\int_0^{h'}A'_d(I-A\theta')B_1w_1^{(k)}(\theta')d\theta'$ $(k=1,\cdots,M)$. This leads us to the introduction of $T_j^{[0]}$. By the properties of the $L_\infty[0,h')$ norm, it suffices us to repeat essentially the same arguments on other mappings in (B6)–(B11). Hence, we are eventually led to the introduction of $V^{[0]}$, $T_j^{[0]}$, $\overline{C_1B_1h'}$, $V^{[h']}$, and $T_j^{[h']}$ $(j=0,\cdots,N)$. By the property of $L_\infty[0,h')$ and the definition of $F_{NM\mathrm{k}1}^{-}$, it follows that $\|\mathcal{F}_{NM\mathrm{k}1}^{-}\|$ coincides with the $\infty$-norm of the finite-dimensional matrix $F_{NM\mathrm{k}1}^{-}$ given by (51), with $D_{11}$ removed. The above arguments were based on the assumption that $D_{11}=0$, but the assertion for the case $D_{11}\neq 0$ follows immediately again by the property of $L_\infty[0,h')$, as has been the case with the input approximation arguments in Kim and Hagiwara (2015a).