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Extensive Theoretical/Numerical Comparative Studies on H_2 and Generalized H_2 Norms in Sampled-Data Systems

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This paper is concerned with linear time-invariant (LTI) sampled-data systems (by which we mean sampled-data systems with LTI generalized plants and LTI controllers) and studies their H_2 norms from the viewpoint of impulse responses and generalized H_2 norms from the viewpoint of the induced norms from L_2 to L_∞. A new definition of the H_2 norm of LTI sampled-data systems is first introduced through a sort of intermediate standpoint of those for the existing two definitions. We then establish unified treatment of the three definitions of the H_2 norm through a matrix function $G(\tau)$ defined on the sampling interval $[0, h)$. This paper next considers the generalized H_2 norms, in which two types of the L_∞ norm of the output are considered as the temporal supremum magnitude under the spatial 2-norm and ∞-norm of a vector-valued function. We further give unified treatment of the generalized H_2 norms through another matrix function $F(\theta)$ which is also defined on $[0, h)$. Through a close connection between $G(\tau)$ and $F(\theta)$, some theoretical relationships between the H_2 and generalized H_2 norms are provided. Furthermore, appropriate extensions associated with the treatment of $G(\tau)$ and $F(\theta)$ to the closed interval $[0, h]$ are discussed to facilitate numerical computations and comparisons of the H_2 and generalized H_2 norms. Through theoretical and numerical studies, it is shown that the two generalized H_2 norms coincide with neither of the three H_2 norms of LTI sampled-data systems even though all the five definitions coincide with each other when single-output continuous-time LTI systems are considered as a special class of LTI sampled-data systems. To summarize, this paper clarifies that the five control performance measures are mutually related with each other but they are also intrinsically different from each other.

Keywords: sampled-data systems, H_2 norm, L_∞/L_2-induced norm, impulse response

1. Introduction

Two standard (mutually equivalent) definitions are well known for the H_2 norm of linear time-invariant (LTI) continuous-time systems. If we confine ourselves to single-input single-output (SISO) systems for a while for simplicity, then the first one is the L_2 norm of the impulse response, while the second is based on the root of the integral of squared frequency transfer function. These definitions have been generalized in two conceptually different ways to linear time-invariant (LTI) sampled-data systems Chen and Francis (1991); Bamieh and Pearson (1992a); Khargonekar and Sivashankar (1991); Hagiwara and Araki (1995) (by which we mean sampled-data systems with LTI generalized plants and LTI controllers).

The first definition, which aimed at parallel treatment for the first definition for LTI continuous-time systems, was given in Chen and Francis (1991) as the L_2 norm of the response for the impulse input occurring at an instant at which the sampler takes its action. Although it was a pioneering

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study about the H_2 norm of LTI sampled-data systems, assuming the above specific timing for the impulse input was not natural enough when we take account of the periodically time-varying nature of LTI sampled-data systems.

On the other hand, the second definition Bamieh and Pearson (1992a); Khargonekar and Sivashankar (1991); Hagiwara and Araki (1995) amends this issue and corresponds to the root mean square (RMS) of the L_2 norms of all the τ-dependent responses for the impulse inputs occurring at the instant τ in the sampling interval $[0, h)$. It is worthwhile to note that the second definition was actually a natural consequence of attempting to generalize the second definition for LTI continuous-time systems through some frameworks for describing sampled-data systems in the frequency domain.

As a study revisiting the H_2 problem of sampled-data systems, this paper first provides more thorough arguments by introducing another new (i.e., the third) natural definition for the H_2 norm; we consider the supremum of the L_2 norms of all the impulse responses mentioned above in the second definition. This paper then establishes unified treatment of the three definitions for the H_2 norm in LTI sampled-data systems by deriving their closed-form expressions with a single common matrix function $G(\tau)$ defined for $\tau \in [0, h)$. In particular, the meaning of the first definition is made more transparent through the viewpoint provided by the introduction of the new third definition of the H_2 norm. A relevant topic is also discussed whether the treatment of the supremum over $\tau \in [0, h)$ arising in the third definition may be replaced by the maximum over $\tau \in [0, h]$ by providing an explicit and feasible computation method for the missing $G(h)$ or some other alternative treatment. Our positive answer leads us to ease in numerical computations with guaranteed convergence for the third definition of the H_2 norm.

What has been described above essentially applies also to multi-input multi-output (MIMO) LTI continuous-time and sampled-data systems. If we confine ourselves to multi-input single-output (MISO) LTI continuous-time and sampled-data systems, on the other hand, it is known that we are led to an alternative definition of the H_2 norm without referring to the impulse input nor frequency responses. More precisely, the induced norm of MISO LTI continuous-time systems from L_2 to L_∞ coincides with their H_2 norm Wilson (1989); Rotea (1993); Chellabonia and Haddad (2000); Wilson (1990); Grimble (1990). Even though it is not the case for MIMO LTI continuous-time systems, their generalized H_2 norms have been introduced through the same induced norm viewpoint. More precisely, in the multi-output case, two different spatial norms (i.e., the vector ∞ and 2 norms) are often considered in defining the L_∞ norm of the output, which leads to two different generalized H_2 norms.

A pioneering work of formulating the generalized H_2 norms of MIMO LTI sampled-data systems Bamieh and Pearson (1991) applied the idea of the lifting technique Bamieh and Pearson (1991); Yamamoto (1994); Bamieh and Pearson (1992b); Toivonen (1992), and suggested a brief idea for their analysis. However, the arguments involve some mathematical errors; each of the representations for the two generalized H_2 norms involves an infinite series, taking the supremum of a function over a sampling interval, and an operation on a symmetric matrix such as the maximum eigenvalue computation, but their order is incorrect in the arguments in Bamieh and Pearson (1991), as it turns out by the correct arguments in the present paper. Furthermore, the discussions in Bamieh and Pearson (1991) were actually carried out only with an optimization problem of the generalized H_2 norms in mind for possible further studies, rather than an exact analysis of the norms, and some modification was applied in a basic equation in such a way that (the optimization process would not be affected but) the analysis problem is obviously affected. Explicit analysis methods for the generalized H_2 norms for sampled-data systems were first developed in Zhu and Skelton (1995) (under a little restrictive assumption on the generalized plant).

The present paper further revisits the generalized H_2 norm analysis of LTI sampled-data systems. Our underlying motivation does not lie merely in removing the restrictive assumption in Zhu and Skelton (1995) so that a much wider class of practical problems can be handled. In contrast, we are more interested in some important arguments missing in Zhu and Skelton (1995), i.e., theoretical and numerical studies for revealing the mutual relationships among the three H_2 norms and two generalized H_2 norms for LTI sampled-data systems, where all these norms are known to coincide with each other if we confine ourselves to the restricted case of MISO LTI continuous-time systems. To facilitate the theoretical part of such studies, this paper derives closed-form representations for the generalized H_2 norms with a matrix function $F(\theta)$ defined for $\theta \in [0, h)$. A close connection between $F(\theta)$ and $G(\tau)$ will then be used to study some relationship among some of the three H_2 norms and the two generalized H_2 norms, while their intrinsic difference makes it hard to achieve comprehensive comparisons. It is thus among our interest to carry out such comparisons through numerical examples, and we hence discuss a numerical aspect in the computation of the generalized H_2 norms. In particular, even though the supremum over $\theta \in [0, h)$ must be taken in the theoretical characterizations of the generalized H_2 norms, we show that we can have an explicit and feasible method for replacing the supremum with the maximum over $\theta \in [0, h]$, which again leads to ease in numerical computations with guaranteed convergence.

The organization of this paper is as follows. In Section 2, we give mathematical notations used in this paper. An operator-theoretic approach to sampled-data systems through their lifted representation is given in Section 3. With the lifted representation of sampled-data systems, we review the two existing definitions for the H_2 norm of sampled-data systems Chen and Francis (1991); Bamieh and Pearson (1992a); Khargonekar and Sivashankar (1991); Hagiwara and Araki (1995) and introduce a new (i.e., the third) definition for the H_2 norm in Section 4. In Section 5, we deal with the two generalized H_2 norms (i.e., the induced norms from L_2 to L_∞) for LTI sampled-data systems, derive their closed-form representations, and develop theoretical results associated with some inequality relations among the H_2 norms and generalized H_2 norms. These relations suggest that the two generalized H_2 norms coincide with neither of the three definitions of the H_2 norm for LTI sampled-data systems from the viewpoint of impulse response, which we indeed confirm through numerical examples in Section 6; in that section, we first provide approximate but asymptotically exact methods for computing the third H_2 norm as well as the two generalized H_2 norms, and numerical examples demonstrate the validity of the theoretical results. The numerical results are computed through the replacement of the supremum over $\tau \in [0, h)$ (or $\theta \in [0, h)$) with the maximum over the corresponding closed interval. Validity of such treatment and a relevant convergence property are further confirmed in the numerical examples. These examples further demonstrate that no more theoretical inequalities can hold between the H_2 and generalized H_2 norms beyond what has been derived theoretically in this paper.

Finally, we remark that the arguments in this paper are significant extensions of the partial results presented at the conference Kim and Hagiwara (2015a), in which only the SISO case was considered, no arguments were given about the new third definition of the H_2 norm, and no numerical studies and proofs were provided.

2. Notations

In this paper, we use the notations $\mathbb{N}$, $\mathbb{R}^\nu$ and $\delta(t)$ to denote the set of positive integers, the set of ν-dimensional real vectors and the scalar-valued impulse function (occurring at $t = 0$), respectively. We further use the notation $\mathbb{N}_0$ to imply $\mathbb{N} \cup \{0\}$.

The ∞-norm and 2-norm of a finite-dimensional vector are denoted by $|\cdot|_\infty$ and $|\cdot|_2$, respectively. The notation $\|\cdot\|_{(2,2)}$ is used to mean the (standard) L_2 norm, i.e.,

$$\|w(\cdot)\|_{(2,2)} := \left(\int_0^\infty |w(t)|_2^2 dt \right)^{1/2} \tag{1}$$

for a real-vector-valued function w on $[0, \infty)$ such that the right hand side is well-defined. The class of all such ν-dimensional w is denoted by $(L_2)^\nu$. The notations $\|\cdot\|_{(\infty,p)}$ $(p = \infty, 2)$ are used to

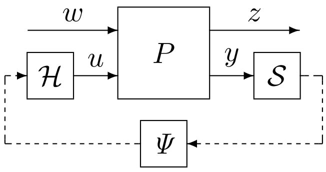


Figure 1. Sampled-data system $\Sigma_{\rm SD}$.

imply the L_∞ norms under the spatial ∞-norm and 2-norm, respectively, i.e.,

$$\|z(\cdot)\|_{(\infty,\infty)} := \operatorname*{ess\,sup}_{0\le t<\infty} |z(t)|_\infty = \operatorname*{ess\,sup}_{0\le t<\infty} \max_{1\le i\le \nu} |z_i(t)| \tag{2}$$

$$\|z(\cdot)\|_{(\infty,2)} := \operatorname*{ess\,sup}_{0\le t<\infty} |z(t)|_2 = \operatorname*{ess\,sup}_{0\le t<\infty} \left(z^T(t)z(t)\right)^{1/2} \tag{3}$$

for a ν-dimensional vector function z on $[0,\infty)$ such that the right hand sides are well-defined (if one is well-defined, then the other is too because the norms for the finite-dimensional vector $z(t)$ are equivalent). The class of all such z is denoted by $(L_\infty)^\nu$. On the other hand, for an operator $\mathbf{T}$ from $(L_2)^{\nu_1}$ to $(L_\infty)^{\nu_2}$, the notations $\|\cdot\|_{(\infty,p)/(2,2)}$ $(p=\infty,2)$ are used to mean the induced norms

$$\|\mathbf{T}\|_{(\infty,p)/(2,2)} := \sup_{w\in(L_2)^{\nu_1}} \frac{\|\mathbf{T}w\|_{(\infty,p)}}{\|w\|_{(2,2)}} \quad (p=\infty,2) \tag{4}$$

The notations $\|w\|_{(2,2)}$ and $\|z\|_{(\infty,p)}$ $(p=\infty,2)$ are used also when w and z are defined on a finite interval $[0,h)$, in which case (1)–(3) are modified accordingly. Similarly, the above notations $\|\mathbf{T}\|_{(\infty,p)/(2,2)}$ $(p=\infty,2)$ are also used, e.g., for $\mathbf{T}:(L_2)^{\nu_1}\to(L_\infty[0,h))^{\nu_2}$ and all these induced norms are called the L_∞/L_2-induced norm, whose distinction will be clear from the context.

We use the notation $l^2_{(L_2[0,h))^\nu}$ to denote the space of all sequences of functions in $(L_2[0,h))^\nu$ whose norms are square summable. Furthermore, we use the notations $\mathrm{tr}(\cdot)$, $\lambda_{\max}(\cdot)$ and $d_{\max}(\cdot)$ to denote the trace, maximum eigenvalue and maximum diagonal entry of a real symmetric matrix, respectively.

3. Sampled-Data Systems and Their Lifted Representation

Consider the stable LTI sampled-data system $\Sigma_{\rm SD}$ shown in Fig. 1, where P denotes the continuous-time LTI generalized plant, while Ψ, $\mathcal{H}$ and $\mathcal{S}$ denote the discrete-time LTI controller, the zero-order hold and the ideal sampler, respectively, operating with sampling period h in a synchronous fashion. Solid lines and dashed lines in Fig. 1 are used to represent continuous-time and discrete-time signals, respectively. Suppose that P and Ψ are described respectively by

$$P:\begin{cases}\dfrac{dx}{dt}=Ax+B_1w+B_2u\\ z=C_1x+D_{12}u\\ y=C_2x\end{cases},\quad \Psi:\begin{cases}\psi_{k+1}=A_\Psi\psi_k+B_\Psi y_k\\ u_k=C_\Psi\psi_k+D_\Psi y_k\end{cases} \tag{5}$$

where $x(t)\in\mathbb{R}^n$, $w(t)\in\mathbb{R}^{n_w}$, $u(t)\in\mathbb{R}^{n_u}$, $z(t)\in\mathbb{R}^{n_z}$, $y(t)\in\mathbb{R}^{n_y}$, $\psi_k\in\mathbb{R}^{n_\Psi}$, $y_k=y(kh)$ and $u(t)=u_k$ $(kh\le t<(k+1)h)$.

To facilitate the arguments of this paper, we review the lifted representation Bamieh and Pearson (1991); Yamamoto (1994); Bamieh and Pearson (1992b); Toivonen (1992) of the sampled-data system $\Sigma_{\rm SD}$. Given $f\in(L_p)^\nu$ for $p=\infty$ or 2, its lifting $\{\widehat{f}_k\}_{k=0}^\infty$ with $\widehat{f}_k\in(L_p[0,h))^\nu$ $(k\in\mathbb{N}_0)$ is defined by $\widehat{f}_k(\theta) = f(kh+\theta) \quad (0 \le \theta < h)$. Then, the lifted representation of the sampled-data system $\Sigma_{\rm SD}$ viewed as an (h-periodic) mapping from $w \in (L_2)^{n_w}$ to $z \in (L_\infty)^{n_z}$ is given by

$$\begin{cases} \xi_{k+1} &= \mathcal{A}\xi_k + \mathcal{B}\widehat{w}_k \\ \widehat{z}_k &= \mathcal{C}\xi_k + \mathcal{D}\widehat{w}_k \end{cases} \tag{6}$$

with $\xi_k := [x_k^T \ \psi_k^T]^T$ $(x_k := x(kh))$, the matrix

$$\mathcal{A} = \begin{bmatrix} A_d + B_{2d}D_\Psi C_{2d} & B_{2d}C_\Psi \\ B_\psi C_{2d} & A_\Psi \end{bmatrix} : \mathbb{R}^{n+n_\Psi} \to \mathbb{R}^{n+n_\Psi} \tag{7}$$

and the operators

$$\mathcal{B} = J_\Sigma \mathbf{B}_1 : (L_2[0,h))^{n_w} \to \mathbb{R}^{n+n_\Psi} \tag{8}$$

$$\mathcal{C} = \mathbf{M}_1 C_\Sigma : \mathbb{R}^{n+n_\Psi} \to (L_\infty[0,h))^{n_z} \tag{9}$$

$$\mathcal{D} = \mathbf{D}_{11} : (L_2[0,h))^{n_w} \to (L_\infty[0,h))^{n_z} \tag{10}$$

where

$$A_d := \exp(Ah),\ B_{2d} := \int_0^h \exp(A\theta)B_2 d\theta,\ C_{2d} := C_2 \tag{11}$$

$$J_\Sigma := \begin{bmatrix} I \\ 0 \end{bmatrix} \in \mathbb{R}^{(n+n_\Psi)\times n},\ \mathbf{B}_1 w = \int_0^h \exp(A(h-\theta))B_1 w(\theta) d\theta \tag{12}$$

$$M_1 := \begin{bmatrix} C_1 & D_{12} \end{bmatrix},\ A_2 := \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix},\ \left(\mathbf{M}_1 \begin{bmatrix} x \\ u \end{bmatrix}\right)(\theta) = M_1 \exp(A_2\theta) \begin{bmatrix} x \\ u \end{bmatrix} \tag{13}$$

$$C_\Sigma := \begin{bmatrix} I & 0 \\ D_\Psi C_{2d} & C_\Psi \end{bmatrix},\ (\mathbf{D}_{11}w)(\theta) = \int_0^\theta C_1 \exp(A(\theta-\tau))B_1 w(\tau) d\tau \tag{14}$$

From the stability assumption of $\Sigma_{\rm SD}$, $\mathcal{A}$ is Schur stable.

Let us introduce the matrix functions

$$B_h(\tau) = J_\Sigma \exp(A(h-\tau))B_1 \tag{15}$$

$$D_\theta(\tau) = C_1 \exp(A(\theta-\tau))B_1 \mathbf{1}(\theta-\tau) \tag{16}$$

(with $\mathbf{1}(\cdot)$ being the unit step function) and the matrix

$$C_\theta = M_1 \exp(A_2\theta) C_\Sigma \tag{17}$$

Then, we can describe the operations of $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ more concisely as follows:

$$\mathcal{B}\widehat{w}_k = \int_0^h B_h(\tau)\widehat{w}_k(\tau)d\tau,\ \left(\mathcal{C}\begin{bmatrix} x \\ u \end{bmatrix}\right)(\theta) = C_\theta \begin{bmatrix} x \\ u \end{bmatrix},\ (\mathcal{D}\widehat{w}_k)(\theta) = \int_0^h D_\theta(\tau)\widehat{w}_k(\tau)d\tau \tag{18}$$

Remark 1: *The H_2 norm has often been associated with the responses for impulse inputs in the studies of sampled-data systems Chen and Francis (1991); Bamieh and Pearson (1992a); Khargonekar and Sivashankar (1991). There is a reason why we nevertheless have viewed the sampled-data system $\Sigma_{\rm SD}$ as a mapping from $(L_2)^{n_w}$ to $(L_\infty)^{n_z}$ in the above. This is because we further study the L_∞/L_2-induced norms $\|\Sigma_{\rm SD}\|_{(\infty,p)/(2,2)} := \sup_{\|w\|_{(2,2)}\le 1} \|z\|_{(\infty,p)}$ $(p = \infty, 2)$ in Section 5.*

Such a study is very relevant and important because when $\Sigma_{\rm SD}$ is actually a continuous-time LTI system with a single output as a special case, these two induced norms coincide with each other and are known to be equivalent to the H_2 norm of the continuous-time system. Hence, these induced norms could lead to a reasonable alternative definition for the H_2 norm of $\Sigma_{\rm SD}$ with a single output, in which the treatment of impulse inputs is completely suppressed. These two induced norms bifurcate for multi-output continuous-time LTI systems and neither of them coincides with the H_2 norm, in general. However, we can regard these two induced norms as generalized H_2 norms of $\Sigma_{\rm SD}$ (with multiple outputs) Rotea (1993); Wilson (1990); Grimble (1990), in which this paper is further interested. It is worth remarking that the assumptions '$D_{11} = 0$' and '$D_{21} = 0$' in (5) are necessary (and sufficient by the stability of $\Sigma_{\rm SD}$) not only for these induced norms (i.e., generalized H_2 norms) but also for the H_2 norms in the impulse input viewpoint (see the following section) to be well-defined and bounded in $\Sigma_{\rm SD}$.

Remark 2: *Even though $w \notin (L_2)^{n_w}$ when we consider the H_2 norms of $\Sigma_{\rm SD}$ through its impulse responses, we follow the convention of the studies on the H_2 problems of sampled-data systems here, and formally allow to take w to be the impulse function such as $\delta_\tau e_i := \delta(t-\tau)e_i$ (with e_i being the ith vector in the natural basis for $\mathbb{R}^{n_w}$) occurring at $t = \tau$. Under this convention, (18) should also be given the associated interpretations, i.e., $\mathcal{B}\delta_\tau e_i = B_h(\tau)e_i$ and $(\mathcal{D}\delta_\tau e_i)(\theta) = D_\theta(\tau)e_i$.*

The equations (15)–(18) together with Remark 2 play important roles in the subsequent arguments not only in the induced norm viewpoint but also in the impulse input viewpoint.

4. H_2 Norms of Sampled-Data Systems from the Impulse Response Viewpoint

This section first reviews, in technical details, the two existing definitions in Chen and Francis (1991) and in Bamieh and Pearson (1992a); Khargonekar and Sivashankar (1991); Hagiwara and Araki (1995) for the H_2 norm of LTI sampled-data systems based on the lifted-representation of $\Sigma_{\rm SD}$. Roughly speaking, the first definition Chen and Francis (1991) considers the L_2 norm of the regulated output $z(t)$ for the impulse input $w(t) = \delta(t)e_i$ occurring at $t = 0$, an instant at which the sampler takes its action. The second definition Bamieh and Pearson (1992a); Khargonekar and Sivashankar (1991); Hagiwara and Araki (1995), on the other hand, considers the root mean square (RMS) of the L_2 norms of all the impulse responses $z(t)$ for the impulse inputs $w(t)$ occurring at any instants in $[0, h)$. Furthermore, this section gives more thorough arguments by introducing another new (i.e., the third) definition for the H_2 norm of $\Sigma_{\rm SD}$ given by modifying the first definition by taking account of the h-periodic nature of $\Sigma_{\rm SD}$; we consider the impulse input $w(t) = \delta_\tau(t)e_i := \delta(t-\tau)e_i$ occurring at $t = \tau \in [0, h)$ and consider the supremum of the corresponding L_2 norms of $z(\cdot)$ with respect to $\tau \in [0, h)$. The implicit assumptions '$D_{11} = 0$' and '$D_{21} = 0$' for the continuous-time generalized plant P in (5) are necessary (and sufficient by the stability of $\Sigma_{\rm SD}$) for these H_2 norms of $\Sigma_{\rm SD}$ to be well-defined/bounded (recall Remark 1).

4.1 *The H_2 Norm Definition through a Single Impulse Input at $t = 0$ for Each Input Channel*

When $w(t) = \delta_\tau(t)e_i$ ($\tau \in [0, h)$), we can formally regard that its lifted representation is given by

$$\widehat{w}_0 = \delta_\tau(\theta)e_i, \quad \widehat{w}_k = 0 \ (k \in \mathbb{N}) \tag{19}$$

By evaluating the square root of the sum of the squared L_2 norms of the corresponding outputs for $i = 1, \cdots, n_w$ under the limit of $\tau \to h - 0$, the first H_2 norm of the LTI sampled-data system

$\Sigma_{\rm SD}$, denoted by $\|\Sigma_{\rm SD}\|_{H_2}^{[0-]}$, is defined in Chen and Francis (1991) as

$$\|\Sigma_{\rm SD}\|_{H_2}^{[0-]} := \lim_{\tau\to h-0}\left(\sum_{i=1}^{n_w}\left\|\left[(\mathcal{D}\delta_\tau e_i)^T\ (\mathcal{CB}\delta_\tau e_i)^T\ (\mathcal{CAB}\delta_\tau e_i)^T\ \cdots\right]^T\right\|_{(2,2)}^2\right)^{1/2} \tag{20}$$

Remark 3: *The superscript* $[0-]$ *in the notation* $\|\Sigma_{\rm SD}\|_{H_2}^{[0-]}$ *is to mean that the impulse input* $\delta_\tau e_i$ *is applied 'at* $t = 0-$*' despite our earlier mention to the limit of* $\tau \to h-0$*. This somewhat confusing situation is explained as follows. In the treatment of Chen and Francis (1991), whose authors studied to apply the impulse 'at* $t = 0$*,' the impulse was interpreted as driving the state of the continuous-time generalized plant* P *from* $x(0) = 0$ *to* $x(0+) = B_1e_i$*, which in turn was interpreted as producing* $y(0+) = C_2B_1e_i$*. It was further interpreted that this output is sampled at '*$t = 0$*' by the sampler to yield the input* y_0 *of the discrete-time controller* Ψ *at* $k = 0$*. We can verify that taking the limit about* $\tau \to h-0$ *in (20) successfully recovers these interpretations. This situation cannot be reflected by simply taking* $\tau = 0$*, and this is precisely why we take the limit about* $\tau \to h-0$ *in (20) rather than simply taking the value for* $\tau = 0$*. By the* h*-periodicity of* $\Sigma_{\rm SD}$*, it is acceptable and reasonable to use the superscript* $[0-]$ *instead of (seemingly more appropriate)* $[h-]$*. The adopted superscript is also convenient in the sense that it would suggest that the impulse is actually applied before the sampler acts 'at* $t = 0$*' as is precisely the case in the interpretation behind the definition in Chen and Francis (1991).*

Although the H_2 norm considered in this subsection could be computed through the arguments in Chen and Francis (1991) (through a special and intricate interpretation of the action of the sampler at '$t = 0$' as stated in the above remark), the following alternative characterization of the same norm $\|\Sigma_{\rm SD}\|_{H_2}^{[0-]}$ is essential. This is because it provides us with improved consistency and unity with the studies of the other two types of the H_2 norms through a common matrix function $G(\tau)$. More precisely, direct computations of (20) readily lead, again by Remark 2, to

$$\begin{aligned}\|\Sigma_{\rm SD}\|_{H_2}^{[0-]} &= \Bigg(\sum_{i=1}^{n_w}\Big(\int_0^h e_i^T(C_\theta B_h(h))^T C_\theta B_h(h)e_i d\theta \\ &\qquad + \int_0^h e_i^T(C_\theta \mathcal{A}B_h(h))^T C_\theta\mathcal{A}B_h(0)e_i d\theta + \cdots\Big)\Bigg)^{1/2} \\ &= {\rm tr}^{1/2}\left(\sum_{j=0}^{\infty}\int_0^h (C_\theta\mathcal{A}^jB_h(h))^T C_\theta\mathcal{A}^jB_h(h)d\theta\right) \qquad (21) \\ &= \lim_{\tau\to h-0} {\rm tr}^{1/2}\left(G(\tau)\right) \qquad (22)\end{aligned}$$

where

$$G(\tau) := \int_0^h D_\theta(\tau)D_\theta^T(\tau)d\theta + \sum_{j=0}^{\infty}\int_0^h C_\theta\mathcal{A}^jB_h(\tau)(C_\theta\mathcal{A}^jB_h(\tau))^T d\theta \tag{23}$$

Remark 4: *Even though (21) does not involve any terms about* $D_\theta(\tau)$*, we introduced the above* $G(\tau)$ *with such a term. This is because it plays an important role in the following discussions. Note for the validity of (22) that* $D_\theta(\tau)$ *vanishes for each* $\theta \in [0, h)$ *as* $\tau \to h-0$*.*

4.2 H_2 Norm Definition through Averaging about Impulse Inputs for Each Input Channel

If we take account of the h-periodicity of $\Sigma_{\rm SD}$, assuming that the impulse input $\delta_\tau e_i$ is applied only at a sampling instant may not seem very natural. The second H_2 norm, denoted by $\|\Sigma_{\rm SD}\|_{H_2}^{[0,h)}$, circumvents this issue and is defined in Bamieh and Pearson (1992a); Khargonekar and Sivashankar (1991); Hagiwara and Araki (1995) as a sort of the RMS of the L_2 norms of $z(t)$ for the impulse inputs $\delta_\tau e_i$ for $\tau \in [0, h)$ as

$$\begin{aligned}\|\Sigma_{\rm SD}\|_{H_2}^{[0,h)} &:= \left(\frac{1}{h}\int_0^h \sum_{i=1}^{n_w} \left\| [(\mathcal{D}\delta_\tau e_i)^T \ (\mathcal{C}\mathcal{B}\delta_\tau e_i)^T \ (\mathcal{C}\mathcal{A}\mathcal{B}\delta_\tau e_i)^T \ \cdots]^T \right\|_{(2,2)}^2 d\tau \right)^{1/2} \\ &= \mathrm{tr}^{1/2}\left(\frac{1}{h}\left(\int_0^h\int_0^h D_\theta(\tau)D_\theta^T(\tau)d\theta d\tau + \int_0^h\int_0^h C_\theta B_h(\tau)(C_\theta B_h(\tau))^T d\theta d\tau \right.\right. \\ &\qquad \left.\left. + \int_0^h\int_0^h C_\theta \mathcal{A} B_h(\tau)(C_\theta \mathcal{A} B_h(\tau))^T d\theta d\tau + \cdots \right)\right) \end{aligned} \tag{24}$$

The above equation admits two alternative further manipulations. The first one proceeds immediately from (23) as follows.

$$\|\Sigma_{\rm SD}\|_{H_2}^{[0,h)} = \mathrm{tr}^{1/2}\left(\frac{1}{h}\int_0^h G(\tau)d\tau\right) \tag{25}$$

Even though this expression suffices for the comparison of the three H_2 norms discussed in this section, we further consider the second manipulation for the discussions in the following section, which is given by

$$\|\Sigma_{\rm SD}\|_{H_2}^{[0,h)} = \mathrm{tr}^{1/2}\left(\frac{1}{h}\int_0^h F(\theta)d\theta\right) \tag{26}$$

where

$$F(\theta) := \int_0^h D_\theta(\tau)D_\theta^T(\tau)d\tau + \sum_{j=0}^{\infty}\int_0^h (C_\theta \mathcal{A}^j B_h(\tau))(C_\theta \mathcal{A}^j B_h(\tau))^T d\tau \tag{27}$$

Remark 5: *It is obvious from the above manipulations that $G(\tau)$ and $F(\theta)$ differ only in the way the same matrix function in the two variables τ and θ is integrated along the axis of one of the two variables. As stated in Remark 1, Section 5 further studies the L_∞/L_2-induced norms (or generalized H_2) norms of $\Sigma_{\rm SD}$, in which the same $F(\theta)$ plays a central role. Hence, the comparison of the H_2 norm $\|\Sigma_{\rm SD}\|_{H_2}^{[0,h)}$ with the generalized H_2 norms $\|\Sigma_{\rm SD}\|_{(\infty,p)/(2,2)}$ ($p = \infty, 2$) will be carried out by using (26) rather than (25).*

4.3 H_2 Norm Definition through a Single Impulse Input at $t = \tau$ for Each Input Channel and Supremum over τ

In this paper, we newly consider the third definition of the H_2 norm of sampled-data systems through a sort of intermediate standpoint of those for the existing two definitions. That is, we consider the impulse inputs $\delta_\tau(t)e_i$ also for $\tau \neq h - 0$ but take the supremum over τ instead of 'average' (i.e., RMS). The new H_2 norm, denoted by $\|\Sigma_{\rm SD}\|_{H_2}^{[\tau^\star]}$, is defined by

$$\|\Sigma_{\rm SD}\|_{H_2}^{[\tau^\star]} := \sup_{0\le\tau<h}\left(\sum_{i=1}^{n_w}\left\|[(\mathcal{D}\delta_\tau e_i)^T\ (\mathcal{C}\mathcal{B}\delta_\tau e_i)^T\ (\mathcal{C}\mathcal{A}\mathcal{B}\delta_\tau e_i)^T\ \cdots]^T\right\|_{(2,2)}^2\right)^{1/2} \tag{28}$$

Roughly speaking, the (non-numeric) symbol $\tau^\star$ in the notation $\|\Sigma_{\rm SD}\|_{H_2}^{[\tau^\star]}$ means considering "$\arg\sup_\tau$" with respect to the supremum on the right hand side of (28). Direct computations readily lead to

$$\|\Sigma_{\rm SD}\|_{H_2}^{[\tau^\star]} = \sup_{0\le\tau<h} {\rm tr}^{1/2}\,(G(\tau)) \tag{29}$$

where $G(\tau)$ is given by (23).

Note that if we consider a (MIMO) continuous-time LTI system as a special class of $\Sigma_{\rm SD}$, direct computations readily show that $G(\tau)$ in (23) becomes a constant matrix[1] on $[0,h)$. This is because $G(\tau)$ equals the infinite integral over the time interval $[0,\infty)$ of the 'squared' impulse response matrix delayed by τ, where the integral is obviously independent of τ. Hence, as is the case with $\|\Sigma_{\rm SD}\|_{H_2}^{[0-]}$ and $\|\Sigma_{\rm SD}\|_{H_2}^{[0,h)}$, too, $\|\Sigma_{\rm SD}\|_{H_2}^{[\tau^\star]}$ reduces to the H_2 norm of the continuous-time LTI system. Therefore, it is reasonable to adopt $\|\Sigma_{\rm SD}\|_{H_2}^{[\tau^\star]}$ as the third definition of the H_2 norm of sampled-data systems.

The following result is obvious from our preceding arguments.

Proposition 1: *$\|\Sigma_{\rm SD}\|_{H_2}^{[0-]}$, $\|\Sigma_{\rm SD}\|_{H_2}^{[0,h)}$ and $\|\Sigma_{\rm SD}\|_{H_2}^{[\tau^\star]}$ are given respectively by*

$$\|\Sigma_{\rm SD}\|_{H_2}^{[0-]} = \lim_{\tau\to h-0} {\rm tr}^{1/2}(G(\tau)) \tag{30}$$

$$\|\Sigma_{\rm SD}\|_{H_2}^{[0,h)} = \left(\frac{1}{h}\int_0^h {\rm tr}(G(\tau))d\tau\right)^{1/2} \tag{31}$$

$$\|\Sigma_{\rm SD}\|_{H_2}^{[\tau^\star]} = \sup_{0\le\tau<h} {\rm tr}^{1/2}(G(\tau)) \tag{32}$$

where $G(\tau)$ is given by (23). Furthermore,

$$\|\Sigma_{\rm SD}\|_{H_2}^{[0-]} \le \|\Sigma_{\rm SD}\|_{H_2}^{[\tau^\star]},\quad \|\Sigma_{\rm SD}\|_{H_2}^{[0,h)} \le \|\Sigma_{\rm SD}\|_{H_2}^{[\tau^\star]} \tag{33}$$

Proof. The first assertion is nothing but (22), (25) and (29). The second assertion is obvious from the first assertion. □

Having supplemented the third reasonable definition missing in the literature is believed to be useful in the subsequent study, in which we aim at comparing, theoretically and numerically, these H_2 norms and generalized H_2 norms defined through the L_∞/L_2-induced norm viewpoint. In the theoretical comparison, it is very important that $\|\Sigma_{\rm SD}\|_{H_2}^{[0,h)}$ admits an alternative representation with $F(\theta)$ rather than $G(\tau)$ as given in (26).

Another important contribution of our paper is that the meaning of the H_2 norm $\|\Sigma_{\rm SD}\|_{H_2}^{[0-]}$ given in Chen and Francis (1991) has been clarified in connection with the newly introduced third H_2 norm $\|\Sigma_{\rm SD}\|_{H_2}^{[\tau^\star]}$. In particular, the former norm is shown to coincide not with ${\rm tr}^{1/2}(G(0))$ but with

[1] A similar comment applies to $F(\theta)$; even though it is less intuitive and requires a little observation, this matrix is also a constant function over $[0,h)$ for continuous-time LTI systems.

$\mathrm{tr}^{1/2}(G(h-))$ despite the arguments in Chen and Francis (1991) in which the impulse inputs are insisted to be applied at '$t = 0$' (Remark 3). This implies that simply extending $\mathrm{tr}^{1/2}(G(\tau))$, which is defined only for $\tau \in [0, h)$, to an h-periodic function through $\mathrm{tr}^{1/2}(G(\tau + h)) = \mathrm{tr}^{1/2}(G(\tau))$ does not necessarily yield a continuous function (see also Figures 2–4 (a) in Section 6 about numerical examples). Hence, such an extension is helpless in our rewriting (32) into the plainer and more convenient expression

$$\|\varSigma_{\mathrm{SD}}\|_{H_2}^{[\tau^\star]} = \max_{0 \leq \tau \leq h} \mathrm{tr}^{1/2}(G(\tau)) \tag{34}$$

Nevertheless, what we have clarified immediately shows that the above expression is indeed justified by defining the value of $\mathrm{tr}^{1/2}(G(\tau))$ at $\tau = h$ as the H_2 norm in the first definition (i.e., in Chen and Francis (1991)), because it then coincides with the limit $\mathrm{tr}^{1/2}(G(h-))$ and thus yields a continuous function over $[0, h]$ (note that continuity is obvious at $\tau \in [0, h)$). An essential point here is that we can then compute $\mathrm{tr}^{1/2}(G(\tau))$ exactly including that for $\tau = h$, i.e., regardless of $\tau \in [0, h]$ (see Chen and Francis (1991) for the numerical computation of $\mathrm{tr}^{1/2}(G(h))$). Hence, a possible technical difficulty in numerically computing the right hand side of (32) involving the supremum would be alleviated by working instead on (34). Specifically, the method through equally spaced sampling over $[0, h]$ is justified because continuous functions on a closed interval are uniformly continuous (as in our ultimate computation method (52) in Section 6, where the reason is also discussed why we introduce an alternative matrix function $\widetilde{G}(\tau)$ instead of $G(\tau)$).

5. Generalized H_2 Norms of Sampled-Data Systems from the L_∞/L_2-Induced Norm Viewpoint and Their Relationship with the H_2 Norms from the Impulse Response Viewpoint

We mentioned in Section 3 that the L_∞/L_2-induced norms of the sampled-data system $\varSigma_{\mathrm{SD}}$ defined by

$$\|\varSigma_{\mathrm{SD}}\|_{(\infty,p)/(2,2)} := \sup_{\|w\|_{(2,2)} \leq 1} \|z\|_{(\infty,p)} \quad (p = \infty, 2) \tag{35}$$

can be regarded as its generalized H_2 norms (see Remark 1). The analysis of these generalized H_2 norms for sampled-data systems was first tackled in Zhu and Skelton (1995) (under the additional assumption that $D_{12} = 0$ in (5)). The technique therein, however, is somewhat restrictive and lacks perspectives in possible extension of the arguments. More or less relevant to such an aspect is the fact that no comparison of these generalized H_2 norms (through the induced norm viewpoint) was made with any of the H_2 norms through the impulse response viewpoint in the preceding section. This section aims at somewhat bridging insufficiency in the theoretical results in the two different viewpoints by providing some inequality relations among the H_2 norms and generalized H_2 norms of the sampled-data system $\varSigma_{\mathrm{SD}}$ (which actually all coincide with each other when $\varSigma_{\mathrm{SD}}$ is a single-output continuous-time LTI system).

Remark 6: *In the pioneering study of the generalized H_2 norms (i.e., the L_∞/L_2-induced norms) of sampled data systems in Zhu and Skelton (1995), discrete-time measurement noises were also considered. The problem formulation in our paper is hence more restrictive in this respect, while it is less restrictive in the sense that $D_{12} = 0$ is not required in (5). The standpoint of our paper is that the generalization in the latter issue is quite important while the restriction in the former issue is not essential in the theoretical development that the paper aims at. The reason is threefold. First, there are quite a large important classes of problems with $D_{12} \neq 0$ because the magnitude of the control input is usually of practical concern. Second, the studies on the H_2 norms of sampled-data systems* *in the impulse response viewpoint, with which we aim at comparing the generalized H_2 norms, are mostly based on the system configuration without discrete-time measurement noises. It would be thus quite distracting or even nonsense to consider discrete-time measurement noises in our following comparison arguments if it were merely because the pioneering study in this direction did so. The third reason is related to what the essential aspect is in the study of sampled-data systems. As stated above, the existing studies on the H_2 norms (from the impulse response viewpoint) do not take the measurement noises into account. More importantly, the same is true for many other important studies on sampled-data systems such as the H_∞ problem (e.g., Bamieh and Pearson (1992b); Toivonen (1992)) and the L_1 problem (e.g., Dullerud and Francis (1992); Bamieh and Dahleh (1993)). This is because the most essential aspect in the study of sampled-data systems is how to handle adequately and feasibly the intersample behavior of continuous-time signals so as to achieve (either completely or almost) precise evaluation and optimization of the associated system norms. In this sense, considering discrete-time measurement noises may be regarded as secondary importance. We further remark that our following arguments on the generalized H_2 norms can be extended to a discretization approach of the continuous-time generalized plant Kim and Hagiwara (2015b, 2016). Once the problems in sampled-data systems are converted into those in discrete-time systems through such an approach, the discrete-time measurement noises in sampled-data systems can be fully recovered in the (almost) equivalent treatment of the associated discretized systems. Since this direction gives no difference from the treatment in Zhu and Skelton (1995) after all, we can conclude that ignoring the discrete-time measurement noises from the outset leads to no loss of generality at all.*

5.1 *Treatment of Generalized H_2 Norms through the Toeplitz Operator Matrix $\mathcal{T}$ and the Relevant Operator $\mathcal{F}$*

To proceed to the comparisons of H_2 and generalized H_2 norms, we begin our arguments by giving alternative characterizations of generalized H_2 (or L_∞/L_2-induced) norms in the lifting-based framework. A crucial step providing a viewpoint quite different from the treatment in Zhu and Skelton (1995) is to describe the closed-loop input/output relation of $\Sigma_{\rm SD}$ obtained by (6) as $\widehat{z} = \mathcal{T}\widehat{w}$, where $\widehat{w} := [\widehat{w}_0^T,\ \widehat{w}_1^T, \cdots]^T$, $\widehat{z} := [\widehat{z}_0^T,\ \widehat{z}_1^T, \cdots]^T$ and

$$\mathcal{T} = \begin{bmatrix} \mathcal{D} & 0 & \cdots & & & \\ \mathcal{CB} & \mathcal{D} & 0 & \cdots & & \\ \mathcal{CAB} & \mathcal{CB} & \mathcal{D} & 0 & \cdots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{36}$$

By defining $\|\widehat{w}\|_{(2,2)} := \left(\sum_{k=0}^{\infty} \|\widehat{w}_k\|_{(2,2)}^2\right)^{1/2}$ and $\|\widehat{z}\|_{(\infty,p)} = \sup_{k\in\mathbb{N}_0} \|\widehat{z}_k\|_{(\infty,p)}$ ($p=\infty,2$), the L_∞/L_2-induced norms of $\Sigma_{\rm SD}$ coincide with those of the operator matrix $\mathcal{T}$. By the Toeplitz structure of $\mathcal{T}$ (i.e., each row in $\mathcal{T}$ is a left-shifted version of the next row) together with the definitions of $\|\cdot\|_{(\infty,p)}$ ($p=\infty,2$), it readily follows that the L_∞/L_2-induced norms of $\Sigma_{\rm SD}$ are given by

$$\|\Sigma_{\rm SD}\|_{(\infty,p)/(2,2)} = \|\mathcal{F}\|_{(\infty,p)/(2,2)} = \sup_{\|\widehat{w}\|_{(2,2)}\le 1} \|\mathcal{F}\widehat{w}\|_{(\infty,p)} \tag{37}$$

for $p=\infty,2$, where $\mathcal{F}$ is essentially the "last" block row of $\mathcal{T}$ with the order of the columns reversed:

$$\mathcal{F} := \begin{bmatrix} \mathcal{D} & \mathcal{CB} & \mathcal{CAB} & \mathcal{CA}^2\mathcal{B} & \cdots \end{bmatrix} \tag{38}$$

This leads to the new and concise arguments for the characterization of generalized H_2 norms of $\Sigma_{\rm SD}$ given in the following two subsections.

Remark 7: *It is implicitly assumed in (38) (as well as (6)) that the sampler takes its action at* $t = 0$*. One could raise a question that if an intersample instant is regarded as* $t = 0$*, the corresponding generalized* H_2 *norms might become different from* $\|\Sigma_{\rm SD}\|_{(\infty,p)/(2,2)}$ $(p = \infty, 2)$ *as similarly to the case of the* H_2 *norm for* $\Sigma_{\rm SD}$*. However, this is not the case as an immediate property of an induced norm as in the* L_2 *or* L_∞*-induced norm because of the* h*-periodic nature of the input-output mapping of* $\Sigma_{\rm SD}$*.*

5.2 *Characterization of Generalized* H_2 *Norm* $\|\Sigma_{\rm SD}\|_{(\infty,\infty)/(2,2)}$

In this subsection, we deal with $p = \infty$, i.e., $\|\Sigma_{\rm SD}\|_{(\infty,\infty)/(2,2)}$. To this end, we first note (37) and represent $\|\mathcal{F}\|_{(\infty,\infty)/(2,2)}$ as

$$\|\mathcal{F}\|_{(\infty,\infty)/(2,2)} = \sup_{0\le\theta<h} \sup_{\|\widehat{w}\|_{(2,2)}\le 1} |(\mathcal{F}\widehat{w})(\theta)|_\infty \tag{39}$$

For each fixed $\theta \in [0, h)$, it follows from (18) that

$$\begin{aligned}(\mathcal{F}\widehat{w})(\theta) &= (\mathcal{D}\widehat{w}_0)(\theta) + \sum_{j=0}^{\infty}(\mathcal{C}\mathcal{A}^j\mathcal{B}\widehat{w}_{j+1})(\theta) \\ &= \int_0^h D_\theta(\tau)\widehat{w}_0(\tau)d\tau + \sum_{j=0}^{\infty}\int_0^h C_\theta\mathcal{A}^j B_h(\tau)\widehat{w}_{j+1}(\tau)d\tau\end{aligned} \tag{40}$$

Applying the triangle and continuous-time Cauchy-Schwarz inequalities to (40) leads to

$$\begin{aligned}|(\mathcal{F}\widehat{w})_i(\theta)| \le &\left(\int_0^h D_{\theta i}(\tau)D_{\theta i}^T(\tau)d\tau\right)^{1/2} \cdot \left(\int_0^h \widehat{w}_0^T(\tau)\widehat{w}_0(\tau)d\tau\right)^{1/2} \\ &+\sum_{j=0}^{\infty}\left(\int_0^h (C_\theta\mathcal{A}^j B_h(\tau))_i(C_\theta\mathcal{A}^j B_h(\tau))_i^T d\tau\right)^{1/2} \cdot \left(\int_0^h \widehat{w}_{j+1}^T(\tau)\widehat{w}_{j+1}(\tau)d\tau\right)^{1/2}\end{aligned} \tag{41}$$

where $(\mathcal{F}\widehat{w})_i(\theta)$, $D_{\theta i}(\tau)$ and $(C_\theta B_h(\tau))_i$ denote the ith element of $(\mathcal{F}\widehat{w})(\theta)$ and the ith rows of $D_\theta(\tau)$ and $C_\theta B_h(\tau)$, respectively. Furthermore, by applying the discrete-time Cauchy-Schwarz inequality to (41), it readily follows that

$$\begin{aligned}|(\mathcal{F}\widehat{w})_i(\theta)| &\le \left(\int_0^h D_{\theta i}(\tau)D_{\theta i}^T(\tau)d\tau + \sum_{j=0}^{\infty}\int_0^h (C_\theta\mathcal{A}^j B_h(\tau))_i(C_\theta\mathcal{A}^j B_h(\tau))_i^T d\tau\right)^{1/2} \\ &=: \rho_{\theta i}\end{aligned} \tag{42}$$

provided that $\|\widehat{w}\|_{(2,2)} \le 1$.

Note that $\rho_{\theta i}^2$ defined in the above equation equals the ith diagonal entry of $F(\theta)$ given by (27). We are in a position to give the following result.

Proposition 2: $\|\Sigma_{\rm SD}\|_{(\infty,\infty)/(2,2)}$ *is given by*

$$\|\Sigma_{\rm SD}\|_{(\infty,\infty)/(2,2)} = \sup_{0\le\theta<h} d_{\max}^{1/2}(F(\theta)) \tag{43}$$

where $F(\theta)$ *is given by (27).*

Proof. We show that $\sup_{\|\widehat{w}\|_{(2,2)}\le 1} |(\mathcal{F}\widehat{w})_i(\theta)| = \rho_{\theta i}$. Once this claim is established, the assertion of the proposition is an immediate consequence of (39) and the definition of the vector ∞-norm, together with the aforementioned interpretation of $\rho_{\theta i}^2$. To show this claim[2], let us take $\widehat{w}$ given by

$$\widehat{w}_0(\tau) = \frac{1}{\rho_{\theta i}} D_{\theta i}^T(\tau), \quad \widehat{w}_i(\tau) = \frac{1}{\rho_{\theta i}} (C_\theta \mathcal{A}^i B_h(\tau))_i^T \ (i \in \mathbb{N}) \tag{44}$$

Then, $\|\widehat{w}\|_{(2,2)} = 1$ and the equalities hold both in (41) and (42). Hence, the claim is established and the proof is completed. □

5.3 *Characterization of Generalized H_2 Norm $\|\Sigma_{\rm SD}\|_{(\infty,2)/(2,2)}$*

We next deal with the L_∞/L_2-induced norm $\|\Sigma_{\rm SD}\|_{(\infty,2)/(2,2)}$ (i.e., $p = 2$). Again, we note (37) and represent $\|\mathcal{F}\|_{(\infty,2)/(2,2)}$ as

$$\|\mathcal{F}\|_{(\infty,2)/(2,2)} = \sup_{0\le\theta<h} \sup_{\|\widehat{w}\|_{(2,2)}\le 1} |(\mathcal{F}\widehat{w})(\theta)|_2 = \sup_{0\le\theta<h} \sup_{\|\widehat{w}\|_{(2,2)}\le 1} |\mathcal{F}_\theta \widehat{w}|_2 \tag{45}$$

where $\mathcal{F}_\theta : l^2_{(L_2[0,h))^{n_w}} \to \mathbb{R}^{n_z}$ is defined by $\mathcal{F}_\theta \widehat{w} = (\mathcal{F}\widehat{w})(\theta)$ and can be regarded as an operator acting on Hilbert spaces. Hence, $\sup_{\|\widehat{w}\|_{(2,2)}\le 1} |\mathcal{F}_\theta \widehat{w}|_2$ for a fixed $\theta \in [0,h)$ can be computed with the adjoint operator $\mathcal{F}_\theta^* : \mathbb{R}^{n_z} \to l^2_{(L_2[0,h))^{n_w}}$ of $\mathcal{F}_\theta$ defined as

$$\mathcal{F}_\theta^* = \begin{bmatrix} D_\theta(\tau) & C_\theta B_h(\tau) & C_\theta \mathcal{A} B_h(\tau) & \cdots \end{bmatrix}^T \tag{46}$$

More precisely, we see that $\left(\sup_{\|\widehat{w}\|_{(2,2)}\le 1} |\mathcal{F}_\theta \widehat{w}|_2 \right)^2 = \lambda_{\max}(\mathcal{F}_\theta \mathcal{F}_\theta^*)$. Since $\mathcal{F}_\theta \mathcal{F}_\theta^* = F(\theta)$ by (27), we are led to the following result.

Proposition 3: $\|\Sigma_{\rm SD}\|_{(\infty,2)/(2,2)}$ *is given by*

$$\|\Sigma_{\rm SD}\|_{(\infty,2)/(2,2)} = \sup_{0\le\theta<h} \lambda_{\max}^{1/2}(F(\theta)) \tag{47}$$

Remark 8: *Since $F(\theta)$ is a symmetric matrix, we can confirm from Propositions 2 and 3 that*

$$\|\Sigma_{\rm SD}\|_{(\infty,2)/(2,2)} \ge \|\Sigma_{\rm SD}\|_{(\infty,\infty)/(2,2)} \tag{48}$$

This is also a straightforward consequence of the obvious fact that $|z(t)|_2 \ge |z(t)|_\infty$. In particular, when $n_z = 1$, the two types of generalized H_2 norms obviously coincide with each other.

[2]We may assume $\rho_{\theta i} \ne 0$ because the claim is obvious otherwise.

5.4 *Relationship among Different Definitions of H_2 and Generalized H_2 Norms*

Based on (43) and (47) for the generalized H_2 norms (L_∞/L_2-induced norms) of $\varSigma_{\rm SD}$, this subsection is devoted to discussing whether either of the two generalized H_2 norms could possibly coincide with one of the three definitions of the H_2 norm of $\varSigma_{\rm SD}$ discussed in Section 4 through the impulse response viewpoint. This is a natural question because for multi-input/single-output (MISO) continuous-time LTI systems, these two types of generalized H_2 norms both coincide with the H_2 norm Wilson (1989); Rotea (1993); Chellabonia and Haddad (2000); Wilson (1990); Grimble (1990); Wilson and Nekoui (1998)

We believe that having supplemented the new H_2 norm $\|\varSigma_{\rm SD}\|_{H_2}^{[\tau^\star]}$ missing in the literature is very meaningful, particularly in a comparison through numerical results. In theoretical comparison, on the other hand, what this paper has clarified is that the three H_2 norms are all characterized by $G(\tau)$ while the two generalized H_2 norms are characterized by the slightly different matrix function $F(\theta)$. We believe that this is a very important theoretical advance in the studies of sampled-data systems. The following theoretical comparison of the H_2 and generalized H_2 norms naturally centers around $\|\varSigma_{\rm SD}\|_{H_2}^{[0,h)}$, because it is the only H_2 norm definition that admits an alternative representation with $F(\theta)$.

Indeed, an obvious relation is that

$$\|\varSigma_{\rm SD}\|_{H_2}^{[0,h)} \le \|\varSigma_{\rm SD}\|_{(\infty,\infty)/(2,2)} = \|\varSigma_{\rm SD}\|_{(\infty,2)/(2,2)}, \quad (\text{if } n_z = 1) \tag{49}$$

which follows immediately from (26), (43) and (47). This observation suggests that the generalized H_2 norms $\|\varSigma_{\rm SD}\|_{(\infty,\infty)/(2,2)}$ and $\|\varSigma_{\rm SD}\|_{(\infty,2)/(2,2)}$ are intrinsically different from the H_2 norm $\|\varSigma_{\rm SD}\|_{H_2}^{[0,h)}$. It is also expected that the H_2 norms $\|\varSigma_{\rm SD}\|_{H_2}^{[0-]}$ and $\|\varSigma_{\rm SD}\|_{H_2}^{[\tau^\star]}$ are intrinsically different from the generalized H_2 norms $\|\varSigma_{\rm SD}\|_{(\infty,p)/(2,2)}\ (p=\infty,2)$ as well as $\|\varSigma_{\rm SD}\|_{H_2}^{[0,h)}$. We can indeed confirm these assertions through numerical examples in Section 6.

Remark 9: *It is not hard to see that*

$$\|\varSigma_{\rm SD}\|_{H_2}^{[0,h)} \le n_z \|\varSigma_{\rm SD}\|_{(\infty,p)/(2,2)} \quad (p = \infty, 2) \tag{50}$$

and (49) can be regarded as a consequence of these inequalities combined with the observation in Remark 8.

Summarizing the above arguments, we could conclude that the generalized H_2 norms $\|\varSigma_{\rm SD}\|_{(\infty,p)/(2,2)}$ $(p = \infty, 2)$ of (even SISO) LTI sampled-data systems can be characterized by neither of the three H_2 norms of sampled-data systems given so far in Chen and Francis (1991); Bamieh and Pearson (1992a); Khargonekar and Sivashankar (1991); Hagiwara and Araki (1995) and in this paper. Taking this into account, we can summarize that any of the five performance measures, i.e., the H_2 norms $\|\varSigma_{\rm SD}\|_{H_2}^{[0-]}$, $\|\varSigma_{\rm SD}\|_{H_2}^{[0,h)}$ and $\|\varSigma_{\rm SD}\|_{H_2}^{[\tau^\star]}$ as well as the generalized H_2 norms $\|\varSigma_{\rm SD}\|_{(\infty,p)/(2,2)}$ $(p = \infty, 2)$, can be meaningful in sampled-data systems. If we are to choose only one of them to avoid multiobjective problems in the controller synthesis problem, then the relations (33), (48) (49) and (50) derived in this paper can be a helpful guideline.

6. Numerical Computation Methods and Numerical Examples

This section studies some numerical examples to confirm the developed theoretical results on the comparison of the H_2 and generalized H_2 norms of sampled-data systems and also to examine further numerical properties that have not been questioned and answered theoretically. Such discussions are preceded by brief arguments on numerical procedures for the computations of these norms.

6.1 *Numerical Computation Methods*

We begin with the computations of the H_2 norms (30)–(32) defined through the impulse response viewpoint. In particular, we consider (32) because this is the new norm introduced in this paper while the computation methods for the other two norms are well known. Although we can easily see that $G(\tau)$ can be computed for each $\tau \in [0, h)$ by solving a discrete-time Lyapunov equation (for computing an infinite series) and calculating a matrix exponential involving its solution (for an integral over $[0, h)$ for θ), neither of the two steps is invariant and common with respect to a different choice of τ. In other words, no "unified" method is available for $\tau \in [0, h)$, and each τ must be handled "separately one by one." This situation is inconvenient, and to circumvent this issue, it is useful to note that $\mathrm{tr}(G(\tau)) = \mathrm{tr}(\widetilde{G}(\tau))$, where

$$\widetilde{G}(\tau) := \int_0^h D_\theta^T(\tau) D_\theta(\tau) d\theta + \sum_{j=0}^{\infty} \int_0^h (C_\theta \mathcal{A}^j B_h(\tau))^T C_\theta \mathcal{A}^j B_h(\tau) d\theta \tag{51}$$

(using $\widetilde{G}(\tau)$ corresponds to reverting to (21) from (22) for a numerical purpose at a sacrifice of obscuring the theoretical relationship with the generalized H_2 norms $\|\varSigma_{\mathrm{SD}}\|_{(\infty,p)/(2,2)}$ $(p = \infty, 2)$, with which $F(\theta)$ is associated; recall Remark 5). We can easily see that computing $\widetilde{G}(\tau)$ for different values of τ requires solving only one common Lyapunov equation; see Appendix for its numerical computation procedure. Another by-product of using $\widetilde{G}(\tau)$ is that its computation procedure allows us to directly compute $\mathrm{tr}(\widetilde{G}(h)) := \lim_{\tau \to h-0} \mathrm{tr}(\widetilde{G}(\tau))$ (because there is no obstacle in applying the procedure directly even for $\tau = h$, while it is obvious that the computation results of $\widetilde{G}(\tau)$ depend continuously on τ). This situation is in sharp contrast with the special handling required for the computation of $G(\tau)$ at $\tau = h$ (recall the last paragraph of Section 4), which can be attributed to the aforementioned lack of unified treatment of $G(\tau)$ for different values of τ.

The above arguments together with (34) immediately lead to

$$\max_{\tau \in \mathcal{K}_M} \mathrm{tr}^{1/2}(\widetilde{G}(\tau)) \to \|\varSigma_{\mathrm{SD}}\|_{H_2}^{[\tau^\star]} \quad (M \to \infty) \tag{52}$$

where $\mathcal{K}_M = \{0, h', \cdots, Mh'\}$ with $h' := h/M$, and the relevant convergence property in M is also among the interest about the numerical examples in the following subsection.

Next, let us consider the generalized H_2 norms defined through the L_∞/L_2-induced norm viewpoint, which are represented with the matrix function $F(\theta)$. Similarly to the case of $G(\tau)$ as discussed at the end of Section 4, it may not be obvious whether we can replace the supremum over $\theta \in [0, h)$ in (43) and (47) with the maximum over $\theta \in [0, h]$, particularly because $F(\theta)$ is defined only for $\theta \in [0, h)$. However, a feature of $F(\theta)$ that is slightly different from $G(\tau)$ (recall Remark 5) immediately allows us to directly define $F(h)$ in such a way that $F(\theta)$ is continuous on $[0, h]$, and this in turn leads us to a positive answer to the above concern. More precisely, given any $\theta \in [0, h)$, we are immediately led to the computation method of $F(\theta)$ shown in Appendix, in which only one common discrete-time Lyapunov equation is required to be solved regardless of θ. In particular, there is no obstacle in applying the method even for $\theta = h$, and defining $F(h)$ by the associated resulting matrix obviously leads to $F(\theta)$ continuous on $\theta \in [0, h]$. Note that this situation is essentially the same as that for $\widetilde{G}(\tau)$, for which $\widetilde{G}(h)$ can be computed directly; recall the arguments earlier in this subsection. It is worth noting that $F(h)$ defined in this way is different from $F(0)$, in general, as observed in Figures 2–4 (b) for the numerical examples in the following subsection.

Summarizing the above arguments leads to

$$\max_{\theta \in \mathcal{K}_M} d_{\max}^{1/2}(F(\theta)) \to \|\Sigma_{\mathrm{SD}}\|_{(\infty,\infty)/(2,2)} \quad (M \to \infty) \tag{53}$$

$$\max_{\theta \in \mathcal{K}_M} \lambda_{\max}^{1/2}(F(\theta)) \to \|\Sigma_{\mathrm{SD}}\|_{(\infty,2)/(2,2)} \quad (M \to \infty) \tag{54}$$

because a continuous function on a closed interval is uniformly continuous. Again, the relevant convergence property in M is among the interest in the following subsection.

6.2 *Numerical Examples*

We first consider three examples of stable SISO sampled-data systems, which are selected to exhibit three different features in the variation of $G(\tau)^{1/2} = \widetilde{G}(\tau)^{1/2}$ with respect to $\tau \in [0, h]$; the first example shows monotonic increase, the second shows monotonic decrease, while the third shows neither of them. We remark that the first two examples clearly show that no theoretical inequality can exist between $\|\Sigma_{\mathrm{SD}}\|_{H_2}^{[0-]}$ and $\|\Sigma_{\mathrm{SD}}\|_{H_2}^{[\tau^\star]}$. Furthermore, the first example readily implies that our preceding arguments could indeed be very meaningful, where the arguments were about whether and how the expression of $\|\Sigma_{\mathrm{SD}}\|_{H_2}^{[\tau^\star]}$ in (32) (or that with $G(\tau)$ replaced by $\widetilde{G}(\tau)$) in terms of the supremum over $\tau \in [0, h)$ can be replaced by the corresponding one in terms of the maximum over $\tau \in [0, h]$ (by appropriately defining $G(h)$ or $\widetilde{G}(h)$ and providing its feasible computation method). In each of these examples, the computation results for the three H_2 norms (from the impulse input viewpoint) as well as the generalized H_2 norm (from the L_∞/L_2-induced norm viewpoint) are shown as a table. More precisely, (a) $\|\Sigma_{\mathrm{SD}}\|_{H_2}^{[0-]}$ and $\|\Sigma_{\mathrm{SD}}\|_{H_2}^{[0,h)}$ are computed by following the arguments in Chen and Francis (1991) and Bamieh and Pearson (1992a); Khargonekar and Sivashankar (1991); Hagiwara and Araki (1995), respectively, while (b) Equations (52) and (53) are used for the computation of $\|\Sigma_{\mathrm{SD}}\|_{H_2}^{[\tau^\star]}$ and $\|\Sigma_{\mathrm{SD}}\|_{(\infty,\infty)/(2,2)}$ ($= \|\Sigma_{\mathrm{SD}}\|_{(\infty,2)/(2,2)}$), respectively. We then consider an example of MIMO stable sampled-data systems, in which the results are also given in the same way as (a) and (b) above, except that both (53) and (54) are used to compute $\|\Sigma_{\mathrm{SD}}\|_{(\infty,\infty)/(2,2)}$ and $\|\Sigma_{\mathrm{SD}}\|_{(\infty,2)/(2,2)}$, respectively.

Remark 10: *The computation methods in Zhu and Skelton (1995) are essentially equivalent to (53) and (54) except that $D_{12} = 0$ was assumed. This restriction, however, prevents us from applying the existing methods to our examples.*

Example 1: *Consider the stable SISO LTI sampled-data system associated with $h = 0.5$ and*

$$A = \begin{bmatrix} 0 & -0.5 \\ 1 & -1.5 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix},\ B_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix},\ C_1 = \begin{bmatrix} 0 & 1.5 \end{bmatrix},\ C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$$
$$D_{12} = 0.5,\ \ A_\Psi = \begin{bmatrix} -0.4888 & 1.6687 \\ 0.0737 & -0.2547 \end{bmatrix},\ B_\Psi = \begin{bmatrix} -3.1180 \\ 0.4701 \end{bmatrix}$$
$$C_\Psi = \begin{bmatrix} -1.6601 & 5.7348 \end{bmatrix},\ D_\Psi = -7.5709 \tag{55}$$

Example 2: *Consider the stable SISO LTI sampled-data system associated with $h = 0.5$ and*

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -2 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix},\ C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix},\ C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix},D_{12} = 0.5$$
$$A_\Psi = \begin{bmatrix} 0.0010 & -0.1780 \\ 0.0043 & -0.7740 \end{bmatrix},\ B_\Psi = \begin{bmatrix} 0.2815 \\ 1.2240 \end{bmatrix}$$
$$C_\Psi = \begin{bmatrix} -0.0305 & 5.4482 \end{bmatrix},\ D_\Psi = -5.0894 \tag{56}$$

Example 3: *Consider the stable SISO LTI sampled-data system associated with $h = 2$ and*

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -2 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$D_{12} = -0.5,\ A_\Psi = \begin{bmatrix} 2.5665 & -2.3760 \\ 3.8256 & -3.5417 \end{bmatrix}, B_\Psi = \begin{bmatrix} 0.0247 \\ 0.0368 \end{bmatrix}$$

$$C_\Psi = \begin{bmatrix} -0.6388 & 0.5914 \end{bmatrix}, D_\Psi = 0.2367 \tag{57}$$

Example 4: *Consider the stable MIMO LTI sampled-data system associated with $h = 2$ and*

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 0.5 \\ -1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}$$

$$A_\Psi = \begin{bmatrix} 0.3933 & 0.4565 & 0.0061 \\ 0.4565 & -0.2216 & 0.0999 \\ -0.0061 & -0.0999 & -0.2650 \end{bmatrix}, \quad B_\Psi = \begin{bmatrix} -0.4747 \\ 0.0923 \\ -0.0083 \end{bmatrix}$$

$$C_\Psi = \begin{bmatrix} 0.4747 & -0.0923 & -0.0083 \end{bmatrix}, \quad D_\Psi = 0.2028 \tag{58}$$

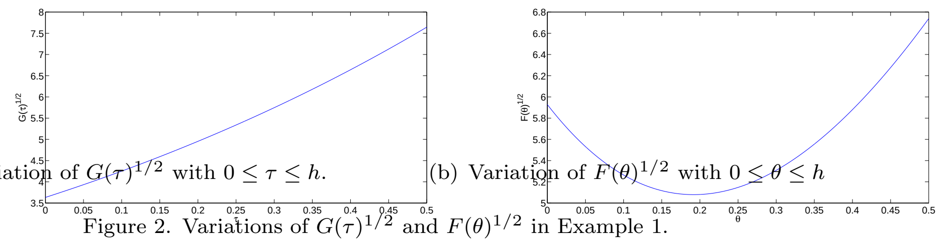
Table 1. Computation results for Example 1.

(a) The first and second definitions of the H_2 norm.

$\|\Sigma_{\mathrm{SD}}\|_{H_2}^{[0-]}$	7.6440
$\|\Sigma_{\mathrm{SD}}\|_{H_2}^{[0,h)}$	5.5595

(b) The third definition of the H_2 norm and generalized H_2 norm.

M	10	20	50	200
$\|\Sigma_{\mathrm{SD}}\|_{H_2}^{[\tau^\star]} = \max_{\tau\in\mathcal{K}_M} G(\tau)^{1/2}$	7.6440	7.6440	7.6440	7.6440
$\|\Sigma_{\mathrm{SD}}\|_{(\infty,\infty)/(2,2)} = \max_{\theta\in\mathcal{K}_M} F(\theta)^{1/2}$	6.7384	6.7384	6.7384	6.7384



(a) Variation of $G(\tau)^{1/2}$ with $0 \le \tau \le h$. (b) Variation of $F(\theta)^{1/2}$ with $0 \le \theta \le h$

Figure 2. Variations of $G(\tau)^{1/2}$ and $F(\theta)^{1/2}$ in Example 1.

The norm computation results for the first three examples of SISO sampled-data systems are shown in Tables 1–3, for which Figures 2–4 are also provided for reference to show the dependence of the associated $G(\tau) = \widetilde{G}(\tau)$ and $F(\theta)$ on τ and θ, respectively. The norm computation results for the last example of a MIMO sampled-data system are shown in Table 4.

First of all, we can see from Tables 1–4 that the convergence asserted in (52)–(54) can indeed be observed, which is modestly fast enough with respect to the increase in M.

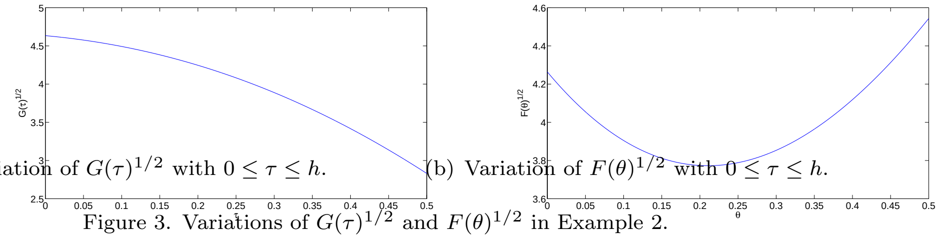
Table 2. Computation results for Example 2.

(a) The first and second definitions of the H_2 norm.

$\lVert \Sigma_{\rm SD} \rVert_{H_2}^{[0-]}$	2.8397
$\lVert \Sigma_{\rm SD} \rVert_{H_2}^{[0,h)}$	4.0000

(b) The third definition of the H_2 norm and generalized H_2 norm.

M	10	20	50	200
$\lVert \Sigma_{\rm SD} \rVert_{H_2}^{[\tau^\star]} = \max_{\tau \in \mathcal{K}_M} G(\tau)^{1/2}$	4.6342	4.6342	4.6342	4.6342
$\lVert \Sigma_{\rm SD} \rVert_{(\infty,\infty)/(2,2)} = \max_{\theta \in \mathcal{K}_M} F(\theta)^{1/2}$	4.5437	4.5437	4.5437	4.5437



(a) Variation of $G(\tau)^{1/2}$ with $0 \le \tau \le h$. (b) Variation of $F(\theta)^{1/2}$ with $0 \le \tau \le h$.

Figure 3. Variations of $G(\tau)^{1/2}$ and $F(\theta)^{1/2}$ in Example 2.

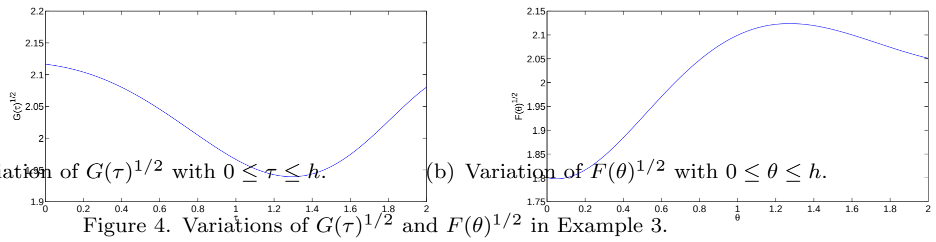
Table 3. Computation results for Example 3.

(a) The first and second definitions of the H_2 norm.

$\lVert \Sigma_{\rm SD} \rVert_{H_2}^{[0-]}$	2.0803
$\lVert \Sigma_{\rm SD} \rVert_{H_2}^{[0,h)}$	2.0191

(b) The third definition of the H_2 norm and generalized H_2 norm.

M	10	20	50	200
$\lVert \Sigma_{\rm SD} \rVert_{H_2}^{[\tau^\star]} = \max_{\tau \in \mathcal{K}_M} G(\tau)^{1/2}$	2.1164	2.1164	2.1164	2.1164
$\lVert \Sigma_{\rm SD} \rVert_{(\infty,\infty)/(2,2)} = \max_{\theta \in \mathcal{K}_M} F(\theta)^{1/2}$	2.1222	2.1236	2.1238	2.1238



(a) Variation of $G(\tau)^{1/2}$ with $0 \le \tau \le h$. (b) Variation of $F(\theta)^{1/2}$ with $0 \le \theta \le h$.

Figure 4. Variations of $G(\tau)^{1/2}$ and $F(\theta)^{1/2}$ in Example 3.

We see from Example 1 that the strict version of the first inequality in (33) cannot hold, in general (even if the case when $\Sigma_{\rm SD}$ is actually a continuous-time LTI system is ruled out). We can confirm from Figures 2–4 (a) that $\lim_{\tau \to h-0} G(\tau) \neq G(0)$ (which obviously implies that $\lim_{\tau \to h-0} \widetilde{G}(\tau) \neq \widetilde{G}(0)$) as discussed in Remark 3. We can further confirm from the same figures that $\lVert \Sigma_{\rm SD} \rVert_{H_2}^{[0-]} =$

Table 4. Computation results for Example 4.

(a) The first and second definitions of the H_2 norm.

$\Vert\Sigma_{\rm SD}\Vert_{H_2}^{[0-]}$	3.0538
$\Vert\Sigma_{\rm SD}\Vert_{H_2}^{[0,h)}$	3.0151

(b) The third definition of the H_2 norm and generalized H_2 norms.

M	10	20	50	200
$\Vert\Sigma_{\rm SD}\Vert_{H_2}^{[\tau^\star]} = \max_{\tau\in\widetilde{\mathcal{K}}_M} {\rm tr}^{1/2}(\widetilde{G}(\tau))$	3.2225	3.2225	3.2233	3.2234
$\Vert\Sigma_{\rm SD}\Vert_{(\infty,\infty)/(2,2)} = \max_{\theta\in\mathcal{K}_M} d_{\max}^{1/2}(F(\theta))$	2.2306	2.2322	2.2322	2.2322
$\Vert\Sigma_{\rm SD}\Vert_{(\infty,2)/(2,2)} = \max_{\theta\in\mathcal{K}_M} \lambda_{\max}^{1/2}(F(\theta))$	2.8117	2.8122	2.8126	2.8127

$\widetilde{G}(h)$ $(= G(h))$, where the right hand side could also be computed directly without referring to the computation method of $\Vert\Sigma_{\rm SD}\Vert_{H_2}^{[0-]}$ in Chen and Francis (1991); recall relevant arguments in the preceding subsection as well as at the end of Section 4.

Next, we can also see from the tables that an existing H_2 norm $\Vert\Sigma_{\rm SD}\Vert_{H_2}^{[0,h)}$ cannot exceed the newly introduced H_2 norm $\Vert\Sigma_{\rm SD}\Vert_{H_2}^{[\tau^\star]}$, which confirms validity of the second inequality in our theoretical results given in (33). Even though we have not provided any theoretical arguments, it would be reasonable to expect that its strict version holds if we rule out the case when $\Sigma_{\rm SD}$ is actually a continuous-time LTI system.

Furthermore, we can see from Tables 1–3 about SISO sampled-data systems that the H_2 norm $\Vert\Sigma_{\rm SD}\Vert_{H_2}^{[0,h)}$ is not larger than the generalized H_2 norm $\Vert\Sigma_{\rm SD}\Vert_{(\infty,\infty)/(2,2)}$ $(= \Vert\Sigma_{\rm SD}\Vert_{(\infty,2)/(2,2)})$, which confirms the theoretical inequality (49). The extension of this inequality to the case of MIMO sampled-data systems is given by (50), which is also confirmed by Example 4.

The above observations have successfully confirmed our theoretical arguments in this paper. On the other hand, one could naturally raise a question whether or not there could exist further theoretical arguments about inequality relations between two quantities for which this paper does not provide any answers. However, we see that the results in Examples 1–4 give a negative answer to the question by showing that no general inequality can hold between such two quantities.

7. Conclusion

This paper first introduced a new definition of the H_2 norm of LTI sampled-data systems by taking a sort of intermediate standpoint between those for the existing two definitions through the viewpoint of impulse responses. The meaning of the H_2 norm in the first definition was then made more transparent through the viewpoint provided by the introduction of the new third definition of the H_2 norm, and unified treatment with the matrix function $G(\tau)$ or $\widetilde{G}(\tau)$ was used for theoretical and numerical comparison of these three definitions of the H_2 norm. This paper next characterized the generalized H_2 norms in LTI sampled-data systems, which are also known as the L_∞/L_2-induced norms with two different spatial norms underlying L_∞. Without making any restrictive assumption on the generalized plant, we first introduced the matrix function $F(\theta)$ to analyze the generalized H_2 norms theoretically and numerically. We then showed that a close connection between this matrix function and $G(\tau)$ (or $\widetilde{G}(\tau)$), together with their appropriate extensions to the closed interval $[0, h]$, can be used to establish some theoretical relationship between the H_2 norms in the impulse input viewpoint and the generalized H_2 norms in the induced-norm viewpoint and facilitate numerical comparisons. Through these theoretical and numerical studies, it was clarified that the two generalized H_2 norms coincide with neither of the three H_2 norms even though all the five definitions coincide with each other when single-output continuous-time LTI systems are considered as a special class of LTI sampled-data systems.

One of the main contributions in this paper can thus be interpreted as clarifying that the above five control performance measures for LTI sampled-data systems are mutually related with each other but that some of them have not been covered by the existing studies on the H_2 problem (or any other problems) of sampled-data systems as far as a synthesis problem is concerned; when the control objective is to minimize either the third H_2 norm or one of the two generalized H_2 norms, what has been clarified in the present paper immediately implies that the direct H_2 sampled-data controller synthesis methods currently available in the control community (Chen and Francis (1991); Bamieh and Pearson (1992a); Khargonekar and Sivashankar (1991); Hagiwara and Araki (1995)) are helpless, although any of such three alternative performance measures could indeed be important in practical applications. In other words, this situation further implies that the associated controller synthesis problems are proved to be open problems. Regarding the case of the generalized H_2 norms, the pioneering work Zhu and Skelton (1995) only deals with their analysis and direct extension of the approach therein to the synthesis problem is very hard. In contrast, we remark that the numerical computation procedures we have developed in this paper for their analysis can be extended in such a way that the procedures reduce to dealing only with discrete-time systems. More precisely, we can further develop a discretization method of the continuous-time generalized plant P Kim and Hagiwara (2015b, 2016), and we have only to compute the l_∞/l_2-induced norm Wilson and Nekoui (1998) of the closed-loop system consisting of the discretized generalized plant (under the approximation parameter M) and the discrete-time controller Ψ. This immediately enables us to develop controller synthesis procedures for minimizing the generalized H_2 norms by only slightly modifying the procedure for the discrete-time H_2 controller synthesis Oliveria and Geromel (2002). In this connection, the theoretical relations derived in this paper about the five performance measures could play important roles if we are to choose only one of the five possible performance measures to avoid multiobjective problem in the sampled-data controller synthesis problem. These implications are believed to shed a new light on the theoretical studies of sampled-data systems relevant to the influence of their hybrid and periodically time-varying nature.

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Appendix A. Computation Methods for $\widetilde{G}(\tau)$ and $F(\theta)$

Computation of $\widetilde{G}(\tau)$:

We readily see that

$$\begin{aligned}\widetilde{G}(\tau) &= B_1^T U_\tau B_1 + B_h^T(\tau)\left(\sum_{i=0}^{\infty}(\mathcal{A}^T)^i C_\Sigma^T V_h C_\Sigma \mathcal{A}^i\right) B_h(\tau)\\ &= B_1^T U_\tau B_1 + B_h^T(\tau) Y_h B_h(\tau) \end{aligned} \tag{A1}$$

where U_τ and V_h are defined by

$$U_\tau := \int_\tau^h \exp(A^T(\theta-\tau)) C_1^T C_1 \exp(A(\theta-\tau)) d\theta \tag{A2}$$

$$V_h := \int_0^h \exp(A_2^T(\theta)) M_1^T M_1 \exp(A_2(\theta)) d\theta \tag{A3}$$

and Y_h is the solution of the discrete-time Lyapunov equation

$$\mathcal{A}^T Y_h \mathcal{A} - Y_h + C_\Sigma^T V_h C_\Sigma = 0 \tag{A4}$$

Computation of $F(\theta)$:

We readily see that

$$\begin{aligned}F(\theta) &= C_1 W_\theta C_1^T + C_\theta \left(\sum_{i=0}^{\infty} \mathcal{A}^i \begin{bmatrix} W_h & 0 \\ 0 & 0 \end{bmatrix} (\mathcal{A}^T)^i\right) C_\theta^T\\ &= C_1 W_\theta C_1^T + C_\theta X_h C_\theta^T \end{aligned} \tag{A5}$$

where W_θ ($\theta \in [0, h]$) is defined by

$$W_\theta := \int_0^\theta \exp(A(\theta - \tau)) B_1 B_1^T \exp(A^T(\theta - \tau)) d\tau \tag{A6}$$

and X_h is the solution of the discrete-time Lyapunov equation

$$\mathcal{A} X_h \mathcal{A}^T - X_h + \begin{bmatrix} W_h & 0 \\ 0 & 0 \end{bmatrix} = 0 \tag{A7}$$