Computing the L_∞-Induced Norm of LTI Systems via Kernel Approximation and Its Comparison with Input Approximation

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Abstract

This paper deals with the L_1 analysis of stable finite-dimensional linear time-invariant (LTI) systems, by which we mean the computation of the L_∞-induced norm of these systems. To compute this norm, we need to integrate the absolute value of the impulse response of the given system, which corresponds to the kernel function in the convolution formula for the input/output relation. However, it is very difficult to compute this integral exactly or even approximately with an explicit upper bound and lower bound. We first review an approach named input approximation, in which the input of the LTI system is approximated by a staircase or piecewise linear function and computation methods for an upper bound and lower bound of the L_∞-induced norm are given. We further develop another approach using an idea of kernel approximation, in which the kernel function in the convolution is approximated by a staircase or piecewise linear function. These approaches are introduced through fast-lifting, by which the interval $[0, h)$ with a sufficiently large h is divided into M subintervals with an equal width. It is then shown that the approximation errors in staircase or piecewise linear approximation are ensured to be reciprocally proportional to M or M^2, respectively. The effectiveness of the proposed methods is demonstrated through numerical examples.

1 Introduction

The L_∞-induced norm of control systems is the peak magnitude of the output for the worst bounded persistent input with a unit peak magnitude. There have been a number of studies on the L_∞-induced norm problem associated with a linear time-invariant (LTI) system [1–3] and a positive system [4, 5] since evaluating the peak magnitude of the output is very important in many control systems. Because this norm corresponds to the L_1 norm of the impulse response of the system in the (strictly causal) finite-dimensional single-input/single-output (SISO) LTI case, the study associated with the treatment of the L_∞-induced norm has been called the L_1 problem. This problem is pertinent to dealing with bounded persistent disturbances such as steps and sinusoids, which are often encountered in control systems. Accurate computation of the L_∞-induced norm associated with an LTI system is very hard since we need to integrate the absolute value (i.e., we need to compute the L_1 norm) of the impulse response of the LTI system, which corresponds to the kernel function in the convolution formula for its input/output relation, and it is very difficult to compute this integral exactly. To the best of authors' knowledge, an exact computation of the integral could be done only when the relevant system is a *positive* finite-dimensional LTI system [6] (for which the impulse response is nonnegative and thus the operation of taking its absolute value may be eliminated, leading to an analytic formula for the integral), and there have been no studies on giving an exact computation of the L_∞-induced norm as well as its upper and lower bounds associated with (*not necessarily positive*) finite-dimensional LTI systems. This paper studies to compute upper and lower bounds in such a way that these bounds can be made as close to each other as one desires.

In this paper, we provide two simple approaches named input approximation and kernel approximation for computing the L_∞-induced norm associated with a stable finite-dimensional LTI system. They are two different approaches in terms of the viewpoint behind approximations but share a common technical feature that they employ a staircase approximation or piecewise linear approximation scheme of functions. In these input and kernel approximation approaches, we first apply a truncation idea, by which the time interval $[0, \infty)$ is divided into $[0, h)$ and $[h, \infty)$ with a sufficiently large constant h. Then, the behavior of the system on the time interval $[0, h)$ is treated as accurately as possible while that on $[h, \infty)$ in a comparatively simple way. This is because the effect of the latter interval on the L_∞-induced norm is very small when h is large enough; this implies that evaluating the effect of the latter interval in a relatively rough way does not cause severe deterioration of the resulting upper and lower bounds for the induced norm, as long as the effect of the former interval is evaluated adequately. Such an accurate evaluation is achieved by first applying to the signals on the former interval the fast-lifting [7] treatment, which has an integer parameter M and simply divides (without applying sampling of signals) the time interval $[0, h)$ into M subintervals with an equal width (in the context of the present paper, the role of fast-lifting is essentially the same as the conventional lifting [8–10] in the studies of sampled-data systems and time-delay systems, except that the original interval $[0, h)$ is finite). With this fast-lifting treatment, the input as well as the kernel function associated with the convolution formula for LTI systems can be dealt with independently on each of the M subintervals. Fast-lifting plays a role in reducing the size of the intervals to be directly dealt with, and provides us with improved accuracy in the approximation of the input and kernel functions. Indeed, it is shown that the staircase and piecewise linear approximation schemes are applicable also to multi-input multi-output (MIMO) systems and lead to approximation errors in the computation of the L_∞-induced norm converging to 0 at the rate of $1/M$ and $1/M^2$, respectively, in the kernel approximation approach. This is a parallel result to that in the input approximation approach, which follows easily by the arguments in [11] associated with the $L_\infty[0, h)$-induced norm computation of compression operators (describing the input-output mapping of the LTI system over the interval $[0, h)$), as long as the convergence rate is concerned. To reveal the mutual connection between the input and kernel approximation approaches, however, we further investigate the relationship between the error bounds in the input and kernel approximation approaches and also give explicit upper and lower bounds for the L_∞-induced norms obtained by these two approaches. Finally, we demonstrate the effectiveness of the resulting four types of computation methods through numerical examples.

In the following, we use the notations $\mathbb{N}$ and $\mathbb{R}^\nu_\infty$ to denote the set of positive integers and the Banach space of ν-dimensional real vectors equipped with vector ∞-norm, respectively. The notation $\|\cdot\|$ is used to mean either the $L_\infty[0, h)$ norm of a vector function, i.e.,

$$\|f(\cdot)\| := \max_j \operatorname*{ess\,sup}_{0 \le t < h} |f_j(t)| \tag{1}$$

(or that with h replaced by h/M or ∞), the $L_\infty[0, h)$-induced norm (or that with h/M or ∞ instead of h) of an operator, or the ∞-norm of a matrix or a vector, i.e., for $T \in \mathbb{R}^{m\times n}_\infty$,

$$\|T\| := \max_{i=1,\ldots,m} \sum_{j=1}^{n} |a_{ij}| \tag{2}$$

whose distinction will be clear from the context.

2 L_∞-Induced Norm and Truncation

Let us consider the stable finite-dimensional linear time-invariant (FDLTI) system

$$\frac{dx}{dt} = Ax + Bw, \quad z = Cx + Dw \tag{3}$$

where $x(t) \in \mathbb{R}^n_\infty$ is the state, $w(t) \in \mathbb{R}^{n_w}_\infty$ is the input and $z(t) \in \mathbb{R}^{n_z}_\infty$ is the output. It is well known that

$$z(t) = \int_0^t C\exp(A(t-\tau))Bw(\tau)d\tau + Dw(t) =: (\mathcal{F}_\infty w)(t) \quad (0 \le t < \infty) \tag{4}$$

where $\mathcal{F}_\infty$ is the operator from $(L_\infty)^{n_w}$ to $(L_\infty)^{n_z}$ associated with the input/output relation of the stable system (3). The L_∞-induced norm of the system (3) is given by

$$\sup_{\|w\|\le 1} \|(\mathcal{F}_\infty w)(\cdot)\| =: \|\mathcal{F}_\infty\| \tag{5}$$

where $\|\cdot\|$ on the left hand side denotes the L_∞ norm. For simplicity, let us assume $D = 0$ for a while; we will return to the general case with $D \neq 0$ before we provide our final results. Then, by noting that $(\mathcal{F}_\infty w)(t)$ is a continuous function, the L_∞-induced norm $\|\mathcal{F}_\infty\|$ is described by

$$\|\mathcal{F}_\infty\| = \sup_{\|w\|\le 1} \sup_t \|(\mathcal{F}_\infty w)(t)\| \tag{6}$$

On the other hand, since the system (3) is LTI, it readily follows from the property of L_∞ that

$$\sup_{\|w\|\le 1} \|(\mathcal{F}_\infty w)(T_1)\| \le \sup_{\|w\|\le 1} \|(\mathcal{F}_\infty w)(T_2)\| \tag{7}$$

whenever $0 \le T_1 < T_2$. This is because for every $w_1 \in L_\infty$ such that $\|w_1\| \le 1$, the function w_2 defined by delaying w_1 as

$$w_2(t) := \begin{cases} 0 & 0 \le t < T_2 - T_1 \\ w_1(t - T_2 + T_1) & t \ge T_2 - T_1 \end{cases} \tag{8}$$

belongs to L_∞, satisfies $\|w_2\| \le 1$, and the corresponding output $\mathcal{F}_\infty w_2$ becomes $\mathcal{F}_\infty w_1$ delayed by $T_2 - T_1$. By combining these arguments, the L_∞-induced norm $\|\mathcal{F}_\infty\|$ can be rearranged as

$$\begin{aligned}
\|\mathcal{F}_\infty\| &= \sup_{\|w\|\le 1} \lim_{t\to\infty} \|(\mathcal{F}_\infty w)(t)\| = \sup_{\|w\|\le 1} \lim_{t\to\infty} \left\| \int_0^t C\exp(A(t-\tau))Bw(\tau)d\tau + Dw(t) \right\| \\
&= \sup_{\|w\|\le 1} \lim_{t\to\infty} \left\| \int_0^t C\exp(A\theta)Bw(t-\theta)d\theta + Dw(t) \right\| \\
&= \sup_{\|u\|\le 1} \lim_{t\to\infty} \left\| \int_0^t C\exp(A\theta)Bu(\theta)d\theta + Du(0) \right\| \\
&=: \sup_{\|u\|\le 1} \lim_{t\to\infty} \|(\mathcal{F}u)(t)\| =: \|\mathcal{F}\|
\end{aligned} \tag{9}$$

where the equality in the third line is validated by letting $\theta := t - \tau$ and considering $u(\theta) := w(t-\theta)$ for $0 \le \theta \le t$. Hence, this paper computes $\|\mathcal{F}_\infty\|$ by computing the L_∞-induced norm $\|\mathcal{F}\|$ instead because of some simplicities in the following arguments.

Remark 1 *Even though we have been assuming for a while that $D = 0$ as mentioned above, note that $\mathcal{F}$ has been defined in (9) for a general D for later purposes.*

To compute $\|\mathcal{F}\|$ when $D = 0$, we first introduce a truncation idea of $\mathcal{F}$. We thus take a sufficiently large h. Without loss of generality (see the limit in the last line of (9)), we then take t larger than h and decompose $\mathcal{F}$ into

$$(\mathcal{F}u)(t) = \mathcal{F}_h^- u + (\mathcal{F}_h^+ u)(t) \tag{10}$$

where

$$\mathcal{F}_h^- u := \int_0^h C\exp(A\theta)Bu(\theta)d\theta, \quad (\mathcal{F}_h^+ u)(t) := \int_h^t C\exp(A\theta)Bu(\theta)d\theta \tag{11}$$

Then, we have

$$\|\mathcal{F}_h^-\| - \|\mathcal{F}_h^+\| \leq \|\mathcal{F}\| \leq \|\mathcal{F}_h^-\| + \|\mathcal{F}_h^+\| \tag{12}$$

where

$$\|\mathcal{F}_h^-\| := \sup_{\|u\|\leq 1} \|\mathcal{F}_h^- u\|, \quad \|\mathcal{F}_h^+\| := \sup_{\|u\|\leq 1} \lim_{t\to\infty} \|(\mathcal{F}_h^+ u)(t)\| \tag{13}$$

It will be explained in Section 4 that $\|\mathcal{F}_h^+\|$ has an upper bound proportional to $\|C\exp(Ah)\|$ and the latter norm becomes arbitrarily small by taking h sufficiently large by the stability assumption of (3). Hence, our approach to computing $\|\mathcal{F}\|$ uses (12), in which $\|\mathcal{F}_h^-\|$ is computed as accurately as possible while the computation of $\|\mathcal{F}_h^+\|$ is treated in a comparatively simple way; we aim at computing upper and lower bounds of the L_∞-induced norm $\|\mathcal{F}\|$ through approximations of $\mathcal{F}_h^-$ and an upper bound computation of $\|\mathcal{F}_h^+\|$. The choice of h (as well as other parameters to be introduced) will be discussed in Section 4.

3 Fast-lifting Treatment of $\mathcal{F}_h^-$ and Computation of $\|\mathcal{F}_h^-\|$

In this section, we suppose that h is given and aim at computing upper and lower bounds of $\|\mathcal{F}_h^-\|$. This is because a closed-form expression for this norm (can readily be obtained but) requires us to compute the integral of the absolute value of each entry of the matrix function $C\exp(A\theta)B$. Since it is very hard to perform such computations exactly, we consider computing the norm approximately but in such a way that its upper and lower bounds are available. To achieve this goal, we introduce input or kernel[1] approximation, where the former is related to $u(\theta)$ while the latter to $C\exp(A\theta)B$. They are two different approaches in terms of the viewpoint behind approximations but share a common technical feature that they use either a staircase approximation or piecewise linear approximation scheme of (either the input or kernel) functions. Furthermore, the associated approximation errors converge to 0 at the rate of $1/M$ and $1/M^2$ in staircase approximation and piecewise linear approximation, respectively. Here, M is the parameter of fast-lifting [7] applied to subdivide the interval $[0, h)$ into M subintervals with an equal width, as a preliminary step to develop such approximation schemes.

To describe the details of the approximate computation methods for $\|\mathcal{F}_h^-\|$, we first review fast-lifting [7] (which in the context of the present paper is nothing but lifting [8–10] applied to

[1]Even though this term should make sense only such a part of (9) relevant to w is referred to, we retain this term with a slight abuse of terminology even when we view such a part of (9) relevant to u.

signals with finite duration). For $M \in \mathbb{N}$ and $h' := h/M$, fast-lifting is defined as the mapping from $f \in (L_\infty[0,h))^\nu$ to $\check{f} := [(f^{(1)})^T \ \cdots \ (f^{(M)})^T]^T \in (L_\infty[0,h'))^{M\nu}$, and is denoted by $\check{f} := \mathbf{L}_M f$, where

$$f^{(i)}(\theta') := f((i-1)h' + \theta') \quad (0 \le \theta' < h') \tag{14}$$

It is easy to see from the fast-lifting treatment of $u(\theta)$ $(0 \le \theta < h)$ that the operation on $\mathcal{F}_h^- u$ is described by

$$\mathcal{F}_h^- u = \sum_{i=1}^{M} C(A'_d)^{i-1}\mathbf{B}' u^{(i)} \tag{15}$$

where

$$\mathbf{B}' u^{(i)} := \int_0^{h'} \exp(A\theta')Bu^{(i)}(\theta')d\theta', \quad A'_d := \exp(Ah') \tag{16}$$

Note that right hand side of (15) corresponds to the expanded representation of $\mathcal{F}_h^- \mathbf{L}_M^{-1}\check{u}$, where $\check{u} = \mathbf{L}_M u = [(u^{(1)})^T, \cdots, (u^{(M)})^T]^T$.

It readily follows that

$$\|\mathcal{F}_h^-\| = \|\mathcal{F}_h^- \mathbf{L}_M^{-1}\| \tag{17}$$

where $\|\cdot\|$ on the right hand side denotes the induced norm from $(L_\infty[0,h'))^{Mn_w}$ to $\mathbb{R}_\infty^{n_z}$. Regarding the right hand side, it follows from (15) that the operator $\mathcal{F}_h^- \mathbf{L}_M^{-1}$ is described by

$$\mathcal{F}_h^- \mathbf{L}_M^{-1} = C'_{dM}\overline{\mathbf{B}'} \tag{18}$$

where

$$C'_{dM} = \begin{bmatrix} C & CA'_d & \cdots & C(A'_d)^{M-1} \end{bmatrix} \tag{19}$$

and $\overline{(\cdot)}$ denotes $\mathrm{diag}[(\cdot), \cdots, (\cdot)]$ consisting of M copies of $(\cdot)$.

As mentioned before, it is difficult to compute $\|\mathcal{F}_h^-\|$ exactly since computing the integral of the absolute value of each entry of the matrix function $C\exp(A\theta)B$ is very hard. We thus aim at its approximate computation, for which the above application of fast-lifting is helpful when we are to compute $\|\mathcal{F}_h^-\|$ by computing $\|\mathcal{F}_h^- \mathbf{L}_M^{-1}\| = \|C'_{dM}\overline{\mathbf{B}'}\|$ instead. This is because the input and kernel function $C\exp(A\theta')B,\ 0 \le \theta \le h'$ associated with the operator $\mathbf{B}'$ are defined on a smaller interval than the interval $[0,h]$ on which $\mathcal{F}_h^-$ is defined. This provides us with a better chance for more accurate approximation. In particular, we aim at computing upper and lower bounds of $\|\mathcal{F}_h^-\|$ through the input or kernel approximation approach.

3.1 Review of Input Approximation Approach

In this subsection, we review the input approximation idea developed in [11,12], in which constant and linear approximations to the input of $\mathbf{B}'$ (which by (18) lead to staircase and piecewise linear approximations to the input of $\mathcal{F}_h^-$) are introduced for computing $\|\mathcal{F}_h^-\|$, as well as the associated convergence rates in M. It is important to note from our preceding arguments that $\|\mathcal{F}_h^-\|$ is nothing but the $L_\infty[0,h)$-induced norm of the compression operator on $[0,h)$ associated with the FDLTI system (3). Hence, the following descriptions in this subsection is nothing but the review of our recent results in [11].

3.1.1 Staircase approximation

Consider the averaging operator $\mathbf{J}_0' : (L_\infty[0,h'))^{n_w} \to (L_\infty[0,h'))^{n_w}$ [11,12] defined by

$$(\mathbf{J}_0' u)(\theta') = \frac{1}{h'} \int_0^{h'} u(\tau') d\tau' \quad (0 \le \theta' < h') \tag{20}$$

and the operator $\mathbf{B}_{\mathrm{i}0}' : (L_\infty[0,h))^{n_w} \to \mathbb{R}_\infty^n$ defined as $\mathbf{B}_{\mathrm{i}0}' := \mathbf{B}'\mathbf{J}_0'$, where the subscript i stands for input approximation. In other words, introducing the operator $\mathbf{B}_{\mathrm{i}0}'$ corresponds to restricting the input of $\mathbf{B}'$ to constant functions and that $\mathbf{B}_{\mathrm{i}0}' u = \mathbf{B}' u$ whenever u is a constant function.

We next consider the operator $\mathcal{F}_{hM\mathrm{i}0}^-$ obtained by replacing $\mathbf{B}'$ with $\mathbf{B}_{\mathrm{i}0}'$ in (18):

$$\mathcal{F}_{hM\mathrm{i}0}^- = C_{dM}' \overline{\mathbf{B}_{\mathrm{i}0}'} \tag{21}$$

It is easy to see that $\mathcal{F}_{hM\mathrm{i}0}^-$ is the fast-lifted counterpart to the staircase approximation of $\mathcal{F}_h^-$ (under the input approximation approach). When we consider $\|\mathcal{F}_{hM\mathrm{i}0}^-\| = \|C_{dM}' \overline{\mathbf{B}_{\mathrm{i}0}'}\|$ as an approximation of $\|\mathcal{F}\|$ (recall (12) and (17)), it is easy to see that the input of $\mathbf{B}_{\mathrm{i}0}'$ may be confined to constant vector functions. In this case, we may replace $C_{dM}' \overline{\mathbf{B}_{\mathrm{i}0}'}$ with $C_{dM}' \overline{B_{0d}'}$ by identifying constant vector functions with constant vectors, where

$$B_{0d}' := \int_0^{h'} \exp(A\theta') B d\theta' \tag{22}$$

This implies that $\|\mathcal{F}_{hM\mathrm{i}0}^-\|$ can be computed exactly and (after the recovery of the treatment of D) leads to the following theorem [11], where

$$F_{hM\mathrm{i}0}^- := \begin{bmatrix} CB_{0d}' & \cdots & C(A_d')^{M-1} B_{0d}' & D \end{bmatrix} \tag{23}$$

Theorem 1 *The inequality*

$$\|F_{hM\mathrm{i}0}^-\| \le \|\mathcal{F}_h^-\| \le \|F_{hM\mathrm{i}0}^-\| + \frac{K_{M\mathrm{i}0}}{M} \tag{24}$$

holds with $K_{M\mathrm{i}0}$ given by

$$K_{M\mathrm{i}0} := \frac{h^2}{M} \|C_{dM}'\| \cdot \|A\| \cdot \|B\| e^{\|A\|h/M} \tag{25}$$

Furthermore, $K_{M\mathrm{i}0}$ has the following uniform upper bound with respect to M:

$$K_{\mathrm{i}0}^U := h^2 \|C\| \cdot \|A\| \cdot \|B\| e^{2\|A\|h} \tag{26}$$

Remark 2 *The second assertion of Theorem 1 can be proved easily if we note from (19) that*

$$\|C_{dM}'\| \le M \|C\| e^{\|A\|h} \tag{27}$$

The same arguments are repeatedly applied to the arguments associated with $K_{\mathrm{i}1}^U$, $K_{\mathrm{k}0}^U$ and $K_{\mathrm{k}1}^U$ in Theorems 2, 3 and 4, respectively.

Theorem 1 implies that an upper bound and a lower bound of $\|\mathcal{F}_h^-\|$ can be computed through matrix ∞-norm computations, and as the fast-lifting parameter M becomes larger, the gap between the upper and lower bounds tends to 0 at no slower convergence rate than $1/M$.

3.1.2 Piecewise linear approximation

Following again the arguments in [11], we next introduce the 'linearizing' operator $\mathbf{J}_1' : (L_\infty[0,h'))^{n_w} \to (L_\infty[0,h'))^{n_w}$ defined as

$$(\mathbf{J}_1' u)(\theta') = \int_0^{h'} f_0(\tau')u(\tau')d\tau' + \theta' \int_0^{h'} f_1(\tau')u(\tau')d\tau' \tag{28}$$

where the scalar-valued functions $f_0(\tau')$ and $f_1(\tau')$ [11,12] are given by

$$f_0(\tau') = -\frac{6}{(h')^2}\tau' + \frac{4}{h'},\ f_1(\tau') = \frac{12}{(h')^3}\tau' - \frac{6}{(h')^2} \tag{29}$$

See [11] for the rationale for taking such specific functions; among important properties of $\mathbf{J}_1'$ is that $\mathbf{J}_1' u = u$ whenever u is a linear function. Let us further introduce the operator $\mathbf{B}_{\mathrm{i}1}' := \mathbf{B}'\mathbf{J}_1'$. Introducing this operator is equivalent to restricting the input of $\mathbf{B}'$ to linear functions, and $\mathbf{B}_{\mathrm{i}1}' u = \mathbf{B}' u$ whenever u is a linear function.

We next consider the operator $\mathcal{F}_{hM\mathrm{i}1}^-$ obtained by replacing $\mathbf{B}'$ with $\mathbf{B}_{\mathrm{i}1}'$ in (18):

$$\mathcal{F}_{hM\mathrm{i}1}^- = C_{dM}' \overline{\mathbf{B}_{\mathrm{i}1}'} \tag{30}$$

It is easy to see that $\mathcal{F}_{hM\mathrm{i}1}^-$ is the fast-lifted counterpart to the piecewise linear approximation of $\mathcal{F}_h^-$ (under the input approximation approach). As discussed in [11], $\|\mathcal{F}_{hM\mathrm{i}1}^-\|$ can also be computed exactly, and (after the recovery of the treatment of D) we are led to Theorem 2 given below, whose statement requires some preparations as follows: Let T_j $(j = 1, \cdots, M)$ be the matrix consisting of the $L_1[0,h')$ norm of each entry of the matrix linear function

$$S_{j0} + S_{j1}\theta' := C(A_d')^{j-1}(G_0 + G_1\theta') \tag{31}$$

where the matrices G_0 and G_1 are defined as

$$G_0 := -\frac{6}{(h')^2}B_{1d}' + \frac{4}{h'}B_{0d}', \quad G_1 := \frac{12}{(h')^3}B_{1d}' - \frac{6}{(h')^2}B_{0d}' \tag{32}$$

through the matrices B_{0d}' and

$$B_{1d}' := \int_0^{h'} \exp(A\theta')\theta' B d\theta' \tag{33}$$

Note that the $L_1[0,h')$ norm of a scalar function f on $[0,h')$ is defined as $\int_0^{h'} |f(t)|dt$. Defining the matrix

$$F_{hM\mathrm{i}1}^- := \begin{bmatrix} T_1 & \cdots & T_M & D \end{bmatrix} \tag{34}$$

we are led to the following theorem [11].

Theorem 2 *The inequality*

$$\|F_{hM\mathrm{i}1}^-\| - \frac{K_{M\mathrm{i}1}}{M^2} \le \|\mathcal{F}_h^-\| \le \|F_{hM\mathrm{i}1}^-\| + \frac{K_{M\mathrm{i}1}}{M^2} \tag{35}$$

holds with $K_{M\mathrm{i}1}$ *given by*

$$K_{M\mathrm{i}1} := \frac{h^3}{2M}\|C_{dM}'\| \cdot \|A\|^2 \cdot \|B\| e^{\|A\|h/M} \tag{36}$$

Furthermore, $K_{M\mathrm{i}1}$ *has the following uniform upper bound with respect to* M*:*

$$K_{\mathrm{i}1}^U := \frac{h^3}{2}\|C\| \cdot \|A\|^2 \|B\| e^{2\|A\|h} \tag{37}$$

Theorem 2 implies that an upper bound and a lower bound of $\|\mathcal{F}_h^-\|$ can be computed through the matrix ∞-norm $\|F_{hM\mathrm{i}1}^-\|$, and as the fast-lifting parameter M becomes larger, the gap between the upper and lower bounds tends to 0 at no slower convergence rate than $1/M^2$.

Remark 3 *We note that FDLTI systems are a special case of sampled-data systems and the input approximation approach in the present paper has been taken also in the L_∞-induced norm computation of sampled-data systems [12] (derived by extending the arguments in [11] developed for the $L_\infty[0,h)$-induced norm computation of compression operators). Hence, Theorems 1 and 2 reviewed above for the input approximation approach can be interpreted to follow from the arguments in [12].*

3.2 Kernel Approximation Approach

This subsection proceeds to one of the main arguments in this paper, in which we develop a new framework for computing $\|\mathcal{F}_h^-\|$ by using an idea of kernel approximation. More precisely, we apply staircase and piecewise linear approximations to the kernel function $C\exp(A\theta)B$ (or more precisely, constant and linear approximations of the kernel function $\exp(A\theta')B,\ 0 \leq \theta' < h'$) and show the associated convergence rates in M.

3.2.1 Staircase approximation

We introduce the operator $\mathbf{B}'_{\mathrm{k}0} : (L_\infty[0,h'))^{n_w} \to \mathbb{R}^n_\infty$ defined as

$$\mathbf{B}'_{\mathrm{k}0} u := \int_0^{h'} Bu(\theta')d\theta' \tag{38}$$

where the subscript k stands for kernel approximation. Introducing the operator $\mathbf{B}'_{\mathrm{k}0}$ corresponds to the zero-order approximation of the kernel function $\exp(A\theta')B = \sum_{i=0}^{\infty} \frac{(A\theta')^i}{i!} B$ of the operator $\mathbf{B}'$.

We next consider the operator $\mathcal{F}_{hM\mathrm{k}0}^-$ obtained by replacing $\mathbf{B}'$ with $\mathbf{B}'_{\mathrm{k}0}$ in (18):

$$\mathcal{F}_{hM\mathrm{k}0}^- := C'_{dM}\overline{\mathbf{B}'_{\mathrm{k}0}} \tag{39}$$

It is easy to see that $\mathcal{F}_{hM\mathrm{k}0}^-$ is the fast-lifted counterpart to the staircase approximation of $\mathcal{F}_h^-$ (under the kernel approximation approach). This paper shows that $\|\mathcal{F}_{hM\mathrm{k}0}^-\|$ can be computed exactly and tends to $\|\mathcal{F}_h^-\|$ as $M \to \infty$. The following two lemmas play important roles in establishing the above facts and the associated convergence rate; we remark that the treatment of D has been recovered in the second lemma.

Lemma 1 *The following inequality holds:*

$$\|\mathbf{B}' - \mathbf{B}'_{\mathrm{k}0}\| \leq \frac{h^2}{2M^2}\|A\| \cdot \|B\| e^{\|A\|h/M} \tag{40}$$

Lemma 2 *$\|\mathcal{F}_{hM\mathrm{k}0}^-\|$ coincides with the ∞-norm of the finite-dimensional matrix $F_{hM\mathrm{k}0}^-$ given by*

$$F_{hM\mathrm{k}0}^- := \begin{bmatrix} CBh' & \cdots & C(A'_d)^{M-1}Bh' & D \end{bmatrix} \tag{41}$$

The proofs of these lemmas are given in Appendix A since they are quite technical. From Lemmas 1 and 2, we can readily obtain the following result.

Theorem 3 *The inequality*

$$\|F_{hM\mathrm{k}0}^{-}\| - \frac{K_{M\mathrm{k}0}}{M} \le \|\mathcal{F}_h^{-}\| \le \|F_{hM\mathrm{k}0}^{-}\| + \frac{K_{M\mathrm{k}0}}{M} \tag{42}$$

holds with $K_{M\mathrm{k}0}$ defined as

$$K_{M\mathrm{k}0} := \frac{h^2}{2M}\|C'_{dM}\| \cdot \|A\| \cdot \|B\| e^{\|A\|h/M} \tag{43}$$

Furthermore, $K_{M\mathrm{k}0}$ has a uniform upper bound with respect to M given by

$$K_{\mathrm{k}0}^{U} := \frac{h^2}{2}\|C\| \cdot \|A\| \cdot \|B\| e^{2\|A\|h} \tag{44}$$

3.2.2 Piecewise linear approximation

We introduce the operator $\mathbf{B}'_{\mathrm{k}1} : (L_\infty[0, h'))^{n_w} \to \mathbb{R}^n_\infty$ defined as

$$\mathbf{B}'_{\mathrm{k}1} u := \int_0^{h'} (I + A\theta')Bu(\theta')d\theta' \tag{45}$$

Introducing the operator $\mathbf{B}'_{\mathrm{k}1}$ is equivalent to the first-order approximation of the kernel function of $\mathbf{B}'$.

We next consider the operator $\mathcal{F}_{hM\mathrm{k}1}^{-}$ obtained by replacing $\mathbf{B}'$ with $\mathbf{B}'_{\mathrm{k}1}$ in (18):

$$\mathcal{F}_{hM\mathrm{k}1}^{-} = C'_{dM}\overline{\mathbf{B}'_{\mathrm{k}1}} \tag{46}$$

It is easy to see that $\mathcal{F}_{hM\mathrm{k}1}^{-}$ is the fast-lifted counterpart to the piecewise linear approximation of $\mathcal{F}_h^{-}$ (under the kernel approximation approach). In the following, we show that $\|\mathcal{F}_{hM\mathrm{k}1}^{-}\|$ can be computed exactly and converges to $\|\mathcal{F}_h^{-}\|$ as $M \to \infty$. The following two lemmas are significant in establishing the above facts together with the associated convergence rate.

Lemma 3 *The following inequality holds:*

$$\|\mathbf{B}' - \mathbf{B}'_{\mathrm{k}1}\| \le \frac{h^3}{6M^3}\|A\|^2 \cdot \|B\| e^{\|A\|h/M} \tag{47}$$

Lemma 4 *Let Y_j $(j = 1, \cdots, M)$ be the matrix consisting of the $L_1[0, h')$ norm of each entry of the matrix linear function $C(A'_d)^{j-1}(I + A\theta')B$ involved in (46). Then, $\|\mathcal{F}_{hM\mathrm{k}1}^{-}\|$ coincides with the ∞-norm of the finite-dimensional matrix $F_{hM\mathrm{k}1}^{-}$ given by*

$$F_{hM\mathrm{k}1}^{-} := \begin{bmatrix} Y_1 & \cdots & Y_M & D \end{bmatrix} \tag{48}$$

The proofs of these lemmas are also given in the appendix. From Lemmas 3 and 4, we can readily obtain the following theorem.

Theorem 4 *The inequality*

$$\|F_{hM\mathrm{k}1}^{-}\| - \frac{K_{M\mathrm{k}1}}{M^2} \leq \|\mathcal{F}_h^{-}\| \leq \|F_{hM\mathrm{k}1}^{-}\| + \frac{K_{M\mathrm{k}1}}{M^2} \tag{49}$$

holds with $K_{M\mathrm{k}1}$ defined as

$$K_{M\mathrm{k}1} := \frac{h^3}{6M}\|C'_{dM}\| \cdot \|A\|^2 \cdot \|B\| e^{\|A\|h/M} \tag{50}$$

Furthermore, $K_{M\mathrm{k}1}$ has a uniform upper bound with respect to M given by

$$K_{\mathrm{k}1}^{U} := \frac{h^3}{6}\|C\| \cdot \|A\|^2 \cdot \|B\| e^{2\|A\|h} \tag{51}$$

We are now in a position to compare effectiveness of the input and kernel approximation approaches in the treatment of $\mathcal{F}_h^{-}$, the compression operator on $[0, h)$. We see that $K_{M\mathrm{k}0}$ and $K_{M\mathrm{k}1}$ relevant to the approximation errors in the kernel approximation approach developed in the present paper are smaller than $K_{M\mathrm{i}0}$ and $K_{M\mathrm{i}1}$, respectively, relevant to those for the existing input approximation approach. More precisely, we can see from (25) and (43) that $K_{M\mathrm{k}0} = K_{M\mathrm{i}0}/2$ for the staircase approximation scheme, while (36) and (50) implies that $K_{M\mathrm{k}1} = K_{M\mathrm{i}1}/3$ for the piecewise linear approximation scheme. If we note that the treatment of the truncated part $\mathcal{F}_h^{+}$ discussed in the following section is common for all the four methods discussed in this section, the following interpretations of these two relations will be justified.

The former relation implies that the gap between the upper and lower bounds in (24) and that in (42) coincide with each other. This could be interpreted as implying that the overall ability is the same for the input and kernel approximation approaches as far as the staircase approximation scheme is taken. For the piecewise linear approximation scheme, on the other hand, the latter relation implies that the gap between the upper and lower bounds in (49) for the kernel approximation approach is one third of that in (35) for the input approximation approach. Meanwhile, it has been (numerically) demonstrated in [11] (dealing with the $L_\infty[0, h)$-induced norm of the compression operator through the input approximation approach) that the piecewise linear approximation scheme is superior to the staircase approximation scheme in the computation of $\|\mathcal{F}_h^{-}\|$ under the input approximation approach. Summarizing these observation clearly indicates an advantage of the method with combined use of the piecewise linear approximation scheme and our new kernel approximation approach over the other three methods.

Remark 4 *Through similar arguments to [12], it is expected that the same combined method can be developed also for the computation of the L_∞-induced norm of sampled-data systems. Such a method is also expected to lead to an improved gap between its upper and lower bounds than the method in [12], which is based on (the piecewise linear approximation scheme and) the input approximation approach.*

4 Upper Bound of $\|\mathcal{F}_h^{+}\|$ and Computation of the L_∞-Induced Norm

This section is dedicated to a computation method for an upper bound of $\|\mathcal{F}_h^{+}\|$, which together with the arguments in the preceding sections leads to methods for computing upper and lower bounds of the L_∞-induced norm $\|\mathcal{F}\|$ of the FDLTI system (3). Theses bounds are ensured to converge to each other as the parameters h and M tends to ∞.

We first note from (13) (with t replaced by $t+h$) and (11) that

$$\|\mathcal{F}_h^+\| \le \lim_{t\to\infty} \sup_{\|u\|\le 1} \left\| \int_0^t \exp(A\theta)Bu(\theta+h)d\theta \right\| \|C\exp(Ah)\| \tag{52}$$

If we take $q>0$ such that $\|\exp(Aq)\|<1$, it easily follows that

$$\begin{aligned}
&\lim_{t\to\infty} \sup_{\|u\|\le 1} \left\| \int_0^t \exp(A\theta)Bu(\theta+h)d\theta \right\| \\
\le &(1+\|\exp(Aq)\|+\|\exp(2Aq)\|+\cdots)\cdot \sup_{\|v\|\le 1} \left\| \int_0^q \exp(A\theta)Bv(\theta)d\theta \right\| \\
\le &\frac{1}{1-\|\exp(Aq)\|}\cdot q e^{\|A\|q}\|B\|
\end{aligned} \tag{53}$$

Summarizing (52) and (53), we can obtain the following result.

Proposition 1 *If we take $q>0$ such that $\|\exp(Aq)\|<1$, then*

$$\|\mathcal{F}_h^+\| \le \frac{q e^{\|A\|q}\|B\|}{1-\|\exp(Aq)\|}\|C\exp(Ah)\| =: K_{hq} \tag{54}$$

and K_{hq} converges to 0 regardless of q as $h\to\infty$.

Combining Theorems 1–4 and Proposition 1 together with (12), we are led to the following main results.

Theorem 5 *If we take $q>0$ such that $\|\exp(Aq)\|<1$, then*

$$\|F^-_{hM\mathrm{i}0}\| - K_{hq} \le \|\mathcal{F}\| \le \|F^-_{hM\mathrm{i}0}\| + \frac{K_{M\mathrm{i}0}}{M} + K_{hq} \tag{55}$$

$$\|F^-_{hM\mathrm{i}1}\| - \frac{K_{M\mathrm{i}1}}{M^2} - K_{hq} \le \|\mathcal{F}\| \le \|F^-_{hM\mathrm{i}1}\| + \frac{K_{M\mathrm{i}1}}{M^2} + K_{hq} \tag{56}$$

$$\|F^-_{hM\mathrm{k}0}\| - \frac{K_{M\mathrm{k}0}}{M} - K_{hq} \le \|\mathcal{F}\| \le \|F^-_{hM\mathrm{k}0}\| + \frac{K_{M\mathrm{k}0}}{M} + K_{hq} \tag{57}$$

$$\|F^-_{hM\mathrm{k}1}\| - \frac{K_{M\mathrm{k}1}}{M^2} - K_{hq} \le \|\mathcal{F}\| \le \|F^-_{hM\mathrm{k}1}\| + \frac{K_{M\mathrm{k}1}}{M^2} + K_{hq} \tag{58}$$

Furthermore, $K_{M\mathrm{i}0}$, $K_{M\mathrm{i}1}$, $K_{M\mathrm{k}0}$ and $K_{M\mathrm{k}1}$ have uniform upper bounds $K^U_{\mathrm{i}0}$, $K^U_{\mathrm{i}1}$, $K^U_{\mathrm{k}0}$ and $K^U_{\mathrm{k}1}$ defined as (26), (37), (44) and (51), respectively, and $K_{M\mathrm{i}0}/M$, $K_{M\mathrm{i}1}/M^2$, $K_{M\mathrm{k}0}/M$ and $K_{M\mathrm{k}1}/M^2$ converge to 0 as $M\to\infty$, while K_{hq} converges to 0 regardless of q as $h\to\infty$.

It should be noted in Theorem 5 that the uniform upper bounds $K^U_{\mathrm{i}0}$, $K^U_{\mathrm{i}1}$, $K^U_{\mathrm{k}0}$ and $K^U_{\mathrm{k}1}$ given in (26), (37), (44) and (51), respectively, depend on h, and increase as h is increased to reduce K_{hq}. However, K_{hq} is bounded from above in the exponential order $e^{\sigma h}$ in h regardless of q, where $\sigma<0$ is the maximum real part of the eigenvalues of A. It is hence expected that K_{hq} can be made small with a modest h and thus we can keep the uniform upper bounds $K^U_{\mathrm{i}0}$, $K^U_{\mathrm{i}1}$, $K^U_{\mathrm{k}0}$ and $K^U_{\mathrm{k}1}$ modest.

Regarding a guideline for taking the parameters h, M and q, we can summarize the above arguments as follows. It may be reasonable to take a relatively small q as long as $\|\exp(Aq)\|<1$; this is to avoid undue increase of K_{hq}, or in particular $e^{\|A\|q}$. Once q is fixed, the next step would be to take an h such that K_{hq} is as small as we wish; this is always possible by taking h sufficiently large. Once h is also fixed, the uniform upper bounds $K_{\rm i0}^U$, $K_{\rm i1}^U$, $K_{\rm k0}^U$ and $K_{\rm k1}^U$ in (26), (37), (44) and (51), respectively, are determined, and thus the last step would be taking an M such that $K_{\rm i0}^U/M$, $K_{\rm i1}^U/M^2$, $K_{\rm k0}^U/M$ and $K_{\rm k1}^U/M^2$ are as small as we wish. It is obvious that following this kind of guideline leads to computation methods for the L_∞-induced norm of the FDLTI system (3) to any degree of accuracy.

5 Numerical Examples

In this section, we study numerical examples and examine effectiveness of the computation methods discussed in this paper.

Let us first consider the stable SISO FDLTI oscillatory system

$$A = \begin{bmatrix} 0 & -2 \\ 2 & -2 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \ D = 1 \tag{59}$$

We compute estimates of its L_∞-induced norm, or equivalently $\|\mathcal{F}\|$, by taking the fast-lifting parameter M ranging from 500 to 5000 on the condition that $h = 25$ and $q = 2$ following the guideline in Section 4, which leads to $K_{hq} = 2.26 \times 10^{-7}$. The results for the upper and lower bounds of $\|\mathcal{F}\|$ obtained by Theorem 5 and the computation times under the staircase approximation scheme are shown in Table 1, while with the piecewise linear approximation scheme are shown in Table 2. We are mainly interested in the comparison between the existing input approximation approach and the kernel approximation approach developed in this paper. Hence, these (and the following) tables consist of Case (a) for the existing approach and Case (b) for the new approach.

We next consider the stable MIMO FDLTI oscillatory system

$$A = \begin{bmatrix} -1 & 0 & 2 & 2 \\ 1 & -1 & 2 & 3 \\ 0 & -2 & -2 & 0 \\ 1 & -1 & -1 & -2 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 0 \\ 1 & -1 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 2 & 1 & -1 & 1 \end{bmatrix}, \ D = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \tag{60}$$

We compute the upper and lower bounds of its L_∞-induced norm by taking the fast-lifting parameter M ranging from 500 to 5000 on the condition that $h = 25$ and $q = 2$, which leads to $K_{hq} = 2.65 \times 10^{-8}$. The results are shown in Tables 3 and 4.

Table 1: Results with staircase approximation scheme in SISO example.

Case (a): Input approximation approach

M	500	1000	2000	5000
$\|F_{hM\rm i0}^-\| + \dfrac{K_{M\rm i0}}{M} + K_{hq}$	3.703542	3.361762	3.215702	3.135198
$\|F_{hM\rm i0}\| - K_{hq}$	3.084104	3.084248	3.084370	3.084370
time (sec)	0.015281	0.030330	0.036423	0.079816

Case (b): Kernel approximation approach

M	500	1000	2000	5000
$\|F_{hM\rm k0}^-\| + \dfrac{K_{M\rm k0}}{M} + K_{hq}$	3.392950	3.222860	3.149950	3.109775
$\|F_{hM\rm k0}^-\| - \dfrac{K_{M\rm k0}}{M} - K_{hq}$	2.773512	2.945347	3.018618	3.058947
time (sec)	0.015093	0.024384	0.036143	0.078988

We can see from these tables that the error bounds for the computation of $\|\mathcal{F}\|$ (i.e., the gaps between the upper and lower bounds) decrease by taking M larger for all estimates. Hence, all the four approximation methods discussed in this paper can be validated as methods for computing the L_∞-induced norm. A more important concern in this paper, however, lies in the effectiveness comparison between (a) the existing input approximation approach and (b) the kernel approximation approach developed in this paper. In this respect, we had an earlier discussion in Section 3, which implies that, under the staircase approximation scheme, the kernel approximation approach can provide no advantage over the input approximation approach in reducing the gap between the computed upper and lower bounds. As seen from Tables 1 and 3, the convergence of this gap (common for the input and kernel approximation schemes) is not fast with respect to M. This suggests us to use the piecewise linear approximation scheme instead, which exhibits much faster convergence as seen from Tables 2 and 4. We can further observe from these tables that once we switch to the piecewise linear approximation scheme, an advantage of the kernel approximation approach over the input approximation approach is prominent. This is because the range between the upper and lower bounds obtained by the kernel approximation approach is always contained in (and thus less conservative than) that by the input approximation approach for the same M.

Table 2: Results with piecewise linear approximation scheme in SISO example.

Case (a): Input approximation approach

M	500	1000	2000	5000
$\|F_{hM\mathrm{i}1}^{-}\| + \dfrac{K_{M\mathrm{i}1}}{M^2} + K_{hq}$	3.146314	3.098249	3.087656	3.084882
$\|F_{hM\mathrm{i}1}^{-}\| - \dfrac{K_{M\mathrm{i}1}}{M^2} - K_{hq}$	3.022426	3.070497	3.081089	3.083865
time (sec)	0.020353	0.035159	0.049186	0.120428

Case (b): Kernel approximation approach

M	500	1000	2000	5000
$\|F_{hM\mathrm{k}1}^{-}\| + \dfrac{K_{M\mathrm{k}1}}{M^2} + K_{hq}$	3.108609	3.089882	3.085686	3.084578
$\|F_{hM\mathrm{k}1}^{-}\| - \dfrac{K_{M\mathrm{k}1}}{M^2} - K_{hq}$	3.067312	3.080631	3.083497	3.084238
time (sec)	0.019011	0.032270	0.038521	0.119789

Table 3: Results with staircase approximation scheme in MIMO example.

Case (a): Input approximation approach

M	500	1000	2000	5000
$\|F_{hM\mathrm{i}0}^{-}\| + \dfrac{K_{M\mathrm{i}0}}{M} + K_{hq}$	18.548486	13.828079	11.996866	11.041614
$\|F_{hM\mathrm{i}0}\| - K_{hq}$	10.458708	10.458788	10.459380	10.459432
time (sec)	0.017563	0.033310	0.048476	0.094044

Case (b): Kernel approximation approach

M	500	1000	2000	5000
$\|F_{hM\mathrm{k}0}^{-}\| + \dfrac{K_{M\mathrm{k}0}}{M} + K_{hq}$	14.576154	12.181119	11.246745	10.758005
$\|F_{hM\mathrm{k}0}^{-}\| - \dfrac{K_{M\mathrm{k}0}}{M} - K_{hq}$	6.486376	8.811828	9.709259	10.175823
time (sec)	0.015604	0.026847	0.040918	0.090371

Furthermore, the computation times in the kernel approximation approach are slightly smaller than those in the input approximation approach under the same parameter M. As an overall evaluation, the kernel approximation approach with the piecewise linear approximation scheme exhibits the smallest range for the L_∞-induced norm estimates with relatively short computation times among the four methods discussed in this paper, and thus can be an effective alternative to the existing methods developed in earlier studies.

6 Conclusion

This paper tackled a difficult problem of accurately computing the L_∞-induced norm associated with a stable FDLTI system, which is very important in many control systems. To this problem, we applied a truncation idea with a sufficiently large h, which mostly reduces the problem to the induced-norm computation of the compression operator defined on the time interval $[0, h)$ (describing the input/output relation of the FDLTI system on that interval). We first reviewed the input approximation approach to the $L_\infty[0, h)$-induced norm computation of the compression operator based on the fast-lifting treatment. We next developed a new approach to the $L_\infty[0, h)$-induced norm computation called the kernel approximation approach, which is also based on fast-lifting. In the latter new approach, we applied two schemes in approximating kernel functions, which are essentially the same as those used in the former existing approach, i.e., the staircase approximation scheme and the piecewise linear approximation scheme. It was then shown that the approximation errors in our new approach converge to 0 at the rates of $1/M$ and $1/M^2$ in the staircase approximation and piecewise linear approximation schemes, respectively, as the fast-lifting parameter M tends to infinity. Even though these convergence rates are qualitatively the same as those in the existing input approximation approach, our detailed analysis showed that the approximation errors through our new kernel approximation approach are smaller than those through the existing input approximation approach. We then gave a method for evaluating the effect on the truncated interval $[h, \infty)$, and this was used commonly in both the input and kernel approximation approaches. Through this evaluation together with the input and kernel approximation approaches, we can compute the L_∞-induced norm of FDLTI systems to any degree of accuracy. Finally, we examined effectiveness of our kernel approximation approach through numerical studies and confirmed that

Table 4: Results with piecewise linear approximation scheme in MIMO example.

Case (a): Input approximation approach

M	500	1000	2000	5000
$\Vert F_{hM\mathrm{i}1}^{-}\Vert + \dfrac{K_{M\mathrm{i}1}}{M^2} + K_{hq}$	11.875157	10.754256	10.526707	10.469630
$\Vert F_{hM\mathrm{i}1}^{-}\Vert - \dfrac{K_{M\mathrm{i}1}}{M^2} - K_{hq}$	9.043735	10.164630	10.392177	10.449254
time (sec)	0.025248	0.038933	0.064876	0.134318

Case (b): Kernel approximation approach

M	500	1000	2000	5000
$\Vert F_{hM\mathrm{k}1}^{-}\Vert + \dfrac{K_{M\mathrm{k}1}}{M^2} + K_{hq}$	10.943846	10.560772	10.482617	10.462958
$\Vert F_{hM\mathrm{k}1}^{-}\Vert - \dfrac{K_{M\mathrm{k}1}}{M^2} - K_{hq}$	10.000038	10.364230	10.437774	10.456166
time (sec)	0.022003	0.038181	0.061540	0.133024

this approach works more effectively than the existing input approximation approach, not only in accuracy but also in computation times, especially when the piecewise linear approximation scheme is taken.

It is expected that the kernel approximation approach developed in this paper can be extended to the computation of the L_∞-induced norm of sampled-data systems (i.e., the L_1 analysis of sampled-data systems) and lead to more accurate estimates than the existing input approximation approach. On the other hand, however, there seems to be an obstacle for the kernel approximation approach to be directly applied to the L_1 optimal controller synthesis problem of sampled-data systems while the input approximation approach can be, with the staircase approximation scheme [13, 14] or the piecewise linear approximation scheme [15]. This is because the preadjoint arguments, which play a crucial role both in the staircase approximation scheme [14] and the piecewise linear approximation scheme [15] in tackling the L_1 optimal controller synthesis problem with the input approximation approach, do not seem applicable in the kernel approximation approach. This, in turn, implies that developing a theoretical basis of the kernel approximation approach for this synthesis problem seems to be a nontrivial issue. This interesting topic is left for future studies.

We further note that constructing the jth-order approximants $\mathbf{B}'_{\mathrm{i}j}$ and $\mathbf{B}'_{\mathrm{k}j}$ to $\mathbf{B}'$ (with desired properties from the jth-order approximation viewpoint) could be carried out even for $j \geq 2$ by following the same line of arguments as in [11] and Subsection 3.2, respectively. However, the overall performance improvement by taking $j \geq 2$ may not be definite since it would take a longer time to compute the $L_1[0, h')$ norms of jth-order polynomials when $j \geq 2$. This is in sharp contrast with the present paper dealing only with $j = 0$ and $j = 1$ (i.e., constant and linear functions) and might govern the overall performance as the fast-lifting parameter M becomes larger. Analyzing such an aspect and developing an effective computation method exploiting a jth-order approximation idea for $j \geq 2$ may be an interesting future topic.

Finally, we remark that for the class of positive finite-dimensional LTI systems [6], the L_∞-induced norm computation reduces to the finite-dimensional matrix ∞-norm computation $\|D - CA^{-1}B\|$ as shown in [4] (for essentially the same reason as that stated in Introduction). More interestingly, this explicit result has been extended to positive LTI systems with distributed delays in [5], where an 'equivalent' *finite-dimensional* LTI positive system has been clarified that possesses the same L_∞-induced norm as the original system with distributed delays. This might suggest that computing upper and lower bounds of the L_∞-induced norm of (not necessarily positive) LTI systems with (distributed) delays may still be a tractable problem. If the L_∞-induced norm of (not necessarily positive) LTI systems with (distributed) delays can also be reduced to the L_∞-induced norm associated with 'equivalent' LTI systems without delays, both the input approximation method and kernel approximation method discussed in the present paper can immediately be applied to the latter systems, but it is nontrivial whether such equivalent systems do exist generically. This may be an interesting topic to study. On the other hand, aiming at direct extension of the kernel approximation method in the present paper to the distributed delay systems, in which the kernel functions associated with distributed delays are approximated, could also be a (quite nontrivial but) interesting future topic.

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A. Proofs of Lemmas

This appendix is concerned with the proofs of Lemmas in Section 3. They are based on the Taylor expansion of the matrix exponential of $A\theta'$ (or Ah'), and the proofs of Lemmas 1 and 2 proceed in essentially the same way as those of Lemmas 3 and 4. Hence, only the proofs of the latter two lemmas are given.

From the Taylor expansion of $\exp(A\theta')$, we can obtain the following inequalities.

$$
\begin{aligned}
&\|(\mathbf{B}' - \mathbf{B}'_{\mathrm{k1}})u\| = \left\| \int_0^{h'} \{\exp(A\theta') - (I + A\theta')\} Bu(\theta') d\theta' \right\| \\
&= \left\| \int_0^{h'} \left(\sum_{j=2}^{\infty} \frac{A^j (\theta')^j}{j!} \right) Bu(\theta') d\theta' \right\| \leq \int_0^{h'} \sum_{j=2}^{\infty} \frac{\|A\|^j (\theta')^j}{j!} d\theta' \cdot \|B\| \cdot \|u\| \\
&\leq \frac{1}{6} (h')^3 \|A\|^2 \cdot \|B\| e^{\|A\| h'} \cdot \|u\| \qquad (61)
\end{aligned}
$$

This completes the proof of Lemma 3.

We next consider the computation of $\|\mathcal{F}_{hM\mathrm{k1}}^{-}\|$ (assuming that $D = 0$). This norm is described by

$$
\|\mathcal{F}_{hM\mathrm{k1}}^{-}\| = \sup_{\|\check{u}\| \leq 1} \|\mathcal{F}_{hM\mathrm{k1}}^{-} \check{u}\| \qquad (62)
$$

where

$$
\mathcal{F}_{hM\mathrm{k1}}^{-} \check{u} = \sum_{j=1}^{M} C (A'_d)^{j-1} \mathbf{B}'_{\mathrm{k1}} u^{(j)}, \quad [(u^{(1)})^T, \ \cdots, \ (u^{(M)})^T]^T := \check{u} \qquad (63)
$$

By the definition of $\mathbf{B}'_{\mathrm{k1}}$, we can see that

$$
C (A'_d)^{j-1} \mathbf{B}'_{\mathrm{k1}} u^{(j)} = \int_0^{h'} C (A'_d)^{j-1} (I + A\theta') B u^{(j)}(\theta') d\theta' \qquad (64)
$$

Note that the integrand involves the function used in defining Y_j. Hence, by the property of $L_\infty[0, h')$ and the definition of $F_{hM\mathrm{k1}}^{-}$, it follows that $\|\mathcal{F}_{hM\mathrm{k1}}^{-}\|$ coincides with the ∞-norm of the finite-dimensional matrix $F_{hM\mathrm{k1}}^{-}$ given by (48) with D removed. Then, the assertion of Lemma 4 for the case $D \neq 0$ follows immediately again by the property of $L_\infty[0, h')$, as has been the case with the input approximation arguments in [11].