Global Attractor for mKdV Equation on $1D$ Torus

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Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements.

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Abstract

The modified Kortweg-de Vries equation (for short, mKdV) models the propagation of nonlinear water waves in the shallow water approximation. We consider the weakly damped and forced mKdV equation under the periodic boundary condition. We prove the existence of the global attractor in H^s, $s > 11/12$ for the weakly damped and forced mKdV on the one dimensional torus. To see the asymptomatic behavior of the solutions of mKdV equation below energy space, the study of global attractor below energy space is important. The existence of global attractor below the energy space has not been known, though the global well-posedness below the energy space is established. We directly apply the I-method to the damped and forced mKdV, because the Miura transformation does not work for the mKdV with damping and forcing terms. We need to make a close investigation into the trilinear estimates involving resonant frequencies, which are different from the bilinear estimates corresponding to the KdV.

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Chapter 1

Introduction

The increase of interest in turbulence and chaos has motivated for new mathematical mechanics and some new concepts like attractors, fractal sets, Feigenbaum cascades etc. This work mainly concentrate on attractors and global attractors.

In last three decades, theory of global attractor has been develop dramatically on semi-groups for infinite dimensional dynamic systems. The dynamic systems arising in the biology, physics or chemistry are often generated by a partial differential equation and therefore, have infinite dimensional underling space. Usually these systems are either conservative or exhibit some dissipation. We can hope to reduce the study to a bounded or compact attracting set (or a global attractor) that contains enough information of the flow and sometimes has the finite diminution character also.

The global attractor of a dynamical system is the unique compact invariant set that attracts the trajectories starting in any bounded set at a uniform rate. A global attractor plays an important role in the study of the behaviour of solution as time goes to infinity.

The main result in this work is about finding the global attractor for weakly damped and forced modified Kortteweg-de Vries equation. In chapter three of our work, we will discuss this problem in details.

We divided this work into three chapters. First chapter include the introduction to global attractor and their basic properties. The most part of this chapter is from the book by Roger Temam [**?**]. For more details on stable and unstable orbits please see Guckenheimer and Holmes [14], for absorbing sets see J.E.Billoti and J.P.La Salle [**?**] and for details on the main existence theorem for global attractor see F.Abergel [1] and in O.A. Ladyzhenskaya [17]. The second chapter is divided into two sections namely the Soblev spaces and Besov spaces. This introductory part of functional spaces is from the book by H. Bahouri, J-Y Chemin and R. Danchin [2]. For inside details on the references for this book please see [2, page 49-50 for Sobolev spaces and page 120-121 for Besov spaces].

This chapter is dedicated to the introduction of global attractor on semigroup. In first section, we give the definition of a global attractor and state the necessary and sufficient condition for the existence of a global attractor. More precisely, we discuss about the absorbing set, invariant set , an attractor and the global attractor. The second section describe the impotence of a global attractor. We describe how the study of global attractor below energy space is useful. Although, showing the existence of a global attractor below energy space is not that easy. We list some of the difficulties in showing the existence of global attractor below energy space and possible way of handling such issues.

1.1 What is the Global Attractor?

In this section, we define the global attractor on a semi-group which is generally defined by the solutions of an ordinary differential equations(ODE) or partial differential equations(PDE). We consider the dynamical system whose state is described by an element $u = u(t)$ on a metric space H where time t varies over $\mathbb{R}$ or on some interval of $\mathbb{R}$. Usually, H is either the Banach or Hilbert space but for the present chapter, we just consider a metric space.

1.1.1 Notations and Definition

We start this subsection with the definition of semigroup formed by the evolution of dynamical system:

Definition 1.1.1. *The solution of the dynamical system is described by the family of operators $(S(t))_{t\geqslant 0}$, that map H into itself and satisfies the usual semigroup properties.*

$$\begin{cases} S(s+t) = S(s)\cdot S(t) \quad \forall s,t \geqslant 0 \\ S(0) = I \quad \textit{Identity in } H. \end{cases} \tag{1.1.1}$$

Remark 1.1.2. *If f is the state of the dynamical system at time s, then $S(s)f$ is the state at time $s+t$ and*

$$u(t) = S(t)u(0), \tag{1.1.2}$$

$$u(t+s) = S(t)u(s) = S(s)u(t), \quad s,t \geqslant 0. \tag{1.1.3}$$

Remark 1.1.3. *In general, we consider the semigroup generated by the solutions of ODE and PDE's. In case of ODE, the general theorem for existence and uniqueness of the solution provides the definition of $S(t)$ but in the case of infinite dimension, we first investigate the existence and uniqueness theorem as there is no general theorem exists to study the dynamical system.*

We atleast assume that $\forall\ t \geqslant 0$, the operators $S(t)$ are continuous from H into itself. In general, the operators $S(t)$ are not injective as the injectivity property for $S(t)$ is equivalent to backward uniqueness of the dynamical system. Nevertheless, if $S(t)$ are one-one for $t > 0$, we define the operators $S(-t)$ as the inverse of $S(t)$ which maps from H to H. Now, we give few definitions as follows:

Definition 1.1.4. *For $u_0 \in H$, the orbit or trajectories starting at u_0 is the set $\bigcup_{t\geqslant 0} S(t)u_0$. It is also known as positive orbits through u_0.*

Definition 1.1.5. *For $u_0 \in H$, the orbit or trajectories ending at u_0 is the set $\bigcup_{t\geqslant 0} S(-t)^{-1}u_0$. We are assuming here that the orbit or trajectories ending at u_0 exist. It is also know as negative orbit through u_0.*

A complete orbit containing u_0 is the union of positive and negative orbit through u_0. Now, we define the $\omega-$limit set.

Definition 1.1.6. *For $u_0 \in H$ or $\mathcal{A} \subset H$, we define the ω-limit set of u_0 (or $\mathcal{A}$), as*

$$\omega(u_0) = \bigcap_{s\geqslant 0} \overline{\bigcup_{t\geqslant s} S(t)u_0}$$

or

$$\omega(\mathcal{A}) = \bigcap_{s\geqslant 0} \overline{\bigcup_{t\geqslant s} S(t)\mathcal{A}},$$

where the closers are taken over H.

We can define the similar ω-limit set for $t < 0$ but from now on we skip the results related to $t < 0$ as it is not relevant for our work. Let us state the following proposition:

Proposition 1.1.7. *$f \in \omega(\mathcal{A})$ if and only if there exists a sequence of elements $f_n \in \mathcal{A}$ and a sequence $t_n \to +\infty$ such that*

$$S(t_n)f_n \to f \quad \text{as } n \to \infty.$$

Definition 1.1.8. *A fixed point, or a stationary point, or an equilibrium point is a point $u_0 \in H$ such that*

$$S(t)u_0 = u_0 \quad \forall t \geqslant 0.$$

Clearly, the orbit and the ω-limit set for such point is $\{u_0\}$.

If u_0 is the stationary point, then we have the following definitions:

Definition 1.1.9. *The stable manifold of u_0, $\mathcal{M}_-(u_0)$ is the set of points v which belongs to the complete orbit $\{u(t), t \in \mathbb{R}\}, v = u(t_0)$ and such that*

$$u(t) = S(t - t_0)v \to u_0 \quad \textit{as } t \to \infty.$$

Definition 1.1.10. *The unstable manifold of u_0, $\mathcal{M}_+(u_0)$ is the set of points v which belongs to the complete orbit $\{u(t), t \in \mathbb{R}\}$ and such that*

$$u(t) \to u_0 \quad t \to -\infty.$$

Stable or unstable manifolds can be empty set. A stationary point u_0 is stable if $\mathcal{M}_+(u_0) = \emptyset$ and unstable otherwise. We also skip the details for discrete case. Although, discrete case are also similar to continuous case. Now, we will discus about the invariant sets:

Definition 1.1.11. *We say that a set A is positively invariant for the semigroup $S(t)$ if*

$$S(t)A \subset A \quad \forall t > 0$$

and negatively invariant if

$$S(t)A \supset A \quad \forall t > 0.$$

A set which is both positively and negatively invariant is known as the invariant set or a functional invariant set i.e.

$$S(t)A = A \quad \forall t \geqslant 0.$$

Examples 1.1.12.

- *If a set A contains a fixed point or a union of fixed points, then it is a trivial example of invariant set.*

- *If it exists, a time periodic orbit is an invariant set. In fact, if for some $u_0 \in H, T > 0$ and $S(T)u_0 = u_0$, then $S(t)u_0$ exists for all $t \in \mathbb{R}$ and*

  $$A = \{S(t)u_0, t \in \mathbb{R}\}$$

  is invariant.

The following lemma generates some special types of invariant sets.

Lemma 1.1.13. *Let $\mathcal{A} \subset H$ and $\mathcal{A} \neq \emptyset$. Assume that for some $t_0 > 0$, the set $\bigcup_{t \geqslant t_0} S(t)\mathcal{A}$ is relatively compact in H. Then, $\omega(\mathcal{A})$ is nonempty, compact and invariant.*

For proving the main assumption that $\bigcup_{t \geqslant 0} S(t)\mathcal{A}$ is relatively compact, we need to show that it is bounded if H is finite dimensional and show that it is bounded in a space W compactly embedded in H for infinite dimension. The lemma is used especially for ω-limit set.
Now, we define one more important ingredient for the definition of global attractor.

Definition 1.1.14. *An attractor is a set $\mathcal{A} \subset H$ that has following properties:*

1. *$\mathcal{A}$ is an invariant set i.e. $S(t)\mathcal{A} = \mathcal{A}, \; t \geqslant 0$.*

2. *$\mathcal{A}$ possesses an open neighbourhood O such that, $\forall \; u_0 \in O$, $S(t)u_0$ converges to $\mathcal{A}$ as $t \to \infty$*

$$d(S(t)u_0, \mathcal{A}) \to 0 \quad \text{as } t \to \infty,$$

where d is the distance between a set and a point given as

$$d(x, \mathcal{A}) = \inf_{y \in \mathcal{A}} d(x, y).$$

Remark 1.1.15. *If $\mathcal{A}$ is an attractor, the largest open set O that satisfies (2) of Definition 1.1.14 is called the basin of the attractor.*

Definition 1.1.16. *We say that $\mathcal{A}$ uniformly attracts the set $B \subset O$ if*

$$d(S(t)B, \mathcal{A}) \to 0 \quad \text{as } t \to \infty,$$

where $d(B_1, B_2)$ is the semidistance defined as

$$d(B_1, B_2) = \sup_{x \in B_1} \inf_{y \in B_2} d(x, y).$$

In infinite dimensions to work with different topologies, we have the following definition:

Definition 1.1.17. *Let $V \subset W$. We say that $\mathcal{A}$ is an attractor in V if $\mathcal{A} \subset V, S(t)\mathcal{A} = \mathcal{A}$ and satisfies the second condition in Definition 1.1.14 with respect to the topology of V.*

Finally, we define the global attractor as follow:

Definition 1.1.18. *We say that $\mathcal{A} \subset H$ is a global attractor for the semigroup $(S(t))_{t\geqslant 0}$ if $\mathcal{A}$ is a compact attractor that attracts the bounded sets of H. The basin of this set is all of H.*

1.1.2 Existence and Uniqueness

In this subsection, we discuss about the conditions for existence of the global attractor. We start this subsection with the definition of the absorbing set:

Definition 1.1.19. *Let $B \subset H$ and O an open set containing B. We say that B is absorbing in O if the orbit of any bounded set of O enters into B after some time i.e.*

$$\begin{cases} \forall B_1 \subset O, \quad B_1 \text{ bounded} \\ \exists\ t_1(B_1) \text{ such that } S(t)B_1 \subset B \ \ \forall\ t \geqslant t_1(B_1). \end{cases}$$

We also says that B absorbs the sets of O.

Before giving the main result, let us assume the following two remarks:

Remark 1.1.20. *For every bounded set B there exists t_1 which may depend on B such that*

$$\bigcup_{t\geqslant t_1} S(t)B \tag{1.1.4}$$

is relatively compact in H. In other words, the operators $S(t)$ are relatively compact for t large.

We can also have the following similar condition:

Remark 1.1.21. *Let H is a Banach space and for every t, $S(t) = L_1(t) + L_2(t)$ where the operators $L_1(\cdot)$ are uniformly compact for t large (i.e.satisfies Equation (1.1.4)) and L_2 are continuous mapping from H into itself such that for every bounded set $A \subset H$, we have*

$$r_c(t) = \sup_{u\in A} |L_2(t)u|_H \to 0 \quad \text{as } t \to \infty. \tag{1.1.5}$$

Remark 1.1.22. *For a Banach space, any family of operator satisfying Equation (1.1.4) also satisfies Equation (1.1.5) with $L_2 = 0$.*

Let us state the main result:

Theorem 1.1.23. *Let H is a metric space. Assume that the operators $S(t)$ satisfies (1.1.1)-(1.1.3) and either (1.1.4) or (1.1.5). Also assume that there exists an open set O and a bounded set B of O such that B is absorbing in O.*

Then, the ω-limit set of B, $\mathcal{A} = \omega(B)$, is a compact attractor which attracts the bounded sets of O. For the inclusion relation, it is the maximal bounded attractor in O.

Moreover, if H is a Banach space, D is convex and the mapping $t \to S(t)u_0$ is continuous from $\mathbb{R}$ into H, for every $u_0 \in H$, then $\mathcal{A}$ is connected too.

Remark 1.1.24. *The assumption in Equation (1.1.4) can be weaker for Theorem 1.1.23 i.e. for some $t_0 > 0, S(t_0)$ is compact.*

Note that there are some weaker version of Theorem 1.1.23 exists but we only need to use the above hypothesis. With this main result, we end this section.

1.2 Why Global Attractor Is Important?

A partial differential equation or system can be written in the form

$$\partial_t u = F(u(t)), \tag{1.2.1}$$

where the operator $F(u)$ includes the partial derivatives of u with respect to spatial variable $x = (x_1, \ldots x_n)$. Dynamic system generated by Equation (1.2.1) can be studied locally and globally. The local theory is quite rich but same is not true for global theory. The long-time behaviour of solution of such system can be adequately described in terms of global attractor of the system. In many equations, the influence of initial data vanishes after a long time. Therefore, permanent regimes are of impotence. The time-independent solution of $F(u) = 0$ can be consider as one of the simplest example.

As discussed in the last section, attractor of a semigroup is the ω-limit set of a neighbourhood of the attractor, which can be called as local attractor. A dynamical system can have many local attractors for example stable periodic solution with different domain of attraction. A global attractor is the maximal operator in the sense of inclusion as it define the domain of attraction as whole Banach space H.

1.2.1 Difficulties In Finding The Global Attractor

An infinite dimensional dynamic system generated by PDE has many technical concerns which are not there in finite dimensional theory. We list few of them as follow:

- Most of the times, the semigroup $S(t)$ is only defined for $t > 0$ and can not be extended for $t \in \mathbb{R}$.
- Infinite dimensional function spaces are not locally compact.

- Solution with bounded energy can blow up in a finite time.
- Uniqueness of the solution may be difficult to establish (3D Navier-Stokes system).
- Expression of characteristics of attractors in terms of physical parameters of the problem.
- Interconnection of spatial properties of solution and there dynamical properties.

We should note that there are obvious similarities between finite and infinite systems. For instant, constructing ω-limit set as global attractor works well for both type of systems.

Hence, the global attractor is a key concept to study the behaviour of dynamical systems mainly infinite dimensional. As listed above, it is not easy to study about infinite systems specially globally. But still there are few techniques which makes our job much easer like a global attractor.

1.2.2 Global Attractor Below Energy Space

It is quite difficult to find a global attractor below energy space as the global attractor is much more than showing the global existence. Indeed, we need to find two operators L_1 and L_2 which satisfies the hypothesis of Theorem 1.1.23. Our main work concentrates on weakly damped and forced modified Korteweg-de Vries equation(mKdV). For mKdV equation, global well-posedness is known in Sobolev space H^s for $s > 1/2$. But the existence of a global attractor is not known. Hence, it seems to be intresting problem to consider the global attractor below energy space. To see the asymptomatic behavior of the solution of mKdV equation below energy space, the study of global attractor below energy space is important.

Chapter 2

Functional Spaces

This chapter is devoted to some functional spaces and their basic properties which will be used throughout this work. These functional space are quite important for the existence and uniqueness of the solution associated to the non-linear partial differential equations. this chapter consist of mainly two sections.

The first section contains a brief introduction on Sobolev spaces. In first part of this section, we introduce Fourier transform on $\mathbb{R}^n$. In the second part, we give some basic inequalities of Real analysis mainly we state the Minkowski's and Hölder's inequality which will be used throughout this work. Then we state convolution inequalities on locally compact group equipped with left-invariant Haar measure.Further, we establish bilinear interpolation-type inequality based on atomic decomposition. At the end, we state few properties of Hardy-Littlewood maximal operator.

The third and forth part of this section contains a brief introduction to homogeneous and nonhomogeneous Sobolev spaces, respectively. In the third part of this section, we give some basic properties of homogeneous Sobolev space. We present embeddings in some space spaces like Lebesgue spaces and bounded mean oscillation spaces. We also state embeddings in Hölder spaces. At the end of third part, we give some refined Sobolev inequalities which are invariant by translation and dilation. In the last part of first section, we mainly concentrate on nonhomogeneous spaces. Trace theorems and compact embedding are some of the main results, we discuss there. We also emphasis on Moser-Trudinger and Hardy inequalities.

We can deal with the functions and distributions easily if they can be split into countable sum of smooth functions with compactly supported Fourier transform. Littlewood-Paley theory provides such a decomposition. The first subsection of second section contains the Bernstein inequality. Then we study the action of heat flow or of a diffeomorphism over spectrally localised functions. The second subsection is devoted to homogeneous Besov spaces with an introduction of Littlewood-Paley decomposition. Subsection three is devoted to paradifferential calculus and its basic properties. Last subsection consist of nonhomogeneous Besov spaces, paralinearization theorem and compact properties of Besov spaces.

2.1 Sobolev Spaces

In this section, we study the basic concepts of real analysis, Sobolev spaces used in the theory of nonlinear partial differential equations.

2.1.1 Fourier Transform on $\mathbb{R}^n$

In this subsection, we discuss about the Fourier transform on $\mathbb{R}^n$ and some basic properties of it. We start this subsection with the definition of Fourier transform.

Definition 2.1.1. *Let $f \in L^1(\mathbb{R}^n)$. Then Fourier transform of f is defined as*

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i(x|\xi)} f(x)dx, \tag{2.1.1}$$

where $(x|\xi)$ denotes the inner product on $\mathbb{R}^n$. We can see that $|\hat{f}(\xi)| \leqslant \|f\|_{L^1}$ which implies that $\mathcal{F}: L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ is a continuous mapping. Now we define Schwartz functions denoted as $\mathcal{S}$ with the help of following notations.

Let f be a function on $\mathbb{R}^n$ and α be a multi-index. Suppose that $x \in \mathbb{R}^n$. Then length of α is defined as $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Also, define $\partial^\alpha f = \partial_1^{\alpha_1} f \cdots \partial_n^{\alpha_n} f$ and $x^\alpha = x^{\alpha_1} \cdots x^{\alpha_n}$.

Definition 2.1.2. *The set of smooth functions u on $\mathbb{R}^n$ is said to be Schwartz space $\mathcal{S}(\mathbb{R}^n)$ if for any $j \in \mathbb{N}$ we have*

$$\|u\|_{j,\mathcal{S}} = \sup_{\substack{|\alpha| \leqslant j \\ x \in \mathbb{R}^n}} (1+|x|)^j |\partial^\alpha u(x)| < \infty.$$

The following theorem explain how the Fourier transform act on Schwartz functions.

Theorem 2.1.3. *For any integer j there exist a constant C and an integer k such that*

$$\forall\ u \in \mathcal{S},\ \ \|\hat{u}\|_{j,\mathcal{S}} \leqslant C\|u\|_{k,\mathcal{S}}.$$

Therefore, Fourier transform is a continuous map on $\mathcal{S}$. Also, $\mathcal{F}$ is an automorphism on $\mathcal{S}$ with inverse $(2\pi)^{-n}\check{\mathcal{F}}$, where $\check{\mathcal{F}}$ denotes $f \to \{\xi \to (\mathcal{F}f)(-\xi)\}$.

Now, we define the tempered distribution and the relation between Fourier transform and distribution.

Definition 2.1.4. *Any continuous linear functional on $\mathcal{S}(\mathbb{R}^n)$ is known as tempered distribution on $\mathbb{R}^n$. The set of tempered distribution is denoted as $\mathcal{S}'(\mathbb{R}^n)$*

Definition 2.1.5. *We say that a sequence $(\phi_n)_{n\in\mathbb{N}}$ of tempered distributions is said to converge to ϕ in $\mathcal{S}'(\mathbb{R}^n)$ if*

$$\forall\ \varphi \in \mathbb{S}(\mathbb{R}^n), \quad \lim_{n\to\infty} \langle \phi_n, \varphi\rangle = \langle \phi, \varphi\rangle.$$

We give a proposition related to the duality.

Proposition 2.1.6. *Let $T : \mathcal{S} \to \mathcal{S}$ is a linear continuous map. Then*

$$\langle {}^tT\phi, \varphi\rangle = \langle \phi, T\varphi\rangle$$

defines a tempered distribution. Moreover, if $(u_n)_{n\in\mathbb{N}}$ is a sequence of distributions which converges to u in $\mathcal{S}'(\mathbb{R}^n)$ then $({}^tTu_n)_{n\in\mathbb{N}}$ converges to ${}^tTu$. Therefore, ${}^tT$ is a continuous and linear map.

Let us list some examples which are consequences of Proposition 2.1.6.

Remark 2.1.7. *Consider an operator $(-\partial)^\alpha$ for some multi-index α. Then for all $\varphi \in \mathcal{S}$, we have*

$$\|(-\partial)^\alpha \varphi\|_{j,\mathcal{S}} \leqslant \|\varphi\|_{j+|\alpha|,\mathcal{S}}.$$

Also, we have the same assertion for $x^\alpha \to x^\alpha\varphi$ i.e.

$$\|x^\alpha \varphi\|_{j,\mathcal{S}} \leqslant \|\varphi\|_{j+|\alpha|,\mathcal{S}}.$$

Remark 2.1.8. *Define*

$$T_A\varphi = \frac{1}{\det A}\ \varphi \circ A^{-1},$$

where A is a linear automorphism on $\mathbb{R}^n$. Clearly, T_L satisfies Proposition 2.1.6.

Remark 2.1.9. *For any $\varphi \in \mathcal{S}$, we have*

$$\|T_\theta \varphi\|_{j,\mathcal{S}} \leqslant C_j \|\theta\|_{j+n+1,\mathcal{S}} \|\varphi\|_{j,\mathcal{S}},$$

wher $\theta \in \mathcal{S}(\mathbb{R}^n)$ *and* $T_\theta \varphi = \check{\theta} * \varphi$.

Remark 2.1.10. *Let* θ_M *be the space of smooth functions on* $\mathbb{R}^n$ *such that, for any integer* j*, an integer* N *exists such that*

$$\sup_{x\in\mathbb{R}^n}(1+|x|^j)^{-N}\sup_{|\alpha|\leqslant j}|\partial^\alpha f(x)|<\infty,$$

then the multiplication by f*, the operator* T_f *satisfies the hypothesis of Proposition 2.1.6.*

Definition 2.1.11. *A tempered distribution* g *is said to be homogeneous of degree* n *if*

$$g_\lambda = \lambda^n g \qquad \forall\, \lambda > 0.$$

Remark 2.1.12. *We can easily see that the notation of convolution for distribution is same that of the Schwartz class* $\mathcal{S}$ *if it is in* L^1.

We list the properties of Fourier transform on $\mathcal{S}$ as the following proposition.

Proposition 2.1.13. *Let* $(u,v)\in\mathcal{S}'\times\mathcal{S},\ \lambda\in\mathbb{R}\setminus\{0\}$ *and* $(\alpha,\beta)\in\mathbb{R}^n\times\mathbb{R}^n$*. Then, we have*

$$\begin{aligned}(i\partial)^\alpha\hat{u} &= \mathcal{F}(x^\alpha u), \qquad (i\xi)^\alpha\hat{u} = \mathcal{F}(\partial^\alpha u),\\ e^{-i(\xi|\alpha)}\hat{u} &= \mathcal{F}(\tau_\alpha f), \qquad \tau_\beta\hat{u} = \mathcal{F}(e^{i(x|\beta)}u),\\ \lambda^{-n}\hat{u}(\lambda^{-1}\xi) &= \mathcal{F}(u(\lambda x)), \qquad \mathcal{F}(u*v) = \hat{v}\hat{u},\end{aligned}$$

where τ_α *stands for the translation operator.*

Now, we state the Plancherel's theorem as follow:

Theorem 2.1.14. *The Fourier transform is an automorphism of* $\mathcal{S}'$ *with inverse* $(2\pi)^{-n}\check{\mathcal{F}}$*. Moreover,* $\mathcal{F}$ *is an automorphism on* $L^2(\mathbb{R}^n)$ *and satisfies* $\|\hat{u}\|_{L^2} = (2\pi)^{n/2}\|u\|_{L^2}$.

We define the following subspace of $\mathcal{S}'(\mathbb{R}^n)$ which is quite useful for us.

Definition 2.1.15. *The space of tempered distributions* u *such that*

$$\lim_{\lambda\to\infty}\|v(\lambda D)u\|_{L^\infty} = 0 \quad \text{for any } v\in\mathcal{D}(\mathbb{R}^n), \tag{2.1.2}$$

is denoted by $\mathcal{S}'_h(\mathbb{R}^n)$*. Here,* $\mathcal{D}(\mathbb{R}^n)$ *denotes the space of smooth compactly supported functions on* $\mathbb{R}^n$.

We can easily get the following equivalent condition for the distributions in $\mathcal{S}'_h(\mathbb{R}^n)$.

Proposition 2.1.16. *$u \in \mathcal{S}'_h(\mathbb{R}^n)$ if and only if there exists some smooth compactly supported function v satisfying the inequality (2.1.2) and $v(0) \neq 0$.*

For better understanding of the space $\mathcal{S}'_h(\mathbb{R}^n)$, we list some of the examples:

Examples 2.1.17.

- *If the Fourier transform of a distribution u is locally integrable near 0, then $u \in \mathcal{S}'_h(\mathbb{R}^n)$. Indeed, the space $\mathcal{E}'$ of compactly supported distributions is included in $\mathcal{S}'_h(\mathbb{R}^n)$.*
- *If u is a tempered distribution such that $v(D)u \in L^p$ for some $p \in [1, \infty)$ and some function v in $\mathcal{D}(\mathbb{R}^n)$ with $v(0) \neq 0$, then $u \in \mathcal{S}'_h(\mathbb{R}^n)$.*
- *A nonzero polynomial P does not belongs to $\mathcal{S}'_h(\mathbb{R}^n)$. This implies that $\mathcal{S}'_h(\mathbb{R}^n)$ is not a close subspace of $\mathcal{S}'$ in weak topology.*

We give Fourier transform for some functions which are not in L^1.

Proposition 2.1.18. *Let w be a nonzero complex number with nonnegative real part. Then*

$$\mathcal{F}\left(e^{-w|\cdot|^2}\right)(\xi) = \left(\frac{\pi}{w}\right)^{\frac{n}{2}} e^{-\frac{|\xi|}{4w}},$$

where $w^{-\frac{n}{2}} = |w|^{-\frac{n}{2}} e^{-i\frac{n}{2}\theta}$ with $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Proposition 2.1.19. *Let $s \in)0, n($. Then $\mathcal{F}(|\cdot|^{-s}) = C_{n,s}|\cdot|^{s-n}$ for some constant $C_{n,s}$ depending only on s and n.*

To end this part of first section, let us state the following lemma:

Lemma 2.1.20. *Let A be a distribution on $\mathbb{R}^n$ supported in $\{0\}$. Suppose that $PA = bA$ for some real number b, where $P = \sum_{j=1}^{>n} x_j \partial_j$. Then, we have the following results:*

1. *If b is not an integer less than or equal to $-n$, Then $A = 0$.*
2. *If b is an integer less than or equal to $-n$,, then there exists some real numbers c_α such that*

$$A = \sum_{|\alpha| = -b-n} c_\alpha \partial^\alpha \delta_0$$

2.1.2 Basic Real Analysis

In this section, we state very basic results of analysis which are very essential for our work. Let us start this subsection with Hölder's inequality.

Proposition 2.1.21. ***Hölder Inequality :*** *Let (X,μ) be a measure space and $(p,q,r) \in [1,\infty]^3$ such that*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

Let $(u,v) \in L^p(X,\mu) \times L^q(X,\mu)$, then $uv \in L^r(X,\mu)$ and

$$\|uv\|_{L^r} \leqslant \|u\|_{L^p}\|v\|_{L^q}.$$

Now, let us state the following lemma:

Lemma 2.1.22. *Let (X,μ) be a measure space and $p \in [1,\infty]$. Suppose that u be a measurable function. If*

$$\sup_{\|v\|_{L^{p'}} \leqslant 1} \int_X |u(x)v(x)| d\mu(x) < \infty,$$

then $u \in L^p$ and

$$\|u\|_{L^p} \leqslant \sup_{\|v\|_{L^{p'}} \leqslant 1} \int_X u(x)v(x) d\mu(x),$$

where p' denotes the conjugate exponent of p defined as

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Proposition 2.1.23. ***Minkowski's Inequality :*** *Let (X,μ) and (Y,ν) be two measure spaces and u be a nonzero measurable function on $X \times Y$. For all $1 \leqslant p \leqslant q \leqslant \infty$, we have*

$$\left\| \|u(\cdot,x_2)\|_{L^p(X,\mu)} \right\|_{L^q(Y,\nu)} \leqslant \left\| \|u(x_1,\cdot)\|_{L^q(Y,\nu)} \right\|_{L^p(X,\mu)}.$$

Now, we give the definition of convolution between two measurable functions defined on some locally compact topological group G equipped with a left-invariant Haar measure ν.

Definition 2.1.24. *Let u and v are two measurable functions on a locally compact topological group G equipped with a left-invariant Haar measure ν. Then*

$$u * v(x) = \int_G u(y)v(y^{-1} \cdot x) d\nu(y).$$

Lemma 2.1.25. ***Young's Inequality:*** *Le G be a locally compact topological group equipped with a left-invariant Haar measure ν. If ν satisfies*

$$\nu(P^{-1}) = \nu(P) \quad \text{for any Borel set } P,$$

then for all $(p,q,r) \in [1,\infty]^3$ with

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1 \tag{2.1.3}$$

and for any $(u,v) \in L^p(G,\nu) \times L^q(G,\nu)$, we have

$$u * v \in L^r(G,\nu) \quad \text{and } \|u * v\|_{L^r(G,\nu)} \leqslant \|u\|_{L^p(G,\nu)}\|v\|_{L^q(G,\nu)}.$$

We now state the refined Yong's inequality.

Theorem 2.1.26. ***Refined Young's Inequality:*** *Let (G,ν) satisfies the same assertion as in Lemma 2.1.25. Suppose that $(p,q,r) \in [1,\infty]^3$ and satisfies (2.1.3). For any $u \in L^p(G,\nu)$ and any measurable function v on G where*

$$\|v\|^q_{L^q_w(G,\nu)} = \sup_{\lambda>0} \lambda^q \nu(|g| > \lambda) < \infty.$$

Then, there exists a constant C such that the function $u * v \in L^r(G,\nu)$, and

$$\|u * v\|_{L^r(G,\nu)} \leqslant C\|u\|_{L^p(G,\nu)}\|v\|_{L^q(G,\nu)}.$$

Theorem 2.1.26 implies the well known Hardy-Littlewood-Sobolev inequality on $\mathbb{R}^{\ltimes}$.

Theorem 2.1.27. ***Hardy-Little-wood-Sobolev Inequality:*** *Let $s \in [0,n]$ and $(p,r) \in [1,\infty]^2$ satisfies*

$$\frac{1}{p} + \frac{s}{n} = 1 + \frac{1}{r}.$$

Then, there exists a constant C such that

$$\||\cdot|^{-s} * u\|_{L^r(\mathbb{R}^n)} \leqslant \|u\|_{L^p(\mathbb{R}^n)}.$$

Proposition 2.1.28. ***Atomic Decomposition:*** *Let (X,ν) be a measure space and $p \in [1,\infty]$. Suppose that $u \in L^p$. Then, there exists a sequence of positive real numbers*

$(a_n)_{n\in\mathbb{Z}}$ and a sequence of nonnagetive functions $(u_n)_{k\in\mathbb{Z}}$ such that

$$u = \sum_{n\in\mathbb{Z}} a_n u_n,$$

where the functions u_n have pairwise disjoint supports and

$$\begin{aligned} \nu(Supp\ u_n) &\leqslant 2^{n+1} \\ \|u_n\|_{L^\infty} &\leqslant 2^{-\frac{n}{p}} \\ \frac{1}{2}\|u\|_{L^p}^p \leqslant \sum_{n\in\mathbb{Z}} a_n^p &\leqslant 2\|u\|_{L^p}^p. \end{aligned}$$

Proposition 2.1.28 describe the atomic decomposition of a function $u \in L^p$ which help us to prove Theorem 2.1.26. We state another application of atomic decomposition namely bilinear interpolation theorem which is very useful in our study.

Theorem 2.1.29. ***Bilinear Interpolation Theorem:*** *Let (X,μ) and (Y,ν) are two measure spaces. Let A be a continuous bilinear functional on $L^2(X;L^{p_j}(Y)) \times L^2(X;L^{q_j}(Y))$ for $j \in \{0,1\}$ and $(p_j,q_j) \in [1,2]^2$ and such that $p_0 \neq p_1$ and $q_0 \neq q_1$. For any $\theta \in [0,1]$, the bilinear functional A is then continuous on $L^2(X;L^{p_\theta}(Y)) \times L^2(X;L^{q_\theta}(Y))$, where*

$$\left(\frac{1}{p_\theta},\frac{1}{q_\theta}\right) = (1-\theta)\left(\frac{1}{p_0},\frac{1}{q_0}\right) + \theta\left(\frac{1}{p_1},\frac{1}{q_1}\right).$$

Definition 2.1.30. *Let (X,d) be a metric space endowed with the Borel measure ν. Indeed, if $u : X \to \mathbb{R}$ is in $L^1_{loc}(X,\nu)$, then we define*

$$\forall x \in X,\ \ Mf(x) = \sup_{\tau>0} \frac{1}{\nu(B(x,\tau))}\int_{B(x,\tau)} |u(y)|d\nu(y).$$

Then, M is known as the maximal function.

The following lemma is useful for the proof of Theorem 2.1.32.

Lemma 2.1.31. *Let (X,ν) be a metric space with the Borel measure ν with the doubling property. Then there exists a constant C_1 such that for any family $(B_i)_{1\leqslant i\leqslant n}$ of balls, there exists a subfamily $(B_{i_j})_{1\leqslant j\leqslant n}$ of pairwise disjoint balls such that*

$$\nu\left(\bigcup_{j=1}^{p} B_{i_j}\right) \geqslant C_1\nu\left(\bigcup_{i=1}^{n} B_i\right).$$

We state the following well-known fundamental result about the maximal function.

Theorem 2.1.32. *Let (X, d) be the metric space with the measure ν has doubling property. Then, there exists a constant C depending only on the doubling constant B such that $\forall\ 1 < p \leqslant \infty$ and $u \in L^p(X, \nu)$, we have $Mf \in L^p(X, \nu)$ and*

$$\|Mf\|_{L^p} \leqslant \frac{p}{p-1} C^{\frac{1}{p}} \|u\|_{L^p}.$$

For the proof of Gagliardo-Nirenberg inequalities, we list the following results:

Proposition 2.1.33. *Let G be a locally compact group with neutral element e, with a distance d such that $d(e, y^{-1}.x) = d(x, y)\ \forall\ (x, y) \in G^2$ and a left-invariant Haar measure satisfies $\nu(B^{-1}) = \nu(B)$ for any Borel set B. Also, for $r > 0$, there exists a positive measure μ, on the sphere $\sum_r = \{x \in G \setminus d(e, x) = r\}$ such that for any L^2 function v on G, we have*

$$\int_G v(x) d\nu(x) = \int_0^{+\infty} \left(\int_{\sum_r} v(x) d\mu_r(x) \right) dr.$$

For all measurable functions w and any L^1 function H on G such that

$$\forall x \in G, H(x) = h(d(e, x))$$

for some nonincreasing function $h : \mathbb{R}^+ \to \mathbb{R}^+$, we then have

$$\forall x \in G, |H * f(x)| \leqslant \|H\|_{L^1(G,\nu)} Mf(x).$$

2.1.3 Homogeneous Sobolev Spaces

This subsection consist of introduction to homogeneous Sobolev spaces and there embedding into Lebesgue, BMO and Hölder spaces. The homogeneous Sobolev spaces plays a vital role in study of nonlinear partial differential equations. Let us start this section with the definition and some basic properties of homogeneous Sobolev spaces.

Definition 2.1.34. *Let $s \in \mathbb{R}$. The homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^n)$ is the space of tempered distributions u over R^n such that $\hat{u} \in L^1_{loc}(\mathbb{R}^n)$ and satisfies*

$$\|u\|^2_{\dot{H}^s} = \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi < \infty.$$

The following proposition explain a kind of interpolation of homogeneous Sobolev spaces:

Proposition 2.1.35. *Let $s_1 \leqslant s \leqslant s_2$. Then, $\dot{H}^{s_1} \cap \dot{H}^{s_2}$ included in $\dot{H}^s$ and for $\theta \in [0,1]$, we have*

$$\|u\|_{\dot{H}^s} \leqslant C\|u\|_{\dot{H}^{s_1}}^{1-\theta}\|u\|_{\dot{H}^{s_2}}^{\theta},$$

where $s = s_1(1-\theta) + s_2\theta$.

Proposition 2.1.36. *Let l be a positive integer. Then, the space $\dot{H}^{-l}(\mathbb{R}^m)$ consist of distributions which are the sums of derivatives of order l of $L^2(\mathbb{R}^n)$ functions.*

Remark 2.1.37. *Proposition 2.1.36, describe the homogeneous Sobolev spaces with negative index. For the positive index, $\dot{H}^s$ is the subset of distributions with locally integrable Fourier transforms such that $\partial^\alpha u \in L^2(\mathbb{R}^n)$ for all multi-index α with length s. We observe that $\dot{H}^0 = L^2$ from Plancherel theorem.*

The following proposition explain the necessary and sufficient property for $\dot{H}^s$ to be Hilbert space:

Proposition 2.1.38. *$\dot{H}^s(\mathbb{R}^n)$ is a Hilbert space if and only if $s < \frac{n}{2}$.*

We define the space $\mathcal{S}_0(\mathbb{R}^n)$ as the collection of functions of $\mathcal{S}(\mathbb{R}^n)$ whose Fourier transform vanish near origin. Based on $\mathcal{S}_0(\mathbb{R}^n)$, we have the following proposition.

Proposition 2.1.39. *$\mathcal{S}_0(\mathbb{R}^n)$ is dense in $\dot{H}^s$ for $s < \frac{n}{2}$.*

Proposition 2.1.40. *Let $s < \frac{n}{2}$. Then the bilinear functional*

$$\mathcal{T} = \begin{cases} \mathcal{S}_0 \times \mathcal{S}_0 \to \mathbb{C} \\ (u,v) \to \int_{\mathbb{R}^n} u(x)v(x)dx \end{cases}$$

can be extended to a bilinear continuous functional on $\dot{H}^s \times \dot{H}^{-s}$. Moreover, if B is a continuous linear functional on $\dot{H}^s$, then there exists a unique distribution v in $\dot{H}^{-s}$ such that

$$\forall u \in \dot{H}^s, \langle B, u\rangle = \mathcal{T}(v,u) \quad \text{and} \quad \|B\|_{(\dot{H}^s)'} = \|v\|_{\dot{H}^{-s}}.$$

In fact, Proposition 2.1.67 states that $\dot{H}^{-s}$ can be considered as dual of $\dot{H}^s$. The following proposition explain that we can describe $\dot{H}^s$ as the finite difference for $s \in (0,1)$.

Proposition 2.1.41. *Let $s \in (0,1)$ and $u \in \dot{H}^s(\mathbb{R}^n)$. Then, $\dot{H}^s$ and*

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x+y) - u(y)|^2}{|y|^{n+2s}} dxdy < \infty.$$

Moreover, there exists a constant C_s such that for any function $u \in \dot{H}^s$, we have

$$\|u\|_{\dot{H}^s} \leqslant C_s \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x+y)-u(y)|^2}{|y|^{n+2s}} dxdy.$$

With Proposition 2.1.41, introduction to $\dot{H}^s$ is completed. Now we state the embeddings of $\dot{H}^s$ spaces in $L^p(\mathbb{R}^{n\cdot})$. We begin with the following well known result:

Proposition 2.1.42. *If $s \in [0, n/2)$, then the space $\dot{H}^s$ is continuously embedded in $L^{\frac{2n}{n-2s}}(\mathbb{R}^n)$.*

Proposition 2.1.43. *Let $p \in (1, 2]$, then $L^p(\mathbb{R}^n)$ is continuously embedded in $\dot{H}^s$ for $s = \frac{n}{2} - \frac{n}{p}$.*

In order to make Proposition 2.1.42 invariant under multiplication by any character $e^i(x|w)$, we shall construct a family of Banach spaces B_s whose norm is invariant under translation and satisfies

$$\|f(\lambda \cdot)\|_{B_s} \sim \lambda^{s-n/2} \|f\|_{B_s}, \;\; g\|f(\lambda \cdot)\|_{B_s} \leqslant C_{s,n} \lambda^{s-n/2} \|f\|_{\dot{H}^s},$$

and for some real number $\theta \in (0, 1)$, we have

$$\|f\|_{L^p} \leqslant C_{s,n} \|f\|_{\dot{H}^s}^{1-\theta} \|f\|_{B_s}^{\theta}.$$

To achieve our goal, let us introduce the following space:

Definition 2.1.44. *Let $\eta \in \mathcal{S}(\mathbb{R}^n)$ such that $\hat{\theta}$ is compactly supported, $0 \leqslant \hat{\eta} \leqslant 1$ and has value 1 near 0. For $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\sigma > 0$, we define*

$$\|u\|_{\dot{B}^{-\sigma}} = \sup_{A>0} A^{n-\sigma} \|\eta(A\cdot) * u\|_{L^\infty}.$$

Remark 2.1.45. *If $u \in \dot{B}^{-\sigma}$ such that $\|u\|_{\dot{B}^{-\sigma}}$ is finite, then $\dot{B}^{-\sigma}$ is a Banach space. Moreover, changing the value of η produces an equivalent norm.*

We give the following relation between $\dot{B}^{\sigma}$ and $\dot{H}^s$.

Proposition 2.1.46. *Let $s < \frac{n}{2}$. The space $\dot{H}^s$ is continuously embedded in $\dot{B}^{s-\frac{n}{2}}$. Moreover, there exists a constant only depending on Supp of $\hat{\eta}$ and n such that*

$$\|u\|_{\dot{B}^{s-\frac{n}{2}}} \leqslant \frac{C}{(\frac{n}{2}-s)^{\frac{1}{2}}} \|u\|_{\dot{H}^s} \quad \forall u \in \dot{H}^s.$$

Proposition 2.1.47. *Let $\sigma \in (0, , n]$ and $(\phi_\epsilon)_{\epsilon>0}(x) = e^{i\frac{x_1}{\epsilon}}\phi(x)$. Then, there exists a constant C such that $\|\phi_\epsilon\|_{\dot{B}^{-\sigma}} \leqslant C\epsilon^\sigma\ \forall \epsilon > 0$.*

Now we state the refined Sobolev inequality.

Theorem 2.1.48. *Let $s \in (0, \frac{n}{2})$. There exists a constant depending on n and $\hat{\eta}$ such that*

$$\|u\|_{L^p} \leqslant \frac{C}{(p-2)^{\frac{1}{p}}}\|u\|^{1-\frac{2}{p}}_{\dot{B}^{s-\frac{n}{2}}}\|u\|^{\frac{2}{p}}_{\dot{H}^s},$$

where $p = \frac{2n}{n-2s}$.

Remark 2.1.49. *Let $s \in (0, \frac{n}{2})$. Then from Proposition 2.1.46 and Theorem 2.1.48 and for any $u \in \dot{H}^s$, we have*

$$\|u\|_{L^p} \leqslant \frac{C}{\sqrt{p-2}}\|u\|_{\dot{H}^s},$$

where $p = \frac{2n}{n-2s}$.

Definition 2.1.50. *Define $Q = [-1/2, 1/2]^n$ and let $x_I = 3/8I$ for any element of $\{-1,1\}^n$. We define the transform A as*

$$T : \begin{cases} \mathcal{D}(Q) \to \mathcal{D}(Q) \\ f \to Af = 2^n \sum\limits_{I\in\{-1,1\}^n} A_I f, \end{cases}$$

where $A_I f(x) = f(4(x - x_I))$. For $B \subset Q$, we define $A_I(B) = x_I + \frac{1}{4}B, T(B) = \bigcup_{I\in\{-1,1\}^n} A_I(B)$ and denote $A_I(Q) = Q_I$. The support of $A_I f$ is included in Q_I as $f \in \mathcal{D}(Q)$ and the fact that if $I \neq I'$ then $Q_I \cap Q_{I'} = \phi$, we get

$$\|Af\|_{L^p} = 2^{n(1-\frac{1}{p})}\|f\|_{L^p}. \tag{2.1.4}$$

For simplification, assume that s is an integer, then we have

$$\partial_j(Af)(x) = 2^n \sum_{I\in\{-1,1\}^n} (4\partial_j f)(4(x - x_I)) = 4A(\partial_j f)(x)$$

and from Equation (2.1.4), we get

$$\|Af\|_{\dot{H}^s} = 2^{2s+\frac{n}{2}}\|f\|_{\dot{H}^s}.$$

The estimate of Af in terms of $\dot{B}^{-\sigma}$ is given by following proposition.

Proposition 2.1.51. *For $\sigma \in (0, n]$, there exists a constant C such that*

$$\|Af\|_{\dot{B}^{-\sigma}} \leqslant 2^{n-2\sigma}\|f\|_{\dot{B}^{-\sigma}} + C\|f\|_{L^1}.$$

As $\dot{H}^{\frac{n}{2}}(\mathbb{R}^n)$ is not included in $L^\infty(\mathbb{R}^n)$, we have to discuss this limiting case separately. We give the following definition:

Definition 2.1.52. *The space $BMO(\mathbb{R}^n)$ of bounded mean oscillations is the set of the locally integrable function f such that*

$$\|f\|_{BMO} = \sup_B \frac{1}{|B|}\int_B |f - f_B|dx < \infty,$$

where $f_B = \frac{1}{|B|}\int_B f dx$. The supremum is taken oveer all Euclidean balls. Note that $\|\cdot\|_{BMO}$ vanishes for all constant functions and hence it is a seminorm. Now, we state one of the important theorem of Sobolev embedding:

Theorem 2.1.53. *The space $L^1_{loc}(\mathbb{R}^n) \cap \dot{H}^{\frac{n}{2}}(\mathbb{R}^n)$ is included in $BMO(\mathbb{R}^n)$. Moreover, there exists a constant c such that*

$$\|u\|_{BMO} \leqslant c\|u\|_{\dot{H}^{\frac{n}{2}}} \quad \forall u \in L^1_{loc}(\mathbb{R}^n) \cap \dot{H}^{\frac{n}{2}}(\mathbb{R}^n).$$

Now, we state the embedding of $\dot{H}^s$ in the Hölder space. First of all let us give the definition of Hölder space.

Definition 2.1.54. *Let $(k, p) \in \mathbb{N} \times (0, 1]$. The Hölder space is the space of C^k functions u on $\mathbb{R}^n$ such that*

$$\|u\|_{C^{k,p}} = \sup_{|\alpha|\leqslant k}\left(\|\partial^\alpha u\|_{L^\infty} + \sup_{x\neq y}\frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^p}\right) < \infty.$$

Remark 2.1.55. *$C^{0,1}$ is the space of bounded Lipschitz functions.*

We state the following embedding between $C^{k,p}$ and $\dot{H}^s$.

Theorem 2.1.56. *Let $s > \frac{n}{2}$ and $s - \frac{n}{2}$ is not an integer. Then, the space $\dot{H}^s$ is included in the Hölder space with index*

$$(k, p) = \left(\left[s - \frac{n}{2}\right], s - \frac{n}{2} - \left[s - \frac{n}{2}\right]\right)$$

and for all $u \in \dot{H}^s$, we have

$$\sup_{|\alpha|=k}\sup_{x\neq y}\frac{|\partial^\alpha u(x)-\partial^\alpha u(y)|}{|x-y|^p}\leqslant C_{n,s}\|u\|_{\dot{H}^s}.$$

2.1.4 Nonhomogeneous Sobolev Spaces

This section is devoted to the introduction to nonhomogeneous Sobolev spaces. We mainly focus on the trace theorem and the Hardy inequality. We start this section with the definition of nonhomogeneous Sobolev space.

Definition 2.1.57. *Let $s \in \mathbb{R}$. The Sobolev space $H^s(\mathbb{R}^n)$(denoted as H^s) consist of distributions u such that $\hat{u} \in L^2_{loc}(\mathbb{R}^n)$ and*

$$\|u\|_{H^s}=\int_{\mathbb{R}^n}(1+|\xi|^2)^s|\hat{u}(\xi)|^2d\xi<\infty.$$

Remark 2.1.58. *As the Fourier transform is an isometric linear operation from H^s to the space $L^2(\mathbb{R}^n;(1+|\xi|^2)^s d\xi)$, the space H^s equipped with the scaler product*

$$(u|v)_{H^s}=\int_{\mathbb{R}^n}(1+|\xi|^2)^s\hat{u}(\xi)\bar{\hat{v}}(\xi)d\xi$$

is a Hilbert space.

Proposition 2.1.59. *Let $s_1 \leqslant s \leqslant s_2$. Then, $H^{s_1}\cap H^{s_2}$ included in H^s and for $\theta\in[0,1]$, we have*

$$\|u\|_{H^s}\leqslant C\|u\|^{1-\theta}_{H^{s_1}}\|u\|^{\theta}_{H^{s_2}},$$

where $s=s_1(1-\theta)+s_2\theta$.

Remark 2.1.60. *It is easy to see that the family of H^s spaces is decreasing with respect to s. For nonnegative integer s, from Plancherel theorem, the space H^s coincides with the set of $L^2(\mathbb{R}^n)$ functions u such that $\partial^\alpha \in L^2$ where α is a multi-index such that $|\alpha|\leqslant s$. For negative integer, we have the following proposition.*

Proposition 2.1.61. *Let s be a positive integer. Then, the space $H^{-s}(\mathbb{R}^m)$ consist of distributions which are the sums of derivatives of order l of $L^2(\mathbb{R}^n)$ functions.*

Remark 2.1.62. *For any $\epsilon>0, \delta_0\in H^{-\frac{n}{2}-\epsilon}$ but does not belongs to $H^{-\frac{n}{2}}$. Also, δ_0 is not in $\dot{H}^s$ for any s.*

Remark 2.1.63. *H^s is included in $\dot{H}^s$ for nonnegative s and opposit happen for negative s. Also, $H^s\neq\dot{H}^s, s\neq 0$.*

Proposition 2.1.64. *Let $H^s_K(\mathbb{R}^n)$ is the space of distributions of H^s which are supported on a compact set K where s is nonnegative. Then, there exists a constant C such that*

$$\frac{1}{C}\|u\|_{H^s} \leqslant \|u\|_{\dot{H}^s} \leqslant C\|u\|_{H^s} \quad \forall u \in H^s_K.$$

For functions supported in small balls,we state the following Poincare-type inequality.

Corollary 2.1.65. *Let $0 \leqslant t \leqslant s$. For any positive δ and any function $u \in H^s(\mathbb{R}^n)$ supported on a ball of radios δ, there exists a constant C such that*

$$\|u\|_{\dot{H}^t} \leqslant C\delta^{s-t}\|u\|_{\dot{H}^s} \quad \textit{and} \ \|u\|_{H^t} \leqslant C\delta^{s-t}\|u\|_{H^s}.$$

Proposition 2.1.66. *The space $\mathcal{S}$ is dense in H^s.*

For duality of H^s, we have the following proposition.

Proposition 2.1.67. *For any real s, the bilinear functional*

$$\mathcal{T} = \begin{cases} \mathcal{S} \times \mathcal{S} \to \mathbb{C} \\ (u, v) \to \int_{\mathbb{R}^n} u(x)v(x)dx \end{cases}$$

can be extended to a bilinear continuous functional on $H^s \times H^{-s}$. Moreover, if B is a continuous linear functional on H^s, then there exists a unique distribution v in H^{-s} such that

$$\forall u \in H^s, \langle B, u\rangle = \mathcal{T}(v, u) \quad \textit{and} \ \|B\|_{(H^s)'} = \|v\|_{H^{-s}}.$$

Proposition 2.1.68. *Let $s = k + \sigma$, where $k \in \mathbb{N}$ and $\sigma \in (0, 1)$. Then, for multi-index α we have*

$$H^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n)/|\alpha| \leqslant m, \partial^\alpha u \in L^2(\mathbb{R}^n) \right\}$$

and for $|\alpha| = m$,

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\partial^\alpha u(x+y) - \partial^\alpha u(x)|^2}{|y|^{n+2\sigma}} dxdx < \infty.$$

Moreover, there exists a constant C such that

$$\frac{1}{C}\|u\|^2_{H^s} \leqslant \sum_{|\alpha|=m} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\partial^\alpha u(x+y) - \partial^\alpha u(x)|^2}{|y|^{n+2\sigma}} dxdx + \sum_{|\alpha| \leqslant m} \|\partial^\alpha u\|^2_{L^2} \leqslant C\|u\|_{H^s}.$$

Definition 2.1.69. *A global k diffeomorphism on $\mathbb{R}^n$ is any C^k diffeomorphism $\eta : \mathbb{R}^n \to \mathbb{R}^n$ whose derivatives of order less than or equal to k are bounded and for some constant* *c and* $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, *we have*

$$|\eta(x) - \eta(y)| \geqslant C|x-y|.$$

Corollary 2.1.70. *Let* η *be a global* k *diffeomorphism on* $\mathbb{R}^n, 0 \leqslant s < k$ *and* $u \in \mathbb{R}^n$, *then* $u \circ \eta \in H^s(\mathbb{R}^n)$.

We also have the following density result.

Proposition 2.1.71. *For any real* s, *the space* $\mathcal{D}(\mathbb{R}^n)$ *is dense in* $H^s(\mathbb{R}^n)$.

Proposition 2.1.72. *Multiplication by a Schwartz function* $\mathcal{S}(\mathbb{R}^n)$ *is a continuous map from* H^s *to itself.*

Note that under multiplication by general C^∞ function, H^s is not stable.

Now, we consider the trace and the trace lifting operator. Consider the hyperplane $x_1 = 0$ in $\mathbb{R}^n$. For u in Lebesgue space, we formally define the trace operator Σ as $\Sigma u(x) = u(0,x)$, but as the Haar measure for the hyperplane is 0, this definition is not good to consider. For example, there are L^2 functions which are continuous for $x_1 \neq 0$ and goes to infinity when $x_1 \to 0$. Hence, we can not give this definition in general. Extending the usual trace operator by continuity gives us the definition for trace operator on $\mathbb{R}^n$ spaces.

Theorem 2.1.73. *Let* $s > 1/2$ *be a real number. The restriction map* Σ *defined by*

$$\Sigma : \begin{cases} \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^{n-1}) \\ u \to \Sigma(u) : (x_2, \dots, x_n) \to u(0, x_2, \dots, x_n) \end{cases}$$

can be extended continuously from $H^s(\mathbb{R}^n)$ *onto* $H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$.

We have the following corollary:

Corollary 2.1.74. *Let* $s > k + \frac{1}{2}$ *with* $k \in \mathbb{N}$. *The map*

$$\gamma : \begin{cases} H^s(\mathbb{R}^n) \to \bigoplus_{j=0}^k H^{s-j-\frac{1}{2}}(\mathbb{R}^{n-1}) \\ u \to (\Sigma_j(u))_{0 \leqslant j \leqslant k}, \end{cases}$$

where $\Sigma_j u = \Sigma(\partial_{x_1}^j u)$ *is continuous and onto.*

Remark 2.1.75. *From Theorem 2.1.73 and Corollary 2.1.74, the spaces* H^s *are local and invariant under the action of diffeomorphism and we may define the trace operator* *for any smooth hypersurface S of $\mathbb{R}^n$. So localizing and straightening S reduce the problem to the study of the trace operator defined in Theorem 2.1.73.*

Now, we state the few results related to embedding of the spaces H^s.

Theorem 2.1.76. *The space H^s embedded continuously in*

1. *the Lebesgue space $L^p(\mathbb{R}^n)$, if $0 \leqslant s \leqslant n/2$ and $2 \leqslant p \leqslant 2n/(n-2s)$*

2. *the Hölder space $C^{k,p}(\mathbb{R}^n)$, if $s \geqslant n/2 + k + p$ for some $k \in \mathbb{N}$ and $p \in (0,1)$.*

Following Moser-Trudinger inequality holds. Although, $H^{\frac{n}{2}}$ fails to be embedded in L^∞.

Theorem 2.1.77. *There exist two constants c_1 and c_2 only depending on the dimension n, such that for any $u \in H^{\frac{n}{2}}(\mathbb{R}^n)$, we have*

$$\int_{\mathbb{R}^n} \left(exp\left(c_1 \left(\frac{|u(x)|}{\|u\|_{H^{\frac{1}{2}}}} \right)^2 \right) - 1 \right) dx \leqslant c_2.$$

Theorem 2.1.78. *Let $t < s$. Multiplication by a function in $\mathcal{S}(\mathbb{R}^n)$ is a compact operator from H^s in H^t.*

From Theorem 2.1.78, we can deduce the following compactness result:

Theorem 2.1.79. *For any compact subset K of $\mathbb{R}^n$ and $s' < s$, the embedding $H^s_K(\mathbb{R}^n)$ into $H^{s'}_K(\mathbb{R}^n)$ is a compact linear operator.*

Before stating the Hardy inequality, we need to give the density result of $\mathcal{D}(\mathbb{R}^n \setminus \{0\})$ in H^s. The density of $\mathcal{D}(\mathbb{R}^n \setminus \{0\})$ plays an important role in the proof of Hardy inequality and the related result.

Theorem 2.1.80. *If $s \leqslant n/2$ (resp., $< n/2$.) Then the space $\mathcal{D}(\mathbb{R}^n \setminus \{0\})$ is dense in $H^{n/2}$ (resp., $\dot{H}^{n/2}$.) If $s > n/2$, then the closer of the space $\mathcal{D}(\mathbb{R}^n \setminus \{0\})$ in H^s is the set of functions u in H^s such that for every multi-index α with $|\alpha| < s - n/2$, $\partial^\alpha u(0) = 0$.*

Remark 2.1.81. *For $n = 1$, the map $u \to u(0)$ can not be extended to $H^{\frac{n}{2}}(\mathbb{R})$ functions. In other words, we can prove that the restriction map Σ on hyperplane $x_1 = 0$ can not be extended to $H^{\frac{1}{2}}(\mathbb{R}^n)$ functions.*

Now we state the Hardy inequality with singular weight in Sobolev spaces.

Theorem 2.1.82. *If $n \geqslant 3$, then*

$$\left(\int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^2} dx\right)^{\frac{1}{2}} \leqslant \frac{2}{n-2} \|\nabla u\|_{L^2} \tag{2.1.5}$$

for any $f \in H^1(\mathbb{R}^n)$.

2.2 Besov Spaces

In this section, we give the details of Besov spaces and there basic properties using Littlewood-Paley decomposition.

2.2.1 Bernstein Lemmas

From littlewood-Paley decomposition, the Fourier multiplier acts almost as homotheties on distributions where Fourier transform supported in a ball or an annulus. We start this subsection with Bernstein lemma.

Let $R > 0$. A set $\{\xi \in \mathbb{R}^n : |\xi| \leqslant R\}$ is called a *ball* and for $r_2 > r_1 > 0$ the set $\{\xi \in \mathbb{R}^n : 0 < r_1 \leqslant |\xi| \leqslant r_2\}$ is called an *annulus*.

Lemma 2.2.1. *Let $\mathcal{C}$ and $\mathcal{B}$ be an annulus and a ball respectively. Assume that k be an arbitrary nonnegative integer, $(p, q) \in [1, \infty]^2$ with $q \geqslant p \geqslant 1$ and u be any function of L^p. Then there exists a constant C such that*

$$\text{Supp } \hat{u} \subset \lambda B \Rightarrow \|D^k u\|_{L^q} = \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \leqslant C^{k+1} \lambda^{k+n(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p},$$

$$\text{Supp } \hat{u} \subset \lambda C \Rightarrow C^{-k-1} \lambda^k \|u\|_{L^p} \leqslant \|D^k u\|_{L^p} \leqslant C^{k+1} \lambda^k \|u\|_{L^p}.$$

Now we state the lemma about the action of Fourier multiplier which behaves like homogeneous function of degree m.

Lemma 2.2.2. *Let $k = 2[1 + n/2]$ (where [r] stands for integer part of r), $m \in \mathbb{R}$ and $\mathcal{C}$ be an annulus. Given σ a k-times differentiable function on $\mathbb{R}^n \setminus \{0\}$ and a multi-index α with $|\alpha\| \leqslant k$, there exits a constant C_α such that*

$$\forall\ \xi \in \mathbb{R}^n,\ |\partial^\alpha \sigma(\xi)| \leqslant C_\alpha |\xi|^{m-|\alpha|}.$$

Suppose that $p \in [1,\infty]$, $\lambda > 0$ *and* u *be a function in* L^p *with Fourier transform supported in* $\lambda\mathbb{C}$. *Then there exits a constant* C *(depending only on* C_α*) such that*

$$\|\sigma(D)u\|_{L^p} \leqslant C\lambda^m \|u\|_{L^p} \text{ with } \sigma(D)u = \mathcal{F}^{-1}(\sigma\hat{u}).$$

We state the Faá da Bruno's formula as follow:

Lemma 2.2.3. *Let* $u : \mathbb{R}^n \to \mathbb{R}^m$ *and* $F : \mathbb{R}^m \to \mathbb{R}$ *be smooth functions. Given a multi-index* α, *we have*

$$\partial^\alpha (F \circ u) = \sum_{\mu,\nu} C_{\mu,\nu} \partial^\mu F \prod_{\substack{1\leqslant|\beta|\leqslant|\alpha| \\ 1\leqslant j\leqslant m}} (\partial^\beta u^j)^{\nu_{\beta_j}},$$

where the coefficients $C_{\mu,\nu}$ *are non-negative integers and the sum is taken over those* μ *and* ν *such that* $1 \leqslant |\mu| \leqslant |\alpha|$, $\nu_{\beta_j} \in \mathbb{N}^*$,

$$\sum_{1\leqslant|\beta|\leqslant|\alpha|} \nu_{\beta_j} = \mu_j \ \forall\ 1 \leqslant j \leqslant m \text{ and } \sum_{\substack{1\leqslant|\beta|\leqslant|\alpha| \\ 1\leqslant j\leqslant m}} \beta\nu_{\beta_j} = \alpha.$$

With Fourier transform supported in an annulus, the action of semigroup of heat equation on distributions is given by following lemma:

Lemma 2.2.4. *Consider an annulus* $\mathcal{C}$. *Let* $p \in [1,\infty]$ *and* (t,λ) *be a couple of positive real numbers. Then there exists positive constants* c *and* C *such that*

$$\text{Supp } \hat{u} \subset \lambda\mathcal{C} \Rightarrow \|e^{t\Delta}u\|_{L^p} \leqslant Ce^{-ct\lambda^2}\|u\|_{L^p}.$$

Definition 2.2.5. *Let* X *be a Banach space,* I *is an interval of* $\mathbb{R}$, $p \in [1,\infty]$ *then* $L^p_I(X)$ *is the Lebesgue measurable functions from* I *to* X *such that* $t \to \|u(t)\|_X \in L^p(I)$. *The space* $L^p_I(X)$ *is embedded with the norm*

$$\|u\|_{L^p_I(X)} = \left(\int_I \|u(t)\|^p_X dt\right)^{\frac{1}{p}}.$$

For $p = \infty$, *we can give the definition in terms of esssup.*

Corollary 2.2.6. *Let* λ *be a positive real number and* $\mathcal{C}$ *be an annulus. Let* u_0 *[resp.* $g = g(t,y)$*] satisfy Supp* $\hat{u}_0 \subset \lambda\mathcal{C}$ *(resp., Supp* $\hat{g}(t) \subset \lambda\mathcal{C}\ \ \forall\ \ t \in [0,T]$*). Let* u *and* v *be solutions of*

$$\partial_t u - \nu\Delta u = 0 \quad \text{and} \quad u|_{t=0} = u_0 \quad \text{and}$$

$$\partial_t v - \nu \Delta v = g \quad and \quad v|_{t=0} = 0,$$

respectively. Choose arbitrary $1 \leqslant a \leqslant b \leqslant \infty$ *and* $1 \leqslant p \leqslant q \leqslant \infty$*. Then there exists positive constants* c *and* C*, depending only on* $\mathcal{C}$*, such that*

$$\begin{aligned}\|u\|_{L^q_T(L^b)} &\leqslant C(\nu\lambda^2)^{-\frac{1}{q}}\lambda^{n(\frac{1}{a}-\frac{1}{b})}\|u_0\|_{L^a},\\ \|v\|_{L^q_T(L^b)} &\leqslant C(\nu\lambda^2)^{-1+(\frac{1}{p}-\frac{1}{q})}\lambda^{n(\frac{1}{a}-\frac{1}{b})}\|g\|_{L^p_T(L^a)}.\end{aligned}$$

Now we explain the action of a diffeomorphism

Lemma 2.2.7. *Let* $\chi \in \mathcal{S}(\mathbb{R}^n)$*. Let* $u \in \mathcal{S}'(\mathbb{R}^n)$ *such that* $\hat{u}$ *is supported in* $\lambda\mathcal{C}$*,* ψ *be* $C^{1,1}$ *global diffeomorphism over* $\mathbb{R}^n$ *with inverse* ϕ*,* $p \in [1,\infty]$ *and* $(\lambda,\mu) \in (0,\infty)^2$*. Then there exists a constant* C *such that*

$$\|\chi(\mu^{-1}D)(u\circ\psi)\|_{L^p} \leqslant C\lambda^{-1}\|J_\phi\|^{\frac{1}{p}}_{L^\infty}\|u\|_{L^p}\Big(\|DJ_\phi\|_{L^\infty}\|J_\psi\|_{L^\infty} + \mu\|D\phi\|_{L^\infty}\Big)$$

where $J_\phi(z) = |det(D\phi(z))|$ *and* $\chi(\mu^{-1}D)(u\circ\psi) = \mathcal{F}^{-1}(\chi(\mu^{-1}\cdot)\mathcal{F}(u\circ\psi))$*.*

Lemma 2.2.8. *Consider a smooth function* θ *with support in an annulus of* $\mathbb{R}^n$*. Let* ψ *be a* $C^{0,1}$ *measure-preserving global diffeomorphism over* $\mathbb{R}^n$ *with inverse* ϕ*,* u *be a distribution with* $\hat{u}$ *supported in* $\lambda\mathcal{C}$*,* $(\lambda,\mu) \in (0,\infty)^2$ *and* $p \in [1,\infty]$*. Then there exists a constant* C *such that*

$$\|\theta(\mu^{-1}D)(u\circ\psi)\|_{L^p} \leqslant C\|u\|_{L_p}\min\Big(\frac{\mu}{\lambda}\|D\phi\|_{L^\infty}, \frac{\lambda}{\mu}\|D\psi\|_{L^\infty}\Big).$$

Now we state the lemma which describe some properties of powers of functions with Fourier support in an annulus.

Lemma 2.2.9. *Consider an annulus* $\mathcal{C}$*. Let* λ *be a positive real number,* p *be a positive integer and* u *be a function in* L^p *whose Fourier transform is supported in* $\lambda\mathcal{C}$*. Then there exists a constant* C *such that*

$$\|u^p\|_{L^2} \leqslant C\lambda^{-1}\|\nabla(u^p)\|_{L^2}.$$

Remark 2.2.10. *If* $\mathcal{F}u$ *is supported in an annulus, then* $\mathcal{F}(u^p)$ *is not supported in an annulus, but rather in a ball. The above lemma guarantees that the* L^2 *norm of* u^p *may be controlled by the* L^2 *norm of its gradient.*

Now, let us discuss about the dyadic partition of unity which is useful in further studies.

Proposition 2.2.11. *Consider an annulus* $\{\xi \in \mathbb{R}^n : \ 3/4 \leqslant |\xi| \leqslant 8/3\}$*. There exist radial functions* $\chi \in \mathcal{D}(B(0,4/3))$ *and* $\varphi \in \mathcal{D}(\mathcal{C})$*, with values in the interval* $[0,1]$*, such that*

$$\forall \xi \in \mathbb{R}^n, \ \chi(\xi) + \sum_{j \geqslant 0} \varphi(2^{-j}\xi) = 1, \tag{2.2.1}$$

$$\forall \xi \in \mathbb{R}^n \setminus \{0\}, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \tag{2.2.2}$$

$$|j - j'| \geqslant 2 \Rightarrow Supp\ \varphi(2^{-j}\cdot) \cap Supp\ \varphi(2^{-j'}\cdot) = \emptyset, \tag{2.2.3}$$

$$j \geqslant 1 \Rightarrow Supp\ \chi \cap Supp\ \varphi(2^{-j}\cdot) = \emptyset, \tag{2.2.4}$$

the set $\tilde{\mathcal{C}} = B(0,2/3) + \mathcal{C}$ *is an annulus, and we have*

$$|j - j'| \geqslant 5 \Rightarrow 2^{j'}\tilde{\mathcal{C}} \cap 2^j\mathcal{C} = \emptyset. \tag{2.2.5}$$

Furthermore,

$$\forall \xi \in \mathbb{R}^n, \frac{1}{2} \leqslant \chi^2(\xi) + \sum_{j \geqslant 0} \varphi^2(2^{-j}\xi) \leqslant 1, \tag{2.2.6}$$

$$\forall \xi \in \mathbb{R}^n \setminus 0, \quad \frac{1}{2} \leqslant \sum_{j \in \mathbb{Z}} \varphi^2(2^{-j}\xi) \leqslant 1. \tag{2.2.7}$$

From now on, assume the following remark.

Remark 2.2.12. *We fix two functions* χ *and* φ *satisfying the assertions* (2.2.1) − (2.2.7)*. Write* $h = \mathcal{F}^{-1}\varphi$ *and* $\tilde{h} = \mathcal{F}^{-1}\chi$*. The non-homogeneous dyadic blocks* Δ_j *are defined by*

$$\Delta_j(u) = 0 \quad if \quad j \leqslant -2, \quad \Delta_{-1}u = \chi(D)u = \int_{\mathbb{R}^n} \tilde{h}(y)u(x-y)dy,$$

$$and \quad \Delta_j(u) = \varphi(2^{-j}D)u = 2^{jn} \int_{\mathbb{R}^n} h(2^j y)u(x-y)dy \quad if \quad j \geqslant 0.$$

The nonhomogeneous low-frequency cut-off operator S_j *is defined by*

$$S_j u = \sum_{j' \leqslant j-1} \Delta_{j'} u.$$

The homogeneous dyadic blocks $\dot{\Delta}_j$ *and the homogeneous low-frequency cut-off operators* $\dot{S}_j$ *are defined for all* $j \in \mathbb{Z}$ *by*

$$\dot{\Delta}_j u = \varphi(2^{-j}D)u = 2^{jn}\int_{\mathbb{R}^n} h(2^j y)u(x-y)dy,$$
$$\dot{S}_j u = \chi(2^{-j}D)u = 2^{jn}\int_{\mathbb{R}^n} \tilde{h}(2^j y)u(x-y)dy.$$

Note that, the operators in Remark 2.2.12 map L^p *into* L^p *with norms independent of* j *and* p. *Also, we can write the below (formal) Littlewood-Paley decompositions:*

$$Id = \sum_j \Delta_j \quad and \quad Id = \sum_j \dot{\Delta}_j. \tag{2.2.8}$$

In the non-homogeneous case, the above decomposition makes sense in $\mathcal{S}'(\mathbb{R}^n)$.

Proposition 2.2.13. *Let* $u \in \mathcal{S}'(\mathbb{R}^d)$. *Then* $u = \lim_{j\to\infty} S_j u$, *in* $\mathcal{S}'(\mathbb{R}^d)$.

Now, we state the convergence result.

Proposition 2.2.14. *Let* $\tilde{\mathcal{C}}$ *be a given annulus. Consider* $(u_j)_{j\in\mathbb{N}}$, *a* L^∞ *sequence of bounded functions such that the Fourier transform of* u_j *is supported in* $2^j\tilde{\mathcal{C}}$. *Assume that, for some integer* N, *the sequence* $(2^{-jN}\|u_j\|_{L^\infty})_{j\in\mathbb{N}}$ *is bounded. Then, the series* $\sum_j u_j$ *converges in* $\mathcal{S}'(\mathbb{R}^n)$.

Proposition 2.2.15. *Let* $\tilde{\mathcal{C}}$ *be a given annulus. Consider* $(u_j)_{j\in\mathbb{N}}$, *a sequence of bounded functions such that the support of* $\hat{u}_j$ *is included in* $2^j\tilde{\mathcal{C}}$. *Let, for some integer* N, *the sequence* $(2^{-jN}\|u_j\|_{L^\infty})_{j\in\mathbb{N}}$ *be bounded and the series* $\sum_{j<0} u_j$ *converges in* L^∞. *Then the series* $\sum_{j\in\mathbb{Z}}$ *converges to some* u *in* $\mathcal{S}'$ *and* $u \in \mathcal{S}'_h$.

2.2.2 Homogeneous Besov Spaces

This subsection is devoted to the introduction to homogeneous Besov spaces. We start this subsection with the definition of homogeneous Besov spaces.

Definition 2.2.16. *Let* $(p,r) \in [1,\infty]^2$ *and* $s \in \mathbb{R}$. *The homogeneous Besov space* $\dot{B}^s_{p,r}$ *consists of those distributions* $u \in \mathcal{S}'_h$ *such that*

$$\|u\|_{\dot{B}^s_{p,r}} = \left(\sum_{j\in\mathbb{Z}} 2^{rjs}\|\dot{\Delta}_j u\|^r_{L^p}\right)^{\frac{1}{r}} < \infty.$$

Proposition 2.2.17. *The space $\dot B^s_{p,r}$ endowed with $\|\cdot\|_{\dot B^s_{p,r}}$ is a normed space.*

Remark 2.2.18. *The definition of the Besov space $\dot B^s_{p,r}$ is independent of the function φ used for defining the blocks $\dot\Delta_j$. Moreover, changing φ yields an equivalent norm. In fact, if $\tilde\varphi$ is another dyadic partition of unity, then there exists an integer N_0 such that $|j-j'|\geqslant N_0 \quad\Rightarrow Supp\ \tilde\varphi(2^{-j}\cdot)\cap Supp\ \varphi(2^{-j'}\cdot)=\emptyset$. Therefore,*

$$\begin{aligned}2^{js}\|\tilde\varphi(2^{-j}D)u\|_{L^p} &= 2^{js}\left\|\sum_{|j-j'|\leqslant N_0}\tilde\varphi(2^{-j}D)\dot\Delta_{j'}u\right\|_{L^p}\\ &\leqslant C2^{N_0|s|}\sum_{j'}1_{[-N_0,N_0]}(j-j')2^{j's}\|\dot\Delta_{j'}u\|_{L^p}.\end{aligned}$$

From Young's inequality we get the result.

We also note that a distribution u of $\mathcal{S}'_h$ belongs to $\dot B^s_{p,r}$ if and only if there exits some non-negative sequence $(c_j)_{j\in\mathbb{Z}}$ and some constant C such that

$$\forall\ j\in\mathbb{Z}, \|\dot\Delta_j u\|_{L^p}\leqslant Cc_j2^{-js}\quad and\quad \|(c_j)\|_{\ell^r}=1.$$

Proposition 2.2.19. *Let N be an integer. Consider a distribution u of $\mathcal{S}'_h$. Then $\|u\|_{\dot B^s_{p,r}}$ is finite if and only if u_N is finite. Furthermore, we have*

$$\|u_N\|_{\dot B^s_{p,r}}=2^{N(s-\frac{n}{p})}\|u\|_{\dot B^s_{p,r}}.$$

Remark 2.2.20. *More generally, if λ is a positive number, then there exists a constant C, depending only on s, such that*

$$C^{-1}\lambda^{s-\frac{n}{p}}\|u\|_{\dot B^s_{p,r}}\leqslant\|u(\lambda\cdot)\|_{\dot B^s_{p,r}}\leqslant C\lambda^{s-\frac{n}{p}}\|u\|_{\dot B^s_{p,r}}.$$

We highlight that having u in some homogeneous Besov space $\dot B^s_{p,r}$ yields information about both low and high frequencies of u. Therefore, if $s_1\neq s_2$, then we can't expect any inclusion between the spaces $\dot B^{s_1}_{p,r}$ and $\dot B^{s_2}_{p,r}$. But, we can state the following theorem, which is comparable with Sobolev embedding theorem.

Theorem 2.2.21. *Let s be any real number. Assume $1\leqslant p_1\leqslant p_2\leqslant\infty$ and $1\leqslant r_1\leqslant r_2\leqslant\infty$. Then, the space $\dot B^s_{p_1,r_1}$ is continuously embedded in $\dot B^{s-n(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$.*

As compared to standard homogeneous Sobolev space or L^p spaces for $p<\infty$, the homogeneous Besov spaces contains more nontrival homogeneous functions. We have such a result as follow:

Proposition 2.2.22. *Consider $\sigma \in (0,n)$. Let $p \in [1,\infty]$. Then, the function $|\cdot|^{-\sigma} \in \dot{B}^{\frac{n}{p}-\sigma}_{p,\infty}$.*

Proposition 2.2.23. *Let s_1 and s_2 are real numbers such that $s_1 < s_2$, $\theta \in (0,1)$, $(p,r) \in [1,\infty]^2$ and $u \in \mathcal{S}'_h$. Then there exists a constant C such that*

$$\|u\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,r}} \leqslant \|u\|^{\theta}_{\dot{B}^{s_1}_{p,r}} \|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,r}} \quad \text{and}$$
$$\|u\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,1}} \leqslant \frac{C}{s_2-s_1}\left(\frac{1}{\theta}+\frac{1}{1-\theta}\right)\|u\|^{\theta}_{\dot{B}^{s_1}_{p,\infty}} \|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,\infty}}.$$

For investigating a series to be in Besov spaces, we have the following lemma.

Lemma 2.2.24. *Consider an annulus $\mathcal{C}'$ and a sequence of functions $(u_j)_{j\in\mathbb{Z}}$ such that*

$$\text{Supp } \hat{u}_j \subset 2^j\mathcal{C}' \quad \text{and} \quad \left\|(2^{js}\|u_j\|_{L^p})_{j\in\mathbb{Z}}\right\|_{\ell^r} < \infty.$$

Let the series $\sum_{j\in\mathbb{Z}} u_j$ converges in $\mathcal{S}'$ to some $u \in \mathcal{S}'_h$. Then, $u \in \dot{B}^s_{p,r}$ and

$$\|u\|_{\dot{B}^s_{p,r}} \leqslant C_s \left\|(2^{js}\|u_j\|_{L^p})_{j\in\mathbb{Z}}\right\|_{\ell^r}.$$

Remark 2.2.25. *The above convergence assumption concerns $(u_j)_{j<0}$. If (s,p,r) satisfies the following condition:*

$$s < \frac{n}{p}, \quad \text{or} \quad s = \frac{n}{p} \quad \text{and} \quad r = 1, \tag{2.2.9}$$

then, by Lemma 2.2.1, we have

$$\lim_{j\to-\infty} \sum_{j'<j} u_{j'} = 0 \quad \text{in} \quad L^\infty.$$

Therefore, $\sum_{j\in\mathbb{Z}} u_j$ converges to some $u \in \mathcal{S}'$ and when $j \to -\infty$, $\dot{S}_j u$ tends to 0. In particular, $u \in \mathcal{S}'_h$.

Theorem 2.2.26. *Consider $(s_1,s_2) \in \mathbb{R}^2$ and $1 \leqslant r_1, r_2, p_1, p_2 \leqslant \infty$. Let (s_1,p_1,r_1) satisfies the condition (2.2.9). Then, the space $\dot{B}^{s_1}_{p_1,r_1} \cap \dot{B}^{s_2}_{p_2,r_2}$ endowed with the norm $\|\cdot\|_{\dot{B}^{s_1}_{p_1,r_1}} + \|\cdot\|_{\dot{B}^{s_2}_{p_2,r_2}}$ is complete. Furthermore, it satisfies the Fatou property which states:*
Let $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence of $\dot{B}^{s_1}_{p_1,r_1} \cap \dot{B}^{s_2}_{p_2,r_2}$. Then there exists an element $u \in$

$\dot{B}^{s_1}_{p_1,r_1} \cap \dot{B}^{s_2}_{p_2,r_2}$ *and a subsequence* $u_{\psi(n)}$ *such that:*

$$\lim_{n\to\infty} u_{\psi(n)} = u \text{ in } \mathcal{S}' \quad \text{and} \quad \|u\|_{\dot{B}^{s_k}_{p_k,r_k}} \leqslant C \lim_{n\to\infty} \inf \|u_{\psi(n)}\|_{\dot{B}^{s_k}_{p_k,r_k}} \quad for \quad k = 1, 2.$$

Remark 2.2.27. *If* $s > n/p$ *(or* $s = n/p$ *and* $r > 1$*), then* $\dot{B}^s_{p,r}$ *is no longer a Banach space. This is because of a breakdown of convergence for low frequencies, which is called infrared divergence.*

There is a way to modify the definition of homogeneous Besov spaces so that we can obtain a Banach space, with any regularity index. This is known as realizing homogeneous Besov spaces. It emerges that realizations coincide with our definition when $s < n/p$*, or* $s = n/p$ *and* $r = 1$*. In the other cases, realizations are defined up to a polynomial whose degree depends on* $s - n/p$ *and* r*. It goes without saying that solving partial differential equations in such spaces is quite difficult.*

Proposition 2.2.28. *Let* r *and* p *are both finite. Then the space* $S_0(\mathbb{R}^n)$ *of functions in* $\mathcal{S}(\mathbb{R}^n)$ *such that there Fourier transforms are supported away from* 0 *is dense in* $\dot{B}^s_{p,r}(\mathbb{R}^n)$*.*

Remark 2.2.29. *When* $r = \infty$*, the closure of* $\mathcal{S}_0$ *for the Besov norm* $\dot{B}^s_{p,r}$ *is the set of distributions in* $\mathcal{S}'_h$ *such that*

$$\lim_{j\to\pm\infty} 2^{js} \|\dot{\Delta}_j u\|_{L^p} = 0.$$

It emerges that Besov spaces have good duality properties. Also, note that in Littlewood-Paley theory, the duality on $\mathcal{S}'_h$ *translates, for* $\phi \in \mathcal{S}$*, into*

$$\langle u, \phi \rangle = \sum_{|j-j'|\leqslant 1} \langle \dot{\Delta}_j u, \dot{\Delta}_{j'} \phi \rangle = \sum_{|j-j'|\leqslant 1} \int_{\mathbb{R}^n} \dot{\Delta}_j u(y) \dot{\Delta}_{j'}(y) \phi(y) dy.$$

For L^p *space, by duality we can estimate the norm in* $\dot{B}^s_{p,r}$*.*

Proposition 2.2.30. *Let* $1 \leqslant p, r \leqslant \infty$ *and* $s \in \mathbb{R}$*. Define a continuous bilinear functional on* $\dot{B}^s_{p,r} \times \dot{B}^{-s}_{p',r'}$ *by:*

$$\begin{cases} \dot{B}^s_{p,r} \times \dot{B}^{-s}_{p',r'} & \longrightarrow \mathbb{R} \\ (u, \phi) & \longrightarrow \sum\limits_{|j-j'|\leqslant 1} \langle \dot{\Delta}_j u, \dot{\Delta}_{j'} \phi \rangle. \end{cases}$$

Denote the set of functions $\phi \in \mathcal{S} \cap \dot{B}^{-s}_{p',r'}$ *such that* $\|\phi\|_{\dot{B}^{-s}_{p',r'}} \leqslant 1$ *by* $Q^{-s}_{p',r'}$*. Let* $u \in \mathcal{S}'_h$*, then we have:*

$$\|u\|_{\dot{B}^s_{p,r}} \leqslant C \sup_{\phi \in Q^{-s}_{p',r'}} \langle u, \phi \rangle.$$

The following proposition describe the action of Fourier multiplier on homogeneous Besov spaces.

Proposition 2.2.31. *Consider a smooth function σ on $\mathbb{R}^n \setminus \{0\}$ which is homogeneous of degree m. Let $(s_k, p_k, r_k) \in \mathbb{R} \times [1,\infty]^2$ (with $k = \{1,2\}$) such that $(s_1 - m, p_1, r_1)$ satisfies the condition 2.2.9. Then the operator $\sigma(D)$ continuously maps $\dot{B}^{s_1}_{p_1,r_1} \cap \dot{B}^{s_2}_{p_2,r_2}$ into $\dot{B}^{s_1-m}_{p_1,r_1} \cap \dot{B}^{s_2-m}_{p_2,r_2}$.*

Remark 2.2.32. *Note that, the proof of above Proposition is very simple compared with the similar result on L^p spaces when $p \in (1,\infty)$. Moreover, Fourier multipliers do not map L^∞ into L^∞ in general. From this point of view Besov spaces are much easier to handle than classical L^p spaces or Sobolev spaces modeled on L^p.*

Corollary 2.2.33. *Assume (s_1, p_1, r_1) and (s_2, p_2, r_2) in $\mathbb{R} \times [1,\infty]^2$. Let $(s_1 + 1, p_1, r_1)$ satisfies the condition (2.2.9). If v is a vector field with components in $\dot{B}^{s_1-1}_{p_1,r_1} \cap \dot{B}^{s_2-1}_{p_2,r_2}$ which is curl free (i.e. $\partial_j v^k = \partial_k v^j \ \ \forall\ \ 1 \leqslant j,\ k \leqslant n$), then there exists a unique function in $\dot{B}^{s_1}_{p_1,r_1} \cap \dot{B}^{s_2}_{p_2,r_2}$ such that $\bigtriangledown(a) = v$ and*

$$C^{-1}\|a\|_{\dot{B}^{s_k}_{p_k,r_k}} \leqslant \|v\|_{\dot{B}^{s_k-1}_{p_k,r_k}} \leqslant C\|a\|_{\dot{B}^{s_k}_{p_k,r_k}} \quad for \quad k = 1,2,$$

where C a positive constant independent of v.

For the negative indicies, we have the characterization of homogeneous Besov spaces in terms of operators $\dot{S}_j$.

Proposition 2.2.34. *Assume $s < 0$ and $1 \leqslant p, r \leqslant \infty$. Let u be a distribution in $\mathcal{S}'_h$. Then,*

$$u \in \dot{B}^s_{p,r} \quad \Longleftrightarrow \quad (2^{js}\|\dot{S}_j u\|_{L^p})_{j\in\mathbb{Z}} \in \ell^r.$$

Furthermore, we have

$$C^{-|s|+1}\|u\|_{\dot{B}^s_{p,r}} \leqslant \left\|(2^{js}\|\dot{S}_j u\|_{L^p})_j\right\|_{\ell^r} \leqslant C\Big(1 + \frac{1}{|s|}\Big)\|u\|_{\dot{B}^s_{p,r}},$$

for some constant C depending only on n.

Now we give the result concerning the characterization of Besov spaces which do not require spectral localization.

Theorem 2.2.35. *Let s be a positive real number and $(p,r) \in [1,\infty]^2$. Then for some constant, we have*

$$C^{-1}\|u\|_{\dot{B}^{-2s}_{p,r}} \leqslant \left\|\|t^s e^{t\Delta} u\|_{L^p}\right\|_{L^r(\mathbb{R}^+,\frac{dt}{t})} \leqslant C\|u\|_{\dot{B}^{-2s}_{p,r}} \quad \forall \quad u \in \mathcal{S}'_h.$$

Lemma 2.2.36. *Let s be any positive number. Then*

$$\sup_{t>0} \sum_{j\in\mathbb{Z}} t^s 2^{2js} e^{-ct2^{2j}} < \infty.$$

Now, we will give a characterization of Besov spaces with positive indices in terms of finite differences. For simplicity of the presentation, we only consider the case where the regularity index $s \in (0,1)$.

Theorem 2.2.37. *For any $s \in (0,1)$, $(p,r) \in [1,\infty]^2$ and $u \in \mathcal{S}'_h$, there exists a constant C such that*

$$C^{-1}\|u\|_{\dot{B}^s_{p,r}} \leqslant \left\| \frac{\|\tau_{-y}u - u\|_{L^p}}{|y|^s} \right\|_{L^r(\mathbb{R}^n;\frac{dy}{|y|^n})} \leqslant C\|u\|_{\dot{B}^s_{p,r}}.$$

For $s = 1$, we have the following characterization.

Theorem 2.2.38. *Let $(p,r) \in [1,\infty]^2$ and $u \in \mathcal{S}'_h$. Then there exists a constant C such that:*

$$C^{-1}\|u\|_{\dot{B}^1_{p,r}} \leqslant \left\| \frac{\|\tau_{-y}u + \tau_y u - 2u\|_{L^p}}{|y|} \right\|_{L^r(\mathbb{R}^n;\frac{dy}{|y|^n})} \leqslant C\|u\|_{\dot{B}^1_{p,r}}.$$

Remark 2.2.39. *When we apply the above theorem for $p = r = \infty$, we get that the space $\dot{B}^1_{\infty,\infty}$ coincides with the Zygmund class of functions u such that:*

$$|u(x+y) + u(x-y) - 2u(x)| \leqslant C|y|.$$

Now, we compare homogeneous Besov spaces with Lebesgue spaces.

Proposition 2.2.40. *Let $(p,q) \in [1,\infty]^2$ such that $p \leqslant q$. Then the space $\dot{B}^{\frac{n}{p}-\frac{n}{q}}_{p,1}$ is continuously embedded in L^q. Futhermore, if p is finite, then $\dot{B}^{\frac{n}{p}}_{p,1}$ is continuously embedded in the space $\mathcal{C}_0$ of continuous functions vanishing at infinity. Moreover, $\forall\ q \in [1,\infty]$, the space L^q is continuously embedded in the space $\dot{B}^0_{q,\infty}$, and the space $\mathcal{M}$ of bounded measures on $\mathbb{R}^n$ is continuously embedded in $\dot{B}^0_{1,\infty}$.*

The comparison between homogeneous Besov spaces with regularity index 0 and third index 2 to Lebesgue space is given by following theorem.

Theorem 2.2.41. *Let $p \in (2,\infty)$. Then, $\dot{B}^0_{p,2}$ is continuously included in L^p and $L^{p'}$ is continuously included in $\dot{B}^0_{p',2}$.*

Theorem 2.2.42. *Let $p \in [1,2]$. Then, $\dot{B}^0_{p,p}$ is continuously included in L^p and $L^{p'}$ is continuously included in $\dot{B}^0_{p',p'}$.*

The generalization of refined Sobolev embedding is given in the next theorem.

Theorem 2.2.43. *For any* $1 \leqslant q < p < \infty$ *and a positive real number* α*, there exists a constant* C *such that*

$$\|f\|_{L^p} \leqslant C\|f\|^{1-\theta}_{\dot{B}^{-\alpha}_{\infty,\infty}} \|f\|^{\theta}_{\dot{B}^{\beta}_{q,q}} \quad \text{with} \;\; \beta = \alpha\Big(\frac{p}{q} - 1\Big) \;\; \text{and} \;\; \theta = \frac{q}{p}.$$

Theorem 2.2.44. *For* $q \in (1, \infty)$ *and* $s \in (0, n/q)$*, there exists a constant* C *such that*

$$\|u\|_{L^p} \leqslant C\|u\|^{s-\frac{n}{q}}_{\dot{B}^{\frac{qs}{n}}_{\infty,\infty}} \|u\|^{1-\frac{qs}{n}}_{\dot{W}^s_q} \quad \text{with} \;\; \|u\|_{\dot{W}^s_q} = \|(-\Delta)^{\frac{s}{2}} u\|_{L^q},$$

where $p = \frac{q}{p} = 1 - \frac{qs}{n}$.

Now, we establish the so-called Gagliardo-Nirenberg inequalities.

Theorem 2.2.45. *For any* $(q, r) \in (1, \infty]^2$ *and* $(\sigma, s) \in (0, \infty)^2$ *with* $\sigma < s$*, there exists a constant* C *such that*

$$\|u\|_{\dot{W}^\sigma_p} \leqslant C\|u\|^{\theta}_{L^q} \|u\|^{1-\theta}_{\dot{W}^s_r} \quad \text{with} \;\; \frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{r} \;\; \text{and} \;\; \theta = 1 - \frac{\sigma}{s}.$$

2.2.3 Paradifferential Calculus

This subsection contains the introduction to paradifferential calculus. We mainly concentrate on product of distribution on $\mathcal{S}'_h$ and their properties. First of all, we define the Bony decomposition and for the same purpose let u and v be tempered distributions in $\mathcal{S}'_h$. We have

$$u = \sum_{j'} \dot{\Delta}_{j'} u \;\; \text{and} \;\; v = \sum_{j} \dot{\Delta}_j v.$$

Definition 2.2.46. *The homogeneous paraproduct of* v *by* u *is defined as follows:*

$$\dot{T}_u v = \sum_j \dot{S}_{j-1} u \; \dot{\Delta}_j v.$$

The homogeneous remainder of u *and* v *is defined by*

$$\dot{R}(u, v) = \sum_{|k-j| \leqslant 1} \dot{\Delta}_k u \; \dot{\Delta}_j v.$$

Remark 2.2.47. *It is easy to note that when* $u, v \in \mathcal{S}'_h$*, then* $\dot{T}_u v$ *makes sense in* $\mathcal{S}'$ *and* $\dot{T} : (u, v) \to \dot{T}_u v$ *is a bilinear operator. Also, the remainder operator* $\dot{R} : (u, v) \to \dot{R}(u, v)$*, when restricted to sufficiently smooth distributions, is also bilinear.*

The main motivation for using the operators $\dot{T}$ and $\dot{R}$ is that, at least formally, the following so-called Bony decomposition holds true:

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u, v). \tag{2.2.10}$$

The following theorem describe the continuity of homogeneous paraproduct operator Y.

Theorem 2.2.48. *Let s be a real number, $(p, r) \in [1, \infty]^2$ and $(u, v) \in L^\infty \times \dot{B}^s_{p,r}$. Then there exists a constant C such that*

$$\|\dot{T}_u v\|_{\dot{B}^s_{p,r}} \leqslant C^{1+|s|} \|u\|_{L^\infty} \|v\|_{\dot{B}^s_{p,r}}.$$

Futhermore, if $(s, t) \in \mathbb{R} \times (-\infty, 0)$, $(p, r_1, r_2) \in [1, \infty]^3$ and $(u, v) \in \dot{B}^t_{\infty, r_1} \times \dot{B}^s_{p,r_2}$, then

$$\|\dot{T}_u v\|_{\dot{B}^{s+t}_{p,r}} \leqslant \frac{C^{1+|s+t|}}{-t} \|u\|_{\dot{B}^t_{\infty,r_1}} \|v\|_{\dot{B}^s_{p,r_2}} \quad \textit{with} \quad \frac{1}{r} = \min\left\{1, \frac{1}{r_1} + \frac{1}{r_2}\right\}.$$

Remark 2.2.49. *By Lemma (2.2.24) and remark (2.2.25), the hypothesis of convegence is satisfied whenever (s, p, r) or $(s + t, p, r)$ satisfies (2.2.9).*

Lemma 2.2.50. *Let B be a ball in $\mathbb{R}^n$. Consider a positive real number s and $(p, r) \in [1, \infty]^2$. Let $(u_j)_{j \in \mathbb{Z}}$ be a sequence of smooth functions such that*

$$\textit{Supp} \quad \hat{u}_j \subset 2^j B \quad \textit{and} \quad \|(2^{js}\|u_j\|_{L^p})_{j\in\mathbb{Z}}\|_{\ell^r} < \infty$$

Let the series $\sum_{j\in\mathbb{Z}} u_j$ converges to $u \in \mathcal{S}'_h$. Then, there exists a constant C such that:

$$u \in \dot{B}^s_{p,r} \quad \textit{and} \quad \|u\|_{\dot{B}^s_{p,r}} \leqslant \frac{C}{s} \left\|(2^{js}\|u_j\|_{L^p})_j\right\|_{\ell^r(\mathbb{Z})}.$$

Remark 2.2.51. *By Lemma (2.2.50), it follows that the hypothesis of convergence is satisfied whenever (s, p, r) satisfies (2.2.9).*

Remark 2.2.52. *The above lemma fails in the limit case $s = 0$. In fact, let us fix a non-zero function $f \in L^p$, spectrally supported in some ball B, and a nonnegative real α such that $\alpha r > 1$. Let*

$$u_j = \begin{cases} j^{-\alpha} & \textit{for} \quad j \geqslant 1 \\ 0 & \textit{otherwise} \end{cases}.$$

It is clear that

$$\forall\ j \in \mathbb{Z},\ Supp\ \hat{u}_j \subset 2^j B \quad and \quad \|(\|u_j\|_{L^p})_{j\in\mathbb{N}}\|_{\ell^r} < \infty.$$

If $r > 1$, then we can additionally set $\alpha < 1$ so that the series $\sum_j u_j$ diverges in $\mathcal{S}'$. If $r = 1$, then the series converges to a nonzero multiple of f. As $\dot{B}^0_{p,1}$ is a strict subspace of L^p, the function f need not be in $\dot{B}^0_{p,1}$. Thus, the lemma also fails in this case.

The following theorem gives the continuity of the reminder operator:

Theorem 2.2.53. *Consider $(s_1, s_2) \in \mathbb{R}^2$ and $(p_1, p_2, r_1, r_2) \in [1, \infty]^4$. Suppose that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leqslant 1 \quad and \quad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \leqslant 1.$$

Then there exists a constant C which satisfies the following inequalities:
Let $(u, v) \in \dot{B}^{s_1}_{p_1,r_1} \times \dot{B}^{s_2}_{p_2,r_2}$ and $s_1 + s_2$ be positive. Then, we have

$$\|\dot{R}(u, v)\|_{\dot{B}^{s_1+s_2}_{p,r}} \leqslant \frac{C^{|s_1+s_2|+1}}{s_1 + s_2}\|u\|_{\dot{B}^{s_1}_{p_1,r_1}}\|v\|_{\dot{B}^{s_2}_{p_2,r_2}}.$$

Let $(u, v) \in \dot{B}^{s_1}_{p_1,r_1} \times \dot{B}^{s_2}_{p_2,r_2}$. If $r = 1$ and $s_1 + s_2 \geqslant 0$, then we have

$$\|\dot{R}(u, v)\|_{\dot{B}^{s_1+s_2}_{p,\infty}} \leqslant C^{|s_1+s_2|+1}\|u\|_{\dot{B}^{s_1}_{p_1,r_1}}\|v\|_{\dot{B}^{s_2}_{p_2,r_2}}.$$

Remark 2.2.54. *By Lemma (2.2.50) and the remark that follows it, the hypothesis of the convergence is satisfied whenever $(s_1 + s_2, p, r)$ or $(s_1 + s_2, p, \infty)$ satisfies (2.2.9).*

We give few examples for product of paraproduct.

Corollary 2.2.55. *Let $(s, p, r) \in (0, \infty) \times [1, \infty]^2$ satisfies (2.2.9). Then $L^\infty \cap \dot{B}^s_{p,r}$ is an algebra. Furthermore, there exists a constant C, depending only on dimension n, such that*

$$\|uv\|_{\dot{B}^s_{p,r}} \leqslant \frac{C^{s+1}}{s}\Big(\|u\|_{L^\infty}\|v\|_{\dot{B}^s_{p,r}} + \|u\|_{\dot{B}^s_{p,r}}\|v\|_{L^\infty}\Big).$$

Corollary 2.2.56. *Let $(s_1, s_2) \in (-n/2, n/2)^2$. If $s_1 + s_2$ is positive, then there exists a constant C such that*

$$\|uv\|_{\dot{B}^{s_1+s_2-\frac{n}{2}}_{2,1}} \leqslant C\|u\|_{\dot{H}^{s_1}}\|v\|_{\dot{H}^{s_2}}.$$

Remark 2.2.57. *The constant in Corollary 2.2.56 may be bounded by*

$$C \min\left\{\frac{1}{n - 2s_1}, \frac{1}{n - 2s_2}, \frac{1}{s_1 + s_2}\right\},$$

where C depends only on dimension n.

We state the following form of Hardy's inequality

Theorem 2.2.58. *Let $s \in [0, \frac{n}{2})$ be a real number. Then, there exists a constant C such that $\forall\ f \in \dot{H}^s(\mathbb{R}^n)$,*

$$\int\limits_{R^n} \frac{|f(x)|^2}{|x|^{2s}} dx \leqslant C\|f\|^2_{\dot{H}^s}. \tag{2.2.11}$$

The following theorem gives the refined Hardy inequality.

Theorem 2.2.59. *Consider a triplet of real numbers (s, p, q) such that*

$$0 < s < \frac{n}{2} \quad and \quad 2 \leqslant q < \frac{2n}{n2s} < p \leqslant \infty.$$

Let $u \in \dot{B}^{s-n(\frac{1}{2}-\frac{1}{q})}_{q,2}$. Then there exists a constant C such that

$$\Big(\int \frac{|u(x)|^2}{|x|^{2s}} dx\Big)^{\frac{1}{2}} \leqslant C\|u\|^{\alpha}_{\dot{B}^{s-n(\frac{1}{2}-\frac{1}{p})}_{p,2}} \|u\|^{1-\alpha}_{\dot{B}^{s-n(\frac{1}{2}-\frac{1}{q})}_{q,2}} \quad with \quad \alpha = \frac{pq}{p-q}\Big(\frac{1}{q} - \frac{1}{2} + \frac{s}{n}\Big).$$

Lemma 2.2.60. *Let the assumptions be same as in Theorem 2.2.59. Let $f, g \in L^p \cap L^q$. Then there exists a constant C such that*

$$\langle |\cdot|^{-2s}, fg\rangle \leqslant C\|f\|^{\alpha}_{L^p}\|g\|^{\alpha}_{L^p}\|f\|^{1-\alpha}_{L^q}\|g\|^{1-\alpha}_{L^q} \quad with \quad \alpha = \frac{pq}{p-q}\Big(\frac{1}{q} - \frac{1}{2} + \frac{s}{n}\Big).$$

Remark 2.2.61. *Theorem 2.2.59 fails for $p = q_c = \frac{2n}{n-2s}$. Since, if it was true, for any function u with Fourier transform supported in $\mathcal{C}$, we would have:*

$$\int\limits_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^{2s}} dx \leqslant C\|u\|^2_{\dot{B}^0_{2q_c,2}} \leqslant C\|u\|^2_{L^{2q_c}(\mathbb{R}^n)}. \tag{2.2.12}$$

Particularly, the above inequality holds when $u \in \mathcal{S}(\mathbb{R}^n)$ satisfies

$$supp\ \hat{u} \subset \mathcal{B}(\xi_0, \varepsilon) \subset \mathcal{C}.$$

Since, the inequality (2.2.12) is invariant under oscillation (i.e., under translation in the Fourier space), we conclude that $\forall\ u \in \mathcal{S}(\mathbb{R}^n)$ such that supp $\hat{u} \subset \mathcal{B}(0, \varepsilon)$, it is true. The invariance under dilation implies that it is true for any function $u \in \mathcal{S}(\mathbb{R}^n)$ such that supp $\hat{u} \subset \mathcal{B}(0, R)$ for any $R > 0$. By density, we obtain (2.2.12) $\forall\ u \in L^{2q_c}(\mathbb{R}^n)$, but this implies that the singular weight $|x|^{-2s}$ belongs to $L^{\frac{n}{2s}}$, which is false.

Theorem 2.2.62. *Assume a smooth function f on $\mathbb{R}$ which vanishes at 0. Let (s_1, s_2) be a couple of positive real numbers and $(p_1, p_2, r_1, r_2) \in [1, \infty]^4$ and (s_1, p_1, r_1) satisfies the condition (2.2.9).*

Let $u \in \dot{B}^{s_1}_{p_1,r_1} \cap \dot{B}^{s_2}_{p_2,r_2} \cap L^\infty$, then the function $f \circ u$ belongs to same space, and we have, for $k = 1$ and $k = 2$,

$$\|f \circ u\|_{\dot{B}^{s_k}_{p_k,r_k}} \leqslant C(f', \|u\|_{L^\infty})\|u\|_{\dot{B}^{s_k}_{p_k,r_k}}. \tag{2.2.13}$$

Lemma 2.2.63. *Let the hypothesis be the same as in Theorem 2.2.62. Define $f_j = f(\dot{S}_{j+1}u) - f(\dot{S}_j u)$. Then the series $\sum\limits_{j\in\mathbb{Z}} f_j$ converges to $f(u)$ in $\mathcal{S}'$ and we have*

$$f_j = m_j \dot{\Delta}_j u \quad \text{with} \quad m_j = \int_0^1 f'(\dot{\mathcal{S}}_j u + t\dot{\Delta}_j u)dt.$$

Lemma 2.2.64. *Consider a smooth function g from $\mathbb{R}^2$ to $\mathbb{R}$. For $j \in \mathbb{Z}$, define*

$$m_j(g) = g(\dot{S}_j u, \dot{\Delta}_j u).$$

Let u be any bounded function. Then for all multi-index α

$$\forall j \in \mathbb{Z}, \ \|\partial^\alpha m_j(g)\|_{L^\infty} \leqslant C_\alpha(g, \|u\|_{L^\infty})2^{j|\alpha|}.$$

Lemma 2.2.65. *Consider a positive real number s and $(p, r) \in [1, \infty]^2$. Let $(u_j)_{j\in\mathbb{Z}}$ is a sequence of smooth functions where $\sum u_j$ converges to some u in $\mathcal{S}'_h$ and*

$$N_s((u_j)_{j\in\mathbb{Z}}) = \left\| \left(\sup_{|\alpha|\in\{0,[s]+1\}} 2^{j(s-|\alpha|)}\|\partial^\alpha u_j\|_{L^p} \right)_j \right\|_{\ell^r(\mathbb{Z})} < \infty.$$

Then there exists a constant C_s such that $u \in \dot{B}^s_{p,r}$ and $\|u\|_{\dot{B}^s_{p,r}} \leqslant C_s N_s(u)$.

Corollary 2.2.66. *Consider a function $f \in C^\infty_b(\mathbb{R})$ such that $f(0) = 0$. Let $(s_1, s_2) \in (0, \infty)^2$ and $(p_1, p_1, r_1, r_2) \in [1, \infty]^4$ such that (s_1, p_1, r_1) satisfies (2.2.9).*

Chapter 3

Global Attractor for Weakly Damped and Forced mKdV Equation

3.1 Introduction

We consider the modified Korteweg-de Vries (in short, mKdV) equation:

$$\partial_t u + \partial_x^3 u \pm 2\partial_x u^3 + \gamma u = f, \quad t > 0, \; x \in \mathbb{T}, \tag{3.1.1}$$

$$u(x,0) = u_0(x) \in \dot{H}^s(\mathbb{T}), \tag{3.1.2}$$

where $\mathbb{T}$ is the one-dimensional torus, $\gamma > 0$ is the damping parameter and $f \in \dot{H}^1(\mathbb{T})$ is the external forcing term which does not depends on t. In equation (3.1.1), "+" and "−" represent the focussing and defocussing cases, respectively. We consider the inhomogeneous Sobolev spaces $H^s = \{f \mid \sum_{k\in\mathbb{Z}} \langle k \rangle^{2s} |\hat{f}(k)|^2 < \infty\}$ where $\langle \cdot \rangle = (1 + |\cdot|)$ and the homogeneous Sobolev spaces $\dot{H}^s = \{f \in H^s | \hat{f}(0) = 0\}$. The mKdV equation models the propagation of nonlinear water waves in the shallow water approximation. We only consider the focussing case as the defocussing case follows with the same assertion. Also, considering inhomogeneous Sobolev norm is very important as for homogeneous Sobolev norm, Proposition 3.3.1 does not hold for more details (see appendix by Nobu Kishimoto). From the arguments in [13], [12] and [15], the existence of global attractor for equations (3.1.1)-(3.1.2) directly follows for $s \geqslant 1$ in H^s. In the present paper, we prove the existence of global attractor below the energy space in $\dot{H}^s(\mathbb{T})$ for $1 > s > 11/12$.

Miura [22],[23] and [24] studied the properties of solutions to the Korteweg-de Vries (KdV) equation and its generalization. Miura [22] established the Miura transformation between the solutions of mKdV and KdV. Indeed, if u satisfies equation (3.1.1) with " + " sign, then the function defined by

$$p = \partial_x u + iu^2$$

satisfies the KdV equation, where $i = \sqrt{-1}$. Colliander, Keel, Staffilani, Takaoka and Tao [8] presented the I-method and proved the existence of global solution for mKdV in the Sobolev space $H^s(\mathbb{T})$ for $s \geqslant 1/2$ by using the Miura transformation. However, the Miura transformation does not work well for the weakly damped and forced mKdV. In fact, if we consider the mKdV and KdV equations with the damping and forcing term and apply the Miura transformation, we get

$$p_t + p_{xxx} - 6ipp_x + \gamma p = (2iu + \partial_x)f - i\gamma u^2. \tag{3.1.3}$$

It is clear from (3.1.3) that the Miura transformation does not transform the solution of mKdV equation to the solution of KdV equation. For this reason, the results of damped and forced KdV can not be directly converted to those of damped and forced mKdV by the Miura transform unlike the case without damping and forcing terms.

The study of global attractor is important as it characterizes the global behaviour of all solutions. The asymptotic behaviour of solutions below the energy space has not been known, though the global well-posedness below the energy space is already proved for the Cauchy problem of (3.1.1)-(3.1.2). To study the asympototic behaviour of the solution of mKdV equation below energy space, we need to study the global attractor below energy space. Chen, Tian and Deng [5] used Sobolev inequalities and *a priori* estimates on u_x, u_{xx} derived by the energy method to show the existence of global attractor in H^2. Dlotko, Kania and Yang [9] considered more generalized KdV equation and showed the existence for global attractor in H^1. It is instructive to look at known results on KdV, since KdV has been more extensively studied than mKdV. Tsugawa [29] proved the existence of global attractor for KdV equation in $\dot{H}^s$ for $0 > s > -3/8$ by using the I-method. Later, Yang [31] closely investigated Tsugawa's argument to bring down the lower bound from $s > -3/8$ to $s \geqslant -1/2$.

Though mKdV has many common properties with KdV, there is a big difference between KdV and mKdV in the structure of resonance. For KdV, we consider the homogeneous Sobolev spaces instead of the inhomogeneous one, which eliminates the resonant frequencies in quadratic nonlinearity (see Bourgain [4]). On the other hand, for the homogeneous mKdV equation, to eliminate the resonant frequencies in cubic nonlinearity, we need to consider the reduced equation (or the renormalized equation)

$$\partial_t u + \partial_x^3 u + 6\left(u^2 - \frac{1}{2\pi}\|u\|_{L^2}^2\right)\partial_x u = 0. \tag{3.1.4}$$

Without damping and forcing terms, the L^2 norm of the solution is conserved. So, the transformation from the original mKdV eqation to the reduced mKdV equation is just the translation with constant velocity. But this is not the case with damped and forced mKdV. The resonant structure of cubic nonlinearity is quite different from that of quadratic nonlinearity. Therefore, in the mKdV case, we need to directly handle the resonant trilinear estimate as well as the non-resonant trilinear estimate. In this respect, it seems difficult to employ the modified energy similar to that used in [29],[31]. Especially, the scaling argument is one of the main ingredient of the I method. So we need to make the dependence of estimates on the scaling parameter λ also. Hence, the following questions naturally arise: How should we treat the nonlinearity of mKdV equation with the damping and forcing terms? When we can not use Miura transformation, how should we treat mKdV equation ? To deal with such issues, we apply the I-method directly to (3.1.5)-(3.1.6) in the present paper and prove the following result:

Theorem 3.1.1. *Assume $11/12 < s < 1$ and $u_0 \in \dot{H}^s$. Let $S(t)$ is the semi-group generated by the solution of mKdV. Then, there exists two operators $L_1(t)$ and $L_2(t)$ such that*

$$\begin{aligned}
&S(t)u_0 = L_1(t)u_0 + L_2(t)u_0,\\
&\sup_{t>T_1} \|L_1(t)u_0\|_{H^1} < K,\\
&\|L_2(t)u_0\|_{H^s} < Kexp(-\gamma(t-T_1)), \quad \forall\, t > T_1,
\end{aligned}$$

where $K = K(\|f\|_{H^1}, \gamma)$ and $T_1 = T_1(\|f\|_{H^1}, \|u_0\|_{H^s}, \gamma)$.

In Theorem 3.1.1, the map L_1 is uniformaly compact and L_2 uniformly convergs to 0 in H^s. Therefore, from [28, Theorem 1.1.1], we get the existence of global attractor. For the proof of Theorem 3.1.1, we consider the following equation:

$$\partial_t u + \partial_x^3 u + 6\left(u^2 - \frac{1}{2\pi}\|u\|_{L^2}^2\right)\partial_x u + \gamma u = F \quad t > 0, x \in \mathbb{T}, \tag{3.1.5}$$

$$u(x,0) = u_0(x) \tag{3.1.6}$$

where

$$F = f\left(x + \int_0^t \|u(\tau)\|_{L^2}^2 d\tau\right).$$

If we put $q(x,t) = u(x + \int_0^t \|u(\tau)\|_{L^2}^2 d\tau,\, t)$, then q satisfies Equations (3.1.5)-(3.1.6).

We divide this paper into six sections. In Section 2, we describe the preliminaries required for the present paper. Section 3 descirbes the proof of trilinear estimate by using the Strichartz estimate for mKdV equation proved by J. Bourgain [4]. Section 4 contains *a priori* estimates. We describe the proof of Theorem 3.1.1 in Section 5. Finally in Section 6, some multilinear estimates are proved.

3.2 Preliminaries

In this section, we present the notations and definitions which are used throughout this article.

3.2.1 Notations

In this subsection, we list the notations which we use throughout this paper. C, c are the various time independent constants which depend on s unless specified. $a+$ and $a-$ represent $a+\epsilon$ and $a-\epsilon$, respectively for arbitrary small $\epsilon > 0$. $A \lesssim B$ denotes the estimate of the form $A \leqslant CB$. Similarly, $A \sim B$ denotes $A \lesssim B$ and $B \gtrsim A$.

Define $(dk)_\lambda$ to be normalized counting measure on $\mathbb{Z}/\lambda$:

$$\int \phi(k)(dk)_\lambda = \frac{1}{\lambda} \sum_{k \in \mathbb{Z}/\lambda} \phi(k).$$

Let $\hat{f}(k)$ and $\tilde{f}(k,\tau)$ denotes the Fourier transform of $f(x,t)$ in x and in x and t, respectively. We define the Sobolev space $H^s([0,\lambda])$ with the norm

$$\|f\|_{H^s} = \|\hat{f}(k)\langle k\rangle^s\|_{L^2((dk)_\lambda)},$$

where $\langle\cdot\rangle = (1 + |\cdot|)$. For details see [8],[29]. We define the space $X^{s,b}$ embedded with the norm

$$\|u\|_{X^{s,b}} = \|\langle k\rangle^s \langle \tau - 4\pi^2 k^3\rangle \tilde{u}(k,\tau)\|_{L^2((dk)_\lambda d\tau)}.$$

We often study the KdV and mKdV equation in $X^{s,\frac{1}{2}}$ space but it hardly contorls the norm $L_t^\infty H_x^s$ see [4],[8],[29]. To ensure the continuity of the solution, we define a slightly smaller space with the norm

$$\|u\|_{Y^s} = \|u\|_{X^{s,\frac{1}{2}}} + \|\langle k\rangle^s \tilde{u}(k,\tau)\|_{L^2((dk)_\lambda)L^1(d\tau)}.$$

Z^s space is defined via the norm

$$\|u\|_{Z^s} = \|u\|_{X^{s,-\frac{1}{2}}} + \|\langle k\rangle^s \langle \tau - 4\pi^2 k^3\rangle^{-1} \tilde{u}(k,\tau)\|_{L^2((dk)_\lambda)L^1(d\tau)}.$$

For the time interval $[t_1, t_2]$, we define the restricted spaces $X^{s,b}$ and Y^s embedded with the norms

$$\begin{aligned}
\|u\|_{X^{s,b}_{([0,\lambda]\times[t_1,t_2])}} &= \inf\{\|U\|_{X^{s,b}} : U|_{([0,\lambda]\times[t_1,t_2])} = u\}, \\
\|u\|_{Y^s_{([0,\lambda]\times[t_1,t_2])}} &= \inf\{\|U\|_{Y^s} : U|_{([0,\lambda]\times[t_1,t_2])} = u\}.
\end{aligned}$$

We state the mean value theorem as follow:

Lemma 3.2.1. *If a is controlled by b and $|k_1| \ll |k_2|$, then*

$$a(k_1 + k_2) - a(k_2) = O\left(|k_1| \frac{b(k_2)}{|k_2|}\right).$$

For details see [8, Section 4].

3.2.2 Rescaling

In this subsection, we rescale the mKdV equation. We can rewrite equations (3.1.5)-(3.1.6) in λ-rescaled form as follow:

$$\partial_t v + \partial_{xxx} v + 6\left(v^2 - \frac{1}{2\pi}\|v\|^2_{L^2}\right)\partial_x v + \lambda^{-3}\gamma v = \lambda^{-3} g, \tag{3.2.1}$$

$$v(x, t_0) = v_{t_0}(x), \tag{3.2.2}$$

where

$$\begin{aligned}
g(x,t) &= \lambda^{-1} F(\lambda^{-1}x, \lambda^{-3}t), \\
v_{t_0}(x) &= \lambda^{-1} u(\lambda^{-1}x, \lambda^{-3}t_0),
\end{aligned}$$

for initial time t_0. If u is the solution of the equations (3.1.5)-(3.1.6), then $v(x,t) = \lambda^{-1}u(\lambda^{-1}x, \lambda^{-3}t)$ is the solution of the equations (3.2.1)-(3.2.2). Rescaling is helpful in proving the local in time result as well as *a priori* estimate.

3.2.3 I-Operator

We define an operator I which plays an important role for the I-method. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth monotone $\mathbb{R}$-valued function defined as:

$$\phi(k) = \begin{cases} 1 & |k| < 1, \\ |k|^{s-1} & |k| > 2. \end{cases}$$

Then, for $m(k) = \phi(\frac{k}{N})$, we define

$$m(k) = \begin{cases} 1 & |k| < N, \\ |k|^{s-1} N^{1-s} & |k| > 2N, \end{cases}$$

where we fix N to be a large cut-off. We define the operator I as:

$$\widehat{Iu}(k) = m(k)\hat{u}(k).$$

We can rescale the operator I as follow:

$$\widehat{I'u}(k) = m'(k)\hat{u}(k),$$

where $m'(\frac{k}{\lambda}) = m(k)$. Let $N' = \frac{N}{\lambda}$. Then

$$m'(k) = \begin{cases} 1 & |k| < N', \\ |k|^{s-1} N'^{(1-s)} & |k| > 2N'. \end{cases}$$

We use the rescaled I-operator for proving the local results for mKdV equation in time. Moreover, proving *a priori* estimate also use the same operator.

3.2.4 Strichartz Estimate

Strichartz estimate plays an important role for the proof of the trilinear estimate. Bourgain in [4], proves the L^4 Strichartz estimate for mKdV equation. In the present article, we use the same estimate. We list the following result:

Proposition 3.2.2. *Let $b > \frac{1}{3}$. Then, we have*

$$\|u\|_{L^4(\mathbb{R}\times\mathbb{T})} \lesssim C\|u\|_{X^{0,b}}.$$

3.2.5 Local-Wellposedness

In this subsection, we state the local result in time which can be proved by using the contraction mapping. Let $\eta(t) \in C_0^\infty$ be a cut-off function such that:

$$\eta(t) = \begin{cases} 1 & \text{if } |t| \leqslant 1, \\ 0 & \text{if } |t| > 2. \end{cases}$$

Suppose that

$$D_\lambda(t)f(x) = \int e^{2i\pi kx} e^{-(2i\pi k)^3 t} \hat{f}(k)(dk)_\lambda.$$

We assume the following well known lemmas:

Lemma 3.2.3.

$$\|\eta(t)D_\lambda(t)w\|_{X^{1,\frac{1}{2}}} \leqslant \|w\|_{H^1}.$$

Lemma 3.2.4. *Let $F \in X^{1,-\frac{1}{2}}$. Then*

$$\|\eta(t)\int_0^t D_\lambda(t-t')F(t')dt'\|_{Y^1} \leqslant \|F\|_{Z^1}.$$

For the proof of Lemmas 3.2.3 and 3.2.4 see [8].

Proposition 3.2.5. *Let $\frac{1}{2} \leqslant s < 1$. Then the IVP (3.2.1)-(3.2.2) is locally well-posed for the initial data v_{t_0} satisfying $I'v_{t_0} \in \dot{H}^1(\mathbb{T})$ and $I'g \in \dot{H}^1(\mathbb{T})$. Moreover, there exists a unique solution on the time interval $[t_0, t_0+\delta]$ with the lifespan $\delta \sim (\|I'v_{t_0}\|_{H^1} + \lambda^{-3}\|I'g\|_{H^1} + \gamma\lambda^{-3})^{-\alpha}$ for some $\alpha > 0$ and the solution satisfies*

$$\|I'v\|_{Y^1} \lesssim \|I'v_{t_0}\|_{H^1} + \lambda^{-3}\|I'g\|_{H^1},$$
$$\sup_{t_0 \leqslant t \leqslant t_0+\delta} \|I'v(t)\|_{H^1} \lesssim \|I'v_{t_0}\|_{H^1} + \lambda^{-3}\|I'g\|_{H^1}.$$

Remark 3.2.6. *Note that*

$$\begin{aligned} g(x,t) &= \lambda^{-1}F(\lambda^{-1}x, \lambda^{-3}t) \\ &= \lambda^{-1}f\left(x + \frac{1}{2\pi}\int_0^t \|I'v\|_{L^2}^2\right) \end{aligned}$$

Proof. The proof of the Proposition 3.2.5 follows along the same lines as for KdV equation given in [29] with the help of trilinear estimate given in Proposition 3.3.8. The only difference arises in the estimate of g as it depends on unknown u. To deal with this issue, we define a new metric. Indeed, let

$$B = \{w \in X^{1,\frac{1}{2}} : \|w\|_{X^{1,\frac{1}{2}}} \lesssim C\left(\|I'v_0\|_{H^1} + \lambda^{-3}\|I'g\|_{H^1}\right)\}$$

and define the metric

$$d(w, w') = \|w - w'\|_{X^{0,\frac{1}{2}}} + \|v - v'\|_{X^{0,\frac{1}{2}}},$$

for $I'v = w$. As $X^{0,\frac{1}{2}}$ is reflexive, the ball B is complete with respect to the metric d for details see [19, 9.14 and Lemma 7.3] . Therefore, it is enough to show

$$\begin{aligned}\|N(v,w) - N(v',w')\|_{Y^0} &\lesssim \|\eta(t)(P(v,w) - P(v',w'))\|_{Z^0}\\ &\lesssim \left(\gamma\lambda^{-3} + \lambda^{0+}\left(\|I'v_0\|_{H^1} + \lambda^{-3}\|I'g\|_{H^1}\right)^2 + \lambda^{-3}\|I'g\|_{H^1}\right)\\ &\left(\|w - w'\|_{X^{0,\frac{1}{2}}} + \|v - v'\|_{X^{0,\frac{1}{2}}}\right),\end{aligned}$$

where

$$N(w) = \eta(t)D_\lambda(t)I'v_0 - \eta(t)\int D_\lambda(t - t')\eta(t')P(t')dt'$$

with

$$P(v, w) = 6I'\left(v^2 - \frac{1}{2\pi}\|v\|_{L^2}^2\right)\partial_x v + \gamma\lambda^{-3}w - \lambda^{-3}I'g.$$

As the metric consist of both w and u terms, we consider the pair of equation as:

$$\partial_t v + \partial_{xxx} v + 6\left(v^2 - \frac{1}{2\pi}\|v\|_{L^2}^2\right)\partial_x v + \lambda^{-3}\gamma v = \lambda^{-3}g, \tag{3.2.3}$$

$$\partial_t w + \partial_{xxx} w + 6I'\left(v^2 - \frac{1}{2\pi}\|v\|_{L^2}^2\right)\partial_x (I')^{-1}w + \lambda^{-3}\gamma w = \lambda^{-3}I'g. \tag{3.2.4}$$

The estimate of v in H^s follows from that of w in H^1 because $\|v\|_{H^s} \lesssim \|w\|_{H^1}$. Therefore, we do not need to assume extra condition on ball for the variable "v" .Let

$$\begin{aligned}g'(x,t) &= \lambda^{-1}F(\lambda^{-1}x, \lambda^{-3}t)\\ &= \lambda^{-1}f\left(x + \frac{1}{2\pi}\int_0^t \|I'v'\|_{L^2}^2\right)\end{aligned}$$

At first, we consider the external forcing term for Equation (3.2.3) as:

$$\begin{aligned}
&\|I'g - I'g'\|_{X^{0,-\frac{1}{2}}} \lesssim \|I'g - I'g'\|_{L^2} \\
=&\left\|\lambda^{-1}I'f\left(\lambda^{-1}x + \int_0^{\lambda^{-3}t}\|\lambda v(\lambda\cdot,\lambda^3\tau)\|_{L^2}^2 d\tau\right) - \right. \\
&\left.\lambda^{-1}I'f\left(\lambda^{-1}x + \int_0^{\lambda^{-3}t}\|\lambda v'(\lambda\cdot,\lambda^3\tau)\|_{L^2}^2 d\tau\right)\right\|_{L^2} \\
\lesssim&\left\|\lambda^{-1}\int_0^1 \frac{d}{d\theta}I'f(\lambda^{-1}x + \theta\alpha(t) + (1-\theta)\beta(t))d\theta\right\|_{L^2}
\end{aligned}$$

where

$$\alpha(t) = \int_0^{\lambda^{-3}t}\|\lambda v(\lambda\cdot,\lambda^3\tau)\|_{L^2}^2 d\tau \quad \text{and} \quad \beta(t) = \int_0^{\lambda^{-3}t}\|\lambda v'(\lambda\cdot,\lambda^3\tau)\|_{L^2}^2 d\tau.$$

Now from mean value theorem and the fact that translation is invariant, we get

$$\|I'g - I'g'\|_{L^2} \lesssim \|I'g\|_{H^1}\|v - v'\|_{X^{0,\frac{1}{2}}}.$$

Similarly for Equation (3.2.4), we get

$$\|g - g'\|_{L^2} \lesssim \|g\|_{H^1}\|v - v'\|_{X^{0,\frac{1}{2}}}.$$

The nonlinear term can be estimated similar to the 4-linear estimate of Lemma 3.4.9. Note that the 4-linear estimate has third order derivative on the other hand the nonlinear term has only one. We can make the similar cases for the nonlinear term as given in Integrals $(1) - (3)$ and prove the estimate. Hence, we can use the contraction principle. This shows that the solution $u \in X^{1,\frac{1}{2}}$. We need to show that the solution belongs to Y^1. But from Proposition 3.3.8, the nonlinear term of the integral equation belongs to Y^1. In the same way, we can verify other two terms of integral equation by using Schwarz inequality. Therefore, the solution $u \in Y^1$. □

3.3 Trilinear Estimate

Define an operator J such that

$$\hat{J}[u,v,w] = i\frac{k}{3}\sum_{\substack{k_1+k_2+k_3=k \\ (k_1+k_2)(k_2+k_3)(k_3+k_1)\neq 0}} \hat{u}(k_1)\hat{v}(k_2)\hat{w}(k_3) - ik\hat{u}(k)\hat{v}(k)\hat{w}(-k). \tag{3.3.1}$$

where $\hat{u}$ and $\tilde{v}$ denote the Fourier transforms in x variable and both x and t variables, respectively. We establish the following trilinear estimate for J:

Proposition 3.3.1. *Let $s \geqslant \frac{1}{2}$ and $u, v, w \in X^{s,\frac{1}{2}}$ are λ-periodic in x variable. Then, we have*

$$\|J[u,v,w]\|_{X^{s,-\frac{1}{2}}} \leqslant C\lambda^{0+}\|u\|_{X^{s,\frac{1}{2}}}\|v\|_{X^{s,\frac{1}{2}}}\|w\|_{X^{s,\frac{1}{2}}}. \tag{3.3.2}$$

Remark 3.3.2. *We note that if u is real valued, then*

$$J[u,u,u] = \left(u^2 - \frac{1}{2\pi}\|u\|_{L^2}^2\right)\partial_x u. \tag{3.3.3}$$

yields the nonlinearity of mKdV. The first term and the second term of (3.3.1) can be estimated in H^s for $s \geqslant \frac{1}{4}$ and $s \geqslant \frac{1}{2}$, respectively. So, the bound $s = \frac{1}{2}$ comes from the second term.

Simple computations yield

$$\begin{aligned}
\left(u^2 - \frac{1}{2\pi}\|u\|_{L^2}^2\right)\partial_x u =& i\sum_{\substack{k_1+k_2+k_3=k\\(k_1+k_2)\neq 0}} \hat{u}(k_1)\hat{u}(k_2)k_3\hat{u}(k_3)\\
=& i\{\sum_{\substack{k_1+k_2+k_3=k\\(k_1+k_2)(k_2+k_3)(k_3+k_1)\neq 0}} \hat{u}(k_1)\hat{u}(k_2)k_3\hat{u}(k_3)\\
&+\sum_{\substack{k_1+k_2+k_3=k\\(k_1+k_2)(k_3+k_1)\neq 0\\(k_2+k_3)=0}} \hat{u}(k_1)\hat{u}(-k_3)k_3\hat{u}(k_3)\\
&+\sum_{\substack{k_1+k_2+k_3=k\\(k_1+k_2)(k_2+k_3)\neq 0\\(k_3+k_1)=0}} \hat{u}(-k_3)\hat{u}(k_2)k_3\hat{u}(k_3)\\
&+\sum_{\substack{k_1+k_2+k_3=k\\(k_1+k_2)\neq 0\\(k_2+k_3)=(k_3+k_1)=0}} k_3\hat{u}(k_1)\hat{u}(-k_3)^2\}\\
=& i\frac{k}{3}\{\sum_{\substack{k_1+k_2+k_3=k\\(k_1+k_2)\neq 0}} \hat{u}(k_1)\hat{u}(k_2)\hat{u}(k_3)\}\\
&- ik|\hat{u}(k)|^2\hat{u}(k).
\end{aligned}$$

Remark 3.3.3. *Note that the right hand side of the above formula is equivalent to $\hat{J}$. Therefore, the nonlinearity of mKdV equation can be control if we prove the Proposition 3.3.1 .*

Remark 3.3.4. *If u is a complex-valued function, then we have only to consider*

$$\left(|u^2| - \frac{1}{2\pi}\|u\|_{L^2}^2\right)\partial_x u - \frac{i}{2\pi} Im\langle \partial_x u, u\rangle_{L^2} u$$

instead of the left hand side of the above equality. This yield the nonlinearity of the complex mKdV.

Proof of Proposition 3.3.1. We first consider the trilinear estimate corresponding to non resonant frequencies. We claim that

$$\left\| i\frac{k}{3} \int_{\substack{k_1+k_2+k_3=k \\ (k_1+k_2)(k_2+k_3)(k_3+k_1)\neq 0}} \hat{u}_1(k_1)\hat{u}_2(k_2)\hat{u}_3(k_3) \right\|_{X^{s,-\frac{1}{2}}} \lesssim \prod_{i=1}^{3} \|u_i\|_{X^{s,\frac{1}{2}}}.$$

From duality, it is enough to show

$$\left| \int_{\substack{k_1+k_2+k_3+k_4=0 \\ (k_1+k_2)(k_2+k_3)(k_3+k_1)\neq 0}} \langle k_1\rangle \int_{\sum_{i=1}^{4}\tau_i=0} \prod_{i=1}^{4} \tilde{u}_i(k_i,\tau_i)(dk_i)_\lambda d\tau_i \right| \lesssim \prod_{i=1}^{3} \|u_i\|_{X^{s,\frac{1}{2}}} \|u_4\|_{X^{-s,\frac{1}{2}}}. \quad (3.3.4)$$

Consider LHS of (3.3.4) and let the region of the first integration to be " $*$ " and region of the second integration is denoted by " $**$". Define $\sigma_i = \tau_i - 4\pi k_i^3$ for $1 \leqq i \leqslant 4$. Multiply and divide by $\langle k_4\rangle^{\frac{1}{2}}\langle \sigma_4\rangle^{\frac{1}{2}}$ to get

$$\left| \int_* \int_{**} \langle k_1\rangle^1 \langle k_4\rangle^s \langle \sigma_4\rangle^{-\frac{1}{2}} \tilde{u}_1\tilde{u}_2\tilde{u}_3(\langle k_4\rangle^{-s}\langle\sigma_4\rangle^{\frac{1}{2}}\tilde{u}_4) \right|. \quad (3.3.5)$$

We divide this estimate into following four cases:

1. Let $|\sigma_4| = \max\{|\sigma_i|$ for $1 \leqslant i \leqslant 4\}$.
2. Let $|\sigma_3| = \max\{|\sigma_i|$ for $1 \leqslant i \leqslant 4\}$.
3. Let $|\sigma_2| = \max\{|\sigma_i|$ for $1 \leqslant i \leqslant 4\}$.
4. Let $|\sigma_1| = \max\{|\sigma_i|$ for $1 \leqslant i \leqslant 4\}$.

From the symmetry and the duality argument, it is enough to show for Case 1 because other cases can be treated in the same way. As we know, $k_1 + k_2 + k_3 + k_4 = 0$ and

$\tau_1 + \tau_2 + \tau_3 + \tau_4 = 0$, from simple calculations, we have

$$\langle \sigma_4 \rangle \gtrsim 3(|k_1 + k_2||k_2 + k_3||k_3 + k_1|) \sim 3(|k_2 + k_3||k_3 + k_4||k_4 + k_2|). \tag{3.3.6}$$

From symmetry, we can assume that $|k_1| \geqslant |k_2| \geqslant |k_3|$. Now we can again subdivide all three cases into four cases:

1a $|k_1| \sim |k_2| \sim |k_3| \sim |k_4|$

1b $|k_1| \sim |k_4| \gg |k_2| \gtrsim |k_3|$

1c $|k_1| \sim |k_4| \sim |k_2| \gtrsim |k_3|$

Remark 3.3.5. *Note that there are other cases also but if we consider $|k_1| \gg |k_4|$, the derivative corresponding to $|k_4|$ get very small and the estimate is easy to verify.*

Lemma 3.3.6. *For **Case** 1a, we give the following proof:*

Proof. Note that we wish to prove

$$\|\partial_x M(u,u,u)\|_{X^{s,-\frac{1}{2}}} \lesssim \|u\|^3_{X^{s,\frac{1}{2}}}, \tag{3.3.7}$$

where

$$\mathcal{F}_x[M(u,v,w)] = \sum_{\substack{k_1+k_2+k_3=k \\ |k_1|\sim|k_2|\sim|k_3|}} \hat{u}(k_1)\hat{v}(k_2)\hat{w}(k_3),$$

and $\mathcal{F}$ denotes the Fourier transform in x variable. Hence,

$$\begin{aligned}\|\partial_x M(u,u,u)\|_{X^{s,-\frac{1}{2}}} \sim& \left(\int_k \langle k \rangle^3 \left(\int_{-\infty}^{\infty} \langle \sigma \rangle^{-1} \left| \mathcal{F}_{x,t}[M(u,u,u)] \right|^2 d\tau \right) (dk)_\lambda \right)^{\frac{1}{2}} \\ \sim& \|(\langle k \rangle^{\frac{1}{2}} |\tilde{u}|)^3 \langle \sigma \rangle^{-\frac{1}{2}} \|_{L^2(\mathbb{T}\times\mathbb{R})},\end{aligned}$$

where $\mathcal{F}_{x,t}$ is the Fourier transform in both x and t variables. Let $\tilde{v}(k,\tau) = \langle k \rangle^{\frac{1}{2}} |\tilde{u}(k,\tau)|$. Hence, we get

$$\begin{aligned}\|(\langle k \rangle^{\frac{1}{2}} |\tilde{u}|)^3 \langle \sigma \rangle^{-\frac{1}{2}} \|_{L^2(\mathbb{T}\times\mathbb{R})} &\lesssim \|v^3\|_{X^{0,-\frac{1}{2}}}, \\ &\lesssim \|v^3\|_{L^{\frac{4}{3}}(\mathbb{T}\times\mathbb{R})},\end{aligned}$$

From the duality of Strichartz's estimate and Proposition 3.2.2, we get

$$\|(\langle k \rangle^{\frac{1}{2}} |\tilde{u}|)^3 \langle \sigma \rangle^{-\frac{1}{2}} \|_{L^2(\mathbb{T}\times\mathbb{R})} \lesssim \|v\|^3_{L^4(\mathbb{T}\times\mathbb{R})},$$

$$\lesssim \lambda^{0+}\|u\|^3_{X^{s,\frac{1}{2}}}.$$

Therefore, we can handle Case $1a$ directly.

Case $1b$. We assume that the size of the Fourier support of u_j satisfies

$$\begin{aligned}
&|k_1| \sim |k_4| \gg |k_2|, |k_3|,\\
&|\sigma_4| \gtrsim |k_2+k_3||k_3+k_4||k_4+k_2|,\\
&\frac{1}{\lambda} \leqslant |k_2+k_3| \leqslant 1.
\end{aligned} \tag{3.3.8}$$

Remark 1. The restriction $k_1+k_2+k_3+k_4=0$ and the assumption imply that $|k_1|\sim|k_4|$. But it does not follow that $|k_2|\sim|k_3|$ unless (3.3.8) additionally assumed.

We prove the following estimate of the quardlinear functional on $\mathbb{R}\times\lambda\mathbb{T}$ with parameter $\lambda\geqslant 1$.

Lemma 3.3.7. *For the above conditions, we have*

$$\begin{aligned}
&\left|\int_*\int_{**}\langle k_1\rangle^1\langle k_4\rangle^s\langle\sigma_4\rangle^{-\frac{1}{2}}\tilde{u}_1\tilde{u}_2\tilde{u}_3(\langle k_4\rangle^{-\frac{1}{2}}\langle\sigma_4\rangle^{\frac{1}{2}}\tilde{u}_4)\right|\\
\lesssim&(1+\lambda^{0+})\min\{\|u_2\|_{X^{1/4+,1/2}}\|u_3\|_{X^{0,1/2}},\|u_2\|_{X^{0,1/2}}\|u_3\|_{X^{1/4+,1/2}}\}\times\|u_1\|_{X^{s,1/2}}\|u_4\|_{X^{-s,1/2}}.
\end{aligned} \tag{3.3.9}$$

Proof. We follow the argument in [8, Case 3 in the proof of Proposition 5 on page 733-734]. We first note that

$$|\sigma_4| \gtrsim |k_2+k_3||k_1|^2. \tag{3.3.10}$$

From the Plancherel theorem, inequality (3.3.10) and the Sobolev embedding, the left side of (3.3.9) can be bounded by the following inequalities.

$$\begin{aligned}
&\left|\int_*\int_{**}\langle k_1\rangle^1\langle k_4\rangle^s\langle\sigma_4\rangle^{-\frac{1}{2}}\tilde{u}_1\tilde{u}_2\tilde{u}_3(\langle k_4\rangle^{-s}\langle\sigma_4\rangle^{\frac{1}{2}}\tilde{u}_4)\right|\\
\lesssim&\int_*\int_{**}\langle k_1\rangle^s|\bar{\tilde{u}}_1(k_1)|(|k_2+k_3|^{-1/2}|\tilde{u}_2(k_2)||\tilde{u}_3(k_3)|)|\sigma_4|^{1/2}|k_4|^{-s}|\tilde{u}_4(k_4)|d\tau\\
\lesssim&\|D_x^s v_1\|_{L^4(\mathbb{R}\times\lambda\mathbb{T})}\|D_x^{-1/2}(v_2v_3)\|_{L^4(\mathbb{R}\times\lambda\mathbb{T})}\|v_4\|_{X^{-s,1/2}}\\
\lesssim&\|v_1\|_{X^{s,1/3+}}\|D_x^{-1/4}(v_2v_3)\|_{L^4(\mathbb{R};L^2(\lambda\mathbb{T}))}\|v_4\|_{X^{-s,1/2}},
\end{aligned} \tag{3.3.11}$$

where $\tilde{v_j} = |\tilde{u_j}|$. Furthermore, by the Plancherel's theorem, $1/\lambda \leqslant |k_2|+|k_3| \leqslant 1$, Schwarz inequality and the Young's inequality, we have

$$\begin{aligned}
\|D_x^{-1/4}(v_2v_3)\|_{L^2(\lambda\mathbb{T})} &\lesssim \int_{1/\lambda\leqslant|k_{23}|\leqslant 1} |k_{23}|^{-1/2} \left| \int_{k_{23}=k_2+k_3} \tilde{v_2}(k_2)\tilde{v_3}(k_3) \right|^2 \\
&\lesssim \left(\int_{1/\lambda\leqslant|k_{23}|\leqslant 1} |k_{23}|^{-1} \right)^{1/2} \left(\int_{1/\lambda\leqslant|k_{23}|\leqslant 1} \left| \int_{k_{23}=k_2+k_3} \tilde{v_2}(k_2)\tilde{v_3}(k_3) \right|^4 \right)^{1/2} \\
&\lesssim (1+\log\lambda)^{1/2} \min\{\|v_2\|^2_{L^2(\lambda\mathbb{T})}\|v_3\|^2_{H^{1/4+}(\lambda\mathbb{T})}, \|v_3\|^2_{L^2(\lambda\mathbb{T})}\|v_2\|^2_{H^{1/4+}(\lambda\mathbb{T})}\}. \qquad (3.3.12)
\end{aligned}$$

The integration in t over $\mathbb{R}$ of the squared left side of (3.3.12) yield

$$\begin{aligned}
\|D_x^{-1/4}(v_2v_3)\|_{L^4(\mathbb{R};L^2(\lambda\mathbb{T}))} &\lesssim (1+\lambda^{0+}) \min\{\|v_2\|^2_{L^8(\mathbb{R};L^2(\lambda\mathbb{T}))} \\
&\qquad \|v_3\|^2_{L^8(\mathbb{R};H^{1/4+}(\lambda\mathbb{T}))}, \|v_3\|^2_{L^8(\mathbb{R};L^2(\lambda\mathbb{T}))}\|v_2\|^2_{L^8(\mathbb{R};H^{1/4+}(\lambda\mathbb{T}))}\} \\
&\lesssim (1+\lambda^{0+}) \min\{\|v_2\|^2_{X^{0,1/2}}\|v_3\|^2_{X^{1/4+,1/2}}, \|v_2\|^2_{X^{0,1/2}}\|v_3\|^2_{X^{1/4+,1/2}}\}. \qquad (3.3.13)
\end{aligned}$$

Accordingly, from (3.3.11)-(3.3.13) we obtained the desire inequality (3.3.9).

Case 1c. Inequality (3.3.10) becomes

$$|\sigma_4| \gtrsim |k_2+k_4||k_1|^2.$$

Therefore, we can estimate case **1c** in the similar way as case **1b**.

For the resonant part (the second term of operator J (3.3.1)), the proof is similar to Lemma 3.3.6 with M defined in the formula (3.3.7) changes to the following:

$$\mathcal{F}_x[M(u,u,u)] = |\hat{u}(k)|^2|\hat{u}(k)|.$$

Now, we prove the trilinear estimate corresponding to the function space Z^s:

Proposition 3.3.8. *For $s \geqslant \frac{1}{2}$ and $u, v, w \in X^{s,\frac{1}{2}}$, we have*

$$\|J[u,v,w]\|_{Z^s} \leqslant C\lambda^{0+}\|u\|_{Y^s}\|v\|_{Y^s}\|w\|_{Y^s}. \qquad (3.3.14)$$

Proof. From Proposition 3.3.1, it is enough to show

$$\|\langle k\rangle^s\langle k\rangle\langle\sigma\rangle^{-1}J[u,v,w]\|_{L^2_{(dk)_k}L^1_{d\tau}} \leqslant C\|u\|_{X^{s,\frac{1}{2}}}\|v\|_{X^{s,\frac{1}{2}}}\|w\|_{X^{s,\frac{1}{2}}}.$$

Similar to Proposition 3.3.1, we also divide this problem into the following four cases.

1. Let $|\sigma| = \max\{|\sigma|, |\sigma_i| \text{ for } 1 \leqslant i \leqslant 3\}$.
2. Let $|\sigma_1| = \max\{|\sigma|, |\sigma_i| \text{ for } 1 \leqslant i \leqslant 3\}$.
3. Let $|\sigma_2| = \max\{|\sigma|, |\sigma_i| \text{ for } 1 \leqslant i \leqslant 3\}$.
4. Let $|\sigma_3| = \max\{|\sigma|, |\sigma_i| \text{ for } 1 \leqslant i \leqslant 3\}$.

Case 1 is the worst one. Indeed, otherwise we have by Schwarz's inequality,

$$\begin{aligned}
&\|\langle k\rangle^s\langle k\rangle\langle\sigma\rangle^{-1}\sum_k \hat{u}_1\hat{u}_2\hat{u}_3\|_{L^2_{(dk)_k}L^1_\tau} \\
&\lesssim \left\|\left(\int_{-\infty}^{\infty}\frac{1}{\langle\sigma\rangle^{2(\frac{1}{2}+\epsilon)}}d\tau\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}\frac{\langle k\rangle^{2s}\langle k\rangle^2}{\langle\sigma\rangle^{2(\frac{1}{2}-\epsilon)}}\left|\sum_k \hat{u}_1\hat{u}_2\hat{u}_3\right|^2 d\tau\right)^{\frac{1}{2}}\right\|_{L^2_{(dk)_\lambda}}. \\
&\lesssim C\left\|\frac{\langle k\rangle\langle k\rangle^s}{\langle\sigma\rangle^{(\frac{1}{2}-\epsilon)}}\sum_k \hat{u}_1\hat{u}_2\hat{u}_3 d\tau\right\|_{L^2_{(dk)_k}L^2_\tau},
\end{aligned}$$

and hence it reduces to the same proof as in Proposition 3.3.1. Therefore, we only have to prove Case 1. From symmetry, assume that $|k_1| \geqslant |k_2| \geqslant |k_3|$. We divide Case 1 into further three cases as follow:

1a. $|k_1| \sim |k_2| \sim |k_3|$.

1b. $|k_1| \gg |k_2| \gtrsim |k_3|$.

1c. $|k_1| \sim |k_2| \gg |k_3|$.

Case 1a. By the Schwarz's inequality, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty}\langle\sigma\rangle^{-1}|\mathcal{F}_{t,x}[M(u,u,u)]|d\tau \\
&\leqslant \left(\int_{-\infty}^{\infty}\langle\sigma\rangle^{-1-\epsilon}d\tau\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}\langle\sigma\rangle^{-1+\epsilon}|\mathcal{F}_{t,x}[M(u,u,u)]|^2 d\tau\right)^{\frac{1}{2}},
\end{aligned}$$

where M is defined in (3.3.7). This case is reduces to Lemma 3.3.6.
Case $1b$. In this case, we can clearly see that $\langle\sigma\rangle \gtrsim |k_2+|k_3||(\langle k\rangle^2+\langle\sigma\rangle)$. Due to symmetry, we can assume that $|k|\sim|k_1|$. By using Schwarz's inequality, we get

$$\left|\sum_k \langle k\rangle^s\langle k\rangle\langle\sigma\rangle^{-1}\hat{u}_1\hat{u}_2\hat{u}_3\right|_{L^2_{(dk)_k}L^1_\tau}$$
$$\lesssim \left\|\sum_k\left(\int_{-\infty}^{\infty}\frac{\langle k\rangle^2}{\langle\sigma\rangle^2}d\tau\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}\langle k\rangle^{2s}\left|\hat{u}_1\hat{u}_2\hat{u}_3\right|^2 d\tau\right)^{\frac{1}{2}}\right\|_{L^2_{(dk)_\lambda}}.$$

As we can see

$$\begin{aligned}
\left(\int_{-\infty}^{\infty}\frac{\langle k\rangle^2}{\langle\sigma\rangle^2}d\tau\right)^{\frac{1}{2}} &\lesssim \left(\int_{-\infty}^{\infty}\frac{\langle k\rangle^2}{(\langle\sigma\rangle+|k_2+k_3|\langle k\rangle^2)^2}d\tau\right)^{\frac{1}{2}},\\
&= \left(\int_{-\infty}^{\infty}\frac{\langle k\rangle^2}{(|\tau-k^3|+|k_1+k_2|\langle k\rangle^2)^2}d\tau\right)^{\frac{1}{2}},\\
&= \left(\int_{-\infty}^{k^3}\frac{\langle k\rangle^2}{(k^3-\tau+|k_2+k_3|\langle k\rangle^2)}d\tau\right)^{\frac{1}{2}} + \left(\int_{k^3}^{\infty}\frac{\langle k\rangle^2}{(\tau-k^3+|k_2+k_3|\langle k\rangle^2)}d\tau\right)^{\frac{1}{2}}\\
&\lesssim C|k_2+k_3|^{-1/2}.
\end{aligned}$$

Hence, from Hölder's inequality, Proposition 3.2.2 and inequality (3.3.13), we get

$$\begin{aligned}
&\left\|\sum_k|k_2+k_3|^{-1/2}\langle k\rangle^s\hat{u}_1\hat{u}_2\hat{u}_3\right\|_{L^2_{(dk)_\lambda}L^2_\tau}\\
&\sim \left\|\sum_k(|k_1|^s\hat{u}_1)(|k_2+k_3|^{-1/2}\hat{u}_2\hat{u}_3)\right\|_{L^2_{(dk)_\lambda}L^2_\tau},\\
&\lesssim \|D_x^s u_1\|_{L^4_{x,t}}\|D_x^{-\frac{1}{2}}(u_2u_3)\|_{L^4_{x,t}}\\
&\lesssim \lambda^{0+}\|u_1\|_{X^{s,\frac{1}{3}+}}\|u_2\|_{X^{\frac{1}{4}+,\frac{1}{2}}}\|u_3\|_{X^{0,\frac{1}{2}}}.
\end{aligned}$$

The estimate for the resonant term follows in the same way as Case $1a$. □

Let $u=u_L+u_H$ where *supp* $\hat{u}_L(k)\subset\{|k|\ll N\}$ and *supp* $\hat{u}_H(k)\subset\{|k|\gtrsim N\}$. We prove the following corollary:

Corollary 3.3.9. *Let* $1\gg\epsilon\geqslant 0$. *Let* $u,v,w\in X^{s,\frac{1}{2}-\epsilon}$. *Then, the following three estimates hold:*

1. If v, u are low and w is high frequency functions, then we have

$$\left\| (u_L v_L - \sum_{l=-\infty}^{\infty} \hat{u}_L(l)\hat{v}_L(-l)) w_H \right\|_{X^{1-2\epsilon, -\frac{1}{2}+\epsilon}}$$
$$\lesssim \lambda^{0+} C \min\{\|u_L\|_{X^{\frac{1}{2}+\epsilon, \frac{1}{2}-\epsilon}} \|v_L\|_{X^{0,\frac{1}{2}-\epsilon}}, \|v_L\|_{X^{\frac{1}{2}+\epsilon, \frac{1}{2}-\epsilon}} \|u_L\|_{X^{0,\frac{1}{2}-\epsilon}}\} \|w_H\|_{X^{0,\frac{1}{2}-\frac{\epsilon}{2}}}.$$

2. If v, w are high and u is low frequency functions, then

$$\left\| (u_L v_H - \sum_{l=-\infty}^{\infty} \hat{u}_L(l)\hat{v}_H(-l)) w_H \right\|_{X^{1-2\epsilon, -\frac{1}{2}+\epsilon}}$$
$$\lesssim \lambda^{0+} C \min\{\|u_L\|_{X^{\frac{1}{2}+\epsilon, \frac{1}{2}-\epsilon}} \|v_H\|_{X^{0,\frac{1}{2}-\epsilon}}, \|v_H\|_{X^{\frac{1}{2}+\epsilon, \frac{1}{2}-\epsilon}} \|u_L\|_{X^{0,\frac{1}{2}-\epsilon}}\} \|w_H\|_{X^{0,\frac{1}{2}-\frac{\epsilon}{2}}}.$$

3. If u, v and w all are high frequency functions, then

$$\left\| (u_H v_H - \sum_{l=-\infty}^{\infty} \hat{u}_H(l)\hat{v}_H(-l)) w_H \right\|_{X^{-2\epsilon, \frac{1}{2}+\epsilon}}$$
$$\lesssim \lambda^{0+} \|u_H\|_{X^{0,\frac{7}{18}+\epsilon}} \|v_H\|_{X^{0,\frac{7}{18}+\epsilon}} \|w_H\|_{X^{0,\frac{7}{18}+\epsilon}}.$$

Proof. **1.** We know that

$$\mathcal{F}_x \left[(u_L v_L - \sum_{l=-\infty}^{\infty} \hat{u}_L(l)\hat{v}_L(-l)) w_H \right] = \sum_{\substack{k_1+k_2+k_3+k_4=0 \\ k_1+k_2\neq 0 \\ (k_1+k_2)(k_2+k_3)(k_3+k_1)\neq 0}} \hat{u}_L(k_1)\hat{v}_L(k_2)\hat{w}_H(k_3),$$

where $\mathcal{F}_x$ denotes the Fourier transform in the x variable. Hence, we need to show that

$$\left\| \sum_k e^{ikx} \int_{\substack{k_1+k_2+k_3=k \\ (k_1+k_2)(k_2+k_3)(k_3+k_1)\neq 0}} \langle k_1 \rangle^{1-2\epsilon} \hat{u}_L(k_1)\hat{v}_L(k_2)\hat{w}_H(k_3) \right\|_{X^{0,-\frac{1}{2}+\epsilon}}$$
$$\lesssim C \min\{\|u_L\|_{X^{\frac{1}{2}+\epsilon, \frac{1}{2}-\epsilon}} \|v_L\|_{X^{0,\frac{1}{2}-\epsilon}}, \|v_L\|_{X^{\frac{1}{2}+\epsilon, \frac{1}{2}-\epsilon}} \|u_L\|_{X^{0,\frac{1}{2}-\epsilon}}\} \|w_H\|_{X^{0,\frac{1}{2}-\frac{\epsilon}{2}}}.$$

From duality, it is enough to show

$$\left| \int_{\substack{k_1+k_2+k_3=k \\ (k_1+k_2)(k_2+k_3)(k_3+k_1)\neq 0}} \int_{\sum_{i=1}^{4}\tau_i=0} \langle k_4\rangle^{1-2\epsilon}\tilde{u}_1(k_1)\tilde{u}_2(k_2)\tilde{u}_3(k_3)\tilde{u}_4(k_4) \right| \tag{3.3.15}$$
$$\lesssim C\min\{\|u_L\|_{X^{\frac{1}{2}+\epsilon,\frac{1}{2}-\epsilon}}\|v_L\|_{X^{0,\frac{1}{2}-\epsilon}}, \|v_L\|_{X^{\frac{1}{2}+\epsilon,\frac{1}{2}-\epsilon}}\|u_L\|_{X^{0,\frac{1}{2}-\epsilon}}\}\|w_H\|_{X^{0,\frac{1}{2}-\frac{\epsilon}{2}}}.$$

where $u_1 = u_L,\ u_2 = v_L,\ u_3 = w_H$ and let $u_4 = u_L+u_H$. Let $\sigma_i = \tau_i-4\pi^2k_i^3$ for $1 \leqslant i \leqslant 4$. We divide the proof into the following four cases:

1. Let $|\sigma_4| = \max\{|\sigma_i| \text{ for } 1 \leqslant i \leqslant 4\}$.
2. Let $|\sigma_1| = \max\{|\sigma_i| \text{ for } 1 \leqslant i \leqslant 4\}$.
3. Let $|\sigma_2| = \max\{|\sigma_i| \text{ for } 1 \leqslant i \leqslant 4\}$.
4. Let $|\sigma_3| = \max\{|\sigma_i| \text{ for } 1 \leqslant i \leqslant 4\}$.

It is enough to prove for Case 1 because other cases can be treated in the same way. According to the given conditions, we have $|k_1|, |k_2| \ll N'$ and $|k_3| \sim |k_4| \gtrsim N'$. So, from (3.3.6), $\langle\sigma_4\rangle \gtrsim \langle k_4\rangle^2|k_3 + |k_4|$ and $1/\lambda \leqslant |k_3 + k_4| \leqslant 1$. Let the region for the first integration is denoted as " $*$ " and the region of second integration is denoted as " $**$". By using Plancherel's theorem, Hölder's inequality, for the term (3.3.15), we get

$$\begin{aligned}
&\left|\int_*\int_{**}\langle k_4\rangle^{1-2\epsilon}\tilde{u}_1\tilde{u}_2\tilde{u}_3\tilde{u}_4\right| \\
&\lesssim \left|\int_*\int_{**}\langle k_4\rangle^{1-2\epsilon}\langle k_4\rangle^{-1+2\epsilon}(|k_1+k_2|^{-1/2}|\tilde{u}_1||\tilde{u}_2|)|\tilde{u}_3|(|\tilde{u}_4|\langle\sigma_4\rangle^{\frac{1}{2}-2\epsilon})\right|, \\
&\lesssim \|D_x^{-1/2}(v_1v_2)\|_{L^4_{x,t}}\|v_3\|_{L^4_{x,t}}\|\tilde{v}_4\langle\sigma_4\rangle^{\frac{1}{2}-2\epsilon}\|_{L^2_{k,\tau}}.
\end{aligned}$$

for $v_j = |u_j|$. From Sobolev embedding, inequality (3.3.13) and Proposition 3.2.2, we get the desired inequality.

2. We can prove this case along the similar line.

3. Form duality argument and Proposition 3.2.2, we get the desire estimate. □

Lemma 3.3.10.

$$\|u\|_{L^\infty_{x,t}} \lesssim \|u\|_{X^{\frac{1}{2}+\epsilon,\frac{1}{2}+\epsilon}}. \tag{3.3.16}$$

Proof.

$$\|u\|^2_{L^\infty_t L^2_x} = \sup_{t\in\mathbb{R}} \|U(-t)u(t)\|^2_{L^2_x},$$

where $U(t) = e^{-t\partial_x^3}$. By Sobolev embedding, we have

$$\begin{aligned}\sup_{t\in\mathbb{R}} \|U(-t)u(t)\|^2_{L^2_x} &\lesssim \int \sup_{t\in\mathbb{R}} |U(-t)u(t)|^2 dx \\ &\lesssim \int \langle\partial_t\rangle^{\frac{1}{2}+\epsilon} |U(-t)u(t)|^2 dx \\ &\sim \|u\|^2_{X^{0,\frac{1}{2}+\epsilon}}.\end{aligned}$$

Hence, we get

$$\|u\|^2_{L^\infty_t L^2_x} \lesssim \|u\|^2_{X^{0,\frac{1}{2}+\epsilon}}.$$

□

3.4 A Priori Estimate

In this section, we show a priori estimate of the solution to the mKdV equation which are needed for the proof of Theorem 3.1.1. The energy for the mKdV equation is given as:

$$E(u) = \int (\partial_x u)^2 - (u)^4 dx. \tag{3.4.1}$$

For the operator I', we have

$$E(I'v) = \int (\partial_x I'v)^2 - (I'v)^4 dx.$$

From equations (3.2.1)-(3.2.2), we obtain

$$\begin{aligned}\frac{\mathrm{d}(E(I'v))}{\mathrm{d}t} =& \left[\int (-\partial_x^2 I'v - (I'v)^3)(-\partial_x^3 I'v - \partial_x I'v^3)\right] \\ &+ \left[\int -\lambda^{-3}\partial_x^2 I'vI'g - \lambda^{-3}(I'v)^3 I'g + \frac{1}{2}(I'v)^4\gamma\lambda^{-3}\right].\end{aligned} \tag{3.4.2}$$

For a Banach space X, we define the space $L^\infty_{T'}X$ via the norm:

$$\|u\|_{L^\infty_{T'}X} = \sup_{t\in[0,T']} \|u(t)\|_X.$$

Multiply equation (3.2.1) by v and take L^2 norm to obtain the following lemma:

Lemma 3.4.1.

$$\|v(t)\|_{L^2}^2 \lesssim \|v_0\|_{L^2}\, exp(-\gamma\lambda^{-3}t) + \frac{\lambda^{-3}}{\gamma}\|g\|_{L_t^\infty L^2}^2(1 - exp(-\gamma\lambda^{-3}t)).$$

We establish the following lemma:

Lemma 3.4.2. *Let v is the solution of IVP (3.2.1)-(3.2.2) for $t \in [0, T']$. Then, we have*

$$\|I'v(T')\|_{L^2}^2 exp(\gamma\lambda^{-3}T') \leqslant C_1(\|v(0)\|_{L^2}^2 + \frac{1}{\gamma}\|g\|_{L^2}^2 exp(\gamma\lambda^{-3}T')) \tag{3.4.3}$$

and

$$\begin{aligned}\|I'v(T')\|_{\dot H^1}^2 exp(\gamma\lambda^{-3}T') \leqslant C_1\Big(&\|I'v(0)\|_{\dot H^1}^2 + \frac{1}{\gamma^2}\|I'g\|_{L_{T'}^\infty \dot H^1}^2 exp(\gamma\lambda^{-3}T') \\ &+\|v(0)\|_{L^2}^6 + \frac{1}{\gamma^4}\|g\|_{L^2}^6 exp(\gamma\lambda^{-3}T')\Big) + \left|\int_0^{T'} M(t)dt\right|,\end{aligned} \tag{3.4.4}$$

where

$$M(t) = exp(\gamma\lambda^{-3}t)\int_{\lambda\mathbb{T}}\{-\partial_x^2 I'v - (I'v)^3\}\{-\partial_x I'v^3 - \partial_x^3 I'v\}.$$

Proof. Similar to Lemma 3.4.1, we have

$$\begin{aligned}\frac{d}{dt}\|v(T')\|_{L^2}^2 exp(\gamma\lambda^{-3}T') &= \Big(-\gamma\lambda^{-3}\|v(t)\|_{L^2} + 2\lambda^{-3}\int_{\lambda\mathbb{T}} v(t)g(t)dx\Big)\, exp(\gamma\lambda^{-3}T') \\ &\leqslant \frac{\lambda^{-3}}{\gamma}\|g\|_{L^2}^2 exp(\gamma\lambda^{-3}T').\end{aligned}$$

Intriguing over $[0, T']$ and from the definition of operator I, we get (3.4.3).

From equations (3.2.1)-(3.2.2), we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(E(I'v(t))exp(\gamma\lambda^{-3}t')\Big) \\ =&\frac{\mathrm{d}}{\mathrm{d}t}E(I'v(t))exp(\gamma\lambda^{-3}t') + \gamma\lambda^{-3}E(I'v(t))exp(\gamma\lambda^{-3}t'), \\ =&\Big[\int\{-\partial_x^2 I'v - (I'v)^3\}\{\lambda^{-3}I'g - \gamma\lambda^{-3}I'v - \partial_x^3 I'v - \partial_x I'v^3\}\Big]\, exp(\gamma\lambda^{-3}t') \\ &+ \gamma\lambda^{-3}exp(\gamma\lambda^{-3}t')\int \frac{1}{2}(\partial_x I'v)^2 - \frac{1}{4}(I'v)^4, \\ =&\Big[\int(-\partial_x^2 I'v - (I'v)^3)(-\partial_x^3 I'v - \partial_x I'v^3)\Big]\, exp(\gamma\lambda^{-3}t') \\ &+\Big[\int(-\partial_x^2 I'v - (I'v)^3)(\lambda^{-3}I'g - \gamma\lambda^{-3}I'v)\Big]\, exp(\gamma\lambda^{-3}t')\end{aligned}$$

$$
\begin{aligned}
&+\gamma\lambda^{-3}exp(\gamma\lambda^{-3}t')\int\frac{1}{2}(\partial_x I'v)^2-\frac{1}{4}(I'v)^4,\\
=&M(t')+\left[\int-\lambda^{-3}\partial_x^2 I'vI'g-\lambda^{-3}(I'v)^3I'g-\frac{1}{2}\gamma\lambda^{-3}(\partial_x I'v)^2+\frac{3}{4}(I'v)^4\gamma\lambda^{-3}\right]exp(\gamma\lambda^{-3}t').
\end{aligned}
$$

Put the value of E, integrate over $[0,T']$, take absolute value on both side and from Gagliardo-Nirenberg inequality, we get

$$
\begin{aligned}
&\left(\|I'v(T')\|_{\dot H^1}^2-\|I'v(T')\|_{L^4}^4\right)exp(\gamma\lambda^{-3}T')\\
=&\|I'v(0)\|_{\dot H^1}^2-\|I'v(0)\|_{L^4}^4+\int_0^{T'}M(t')dt'+\int_0^{T'}\left[\int-\lambda^{-3}\partial_x^2I'vI'g-\lambda^{-3}(I'v)^3I'g\right.\\
&\left.-\frac{1}{2}\gamma\lambda^{-3}(\partial_xI'v)^2+\frac{3}{4}(I'v)^4\gamma\lambda^{-3}\right]exp(\gamma\lambda^{-3}t')dt',\\
\lesssim&\|I'v(0)\|_{\dot H^1}^2-\|I'v(0)\|_{L^4}^4+\left|\int_0^{T'}M(t')dt'\right|+\lambda^{-3}\int_0^{T'}\Big[\|I'g\|_{\dot H^1}\|I'v(t')\|_{\dot H^1}\\
&+\|I'v(t')\|_{\dot H^1}\|I'v(t')\|_{L^2}^2\|I'g\|_{L^2}-\gamma\frac{1}{2}\|I'v(t')\|_{\dot H^1}^2\\
&+\gamma\frac{3}{4}\|I'v(t')\|_{\dot H^1}\|I'v(t')\|_{L^2}^3\Big]exp(\gamma\lambda^{-3}t')dt'.
\end{aligned}
$$

From Young's inequality, we have

$$
\begin{aligned}
&\|I'v(T')\|_{\dot H^1}^2exp(\gamma\lambda^{-3}T')\lesssim\|I'v(0)\|_{\dot H^1}^2+\frac{1}{\gamma^2}\|I'g\|_{L^\infty_{T'}\dot H^1}^2exp(\gamma\lambda^{-3}T')\\
&+\left|\int_0^{T'}M(t')dt'\right|+C_1\|I'v(T')\|_{L^2}^6exp(\gamma\lambda^{-3}T')\\
&\quad+C_1\int_0^{T'}\left(\|I'v(t')\|_{L^2}^6+\frac{1}{\gamma^2}\|I'v(t')\|_{L^2}^4\|I'g\|_{L^2}^2\right)\gamma\lambda^{-3}exp(\gamma\lambda^{-3}t')dt'.
\end{aligned}
$$

From inequality (3.4.3) we get

$$
\left(\|I'v(t')\|_{L^2}^6+\frac{1}{\gamma^2}\|I'v(t')\|_{L^2}^4\|I'g\|_{L^2}^2\right)\lesssim\|I'v(0)\|_{L^2}^6exp(-3\gamma\lambda^{-3}t')+\frac{1}{\gamma^3}\|I'g\|_{L^2}^6.
$$

and hence we obtain inequality (3.4.4). □

Remark 3.4.3. *For mKdV equation, we just consider the half part of damping term in* $exp(\gamma\lambda^{-3}T')$ *as compare to KdV equation.*

We need to state the following Leibnitz rule type lemma:

Lemma 3.4.4.

$$\|f(t)g(x,t)\|_{X^{s,b}} \lesssim \|\hat{f}\|_{L^1}\|g\|_{X^{s,b}} + \|f\|_{H_t^b}\|\langle k\rangle^s \tilde{g}\|_{L^2_{(dk)_\lambda} L^1_{d\tau}}.$$

Proof. Assume that $\tau = \tau_1 + \tau_2$. Let $\sigma = \tau - k^3, \sigma_1 = \tau_1$ and $\sigma_2 = \tau_2 - k^3$. Then

$$\langle\sigma\rangle^b = \langle\tau - k^3\rangle^b \lesssim \langle\tau_1\rangle^b + \langle\tau - \tau_1 - k^3\rangle^b.$$

Hence

$$\langle\sigma\rangle^b\langle k\rangle^s \mathcal{F}[f(t)g(x,t)] = \langle\sigma\rangle^b\langle k\rangle^s \int_{\tau_1} \hat{f}(\tau_1)\tilde{g}(k,\tau-\tau_1)d\tau_1,$$
$$\lesssim \langle k\rangle^s \int_{\tau_1} \langle\tau_1\rangle^b|\hat{f}(\tau_1)\tilde{g}(k,\tau-\tau_1)| + \langle\tau-\tau_1-k^3\rangle^b|\hat{f}(\tau_1)\tilde{g}(k,\tau-\tau_1)|d\tau_1.$$

After summing over k and taking L^2 norm, we get

$$\|\langle\sigma\rangle^b\langle k\rangle^s\mathcal{F}[f(t)g(x,t)]\|_{L^2_{k,\tau}} \leqslant \|\langle k\rangle^s\langle\tau_1\rangle^b\hat{f}*\tilde{g}\|_{L^2_{t,k}} + \|\langle k\rangle^s\langle\tau-\tau_1-k^3\rangle^b\hat{f}*\tilde{g}\|_{L^2_{t,k}}.$$

From Young's inequality in τ, we obtain

$$\|\langle k\rangle^s\langle\tau_1\rangle^b\hat{f}*\tilde{g}\|_{L^2_\tau} + \|\langle k\rangle^s\langle\tau-\tau_1-k^3\rangle^b\hat{f}*\tilde{g}\|_{L^2_\tau} \lesssim \|\hat{f}\|_{L^1}\|g\|_{X^{s,b}} + \|f\|_{H_t^b}\|\langle k\rangle^s\tilde{g}\|_{L^2_{(dk)_\lambda} L^1_{d\tau}}.$$

□

Similar to [29, Proposition 3.1], we finally have the following proposition:

Proposition 3.4.5. *Let $\frac{1}{2} \leqslant s < 1$. Let $T > 0$ is given, $\epsilon > 0$ be sufficiently small and u be a solution of IVP (3.1.5)-(3.1.6) on $[0,T]$. Assume that $N^{\frac{1}{2}(1-\epsilon)} \geqslant \gamma, N^{\epsilon-} \geqslant C_6 T$ and*

$$(\|u(0)\|^2_{L^2} + \frac{1}{\gamma^2}\|f\|^2_{L^2} exp(\gamma T)) \leqslant N^{\frac{1}{6}(1-\epsilon)}C_3$$
$$(\|Iu(0)\|^2_{\dot{H}^1} + \frac{1}{\gamma^2}\|If\|^2_{\dot{H}^1} exp(\gamma T)) \leqslant N^{\frac{1}{6}(1-\epsilon)}C_3.$$

Then, we have

$$\|Iu(T)\|^2_{L^2} exp(\gamma T) \leqslant C_4(\|u(0)\|^2_{L^2} + \frac{1}{\gamma^2}\|f\|^2_{L^2} exp(\gamma T)),$$
$$\|Iu(T)\|^2_{\dot{H}^1} exp(\gamma T) \leqslant C_4(\|Iu(0)\|^2_{\dot{H}^1} + \|u(0)\|^6_{L^2} + \frac{1}{\gamma^4}\|f\|^6_{L^2} exp(\gamma T)$$
$$+ \frac{1}{\gamma^2}\|If\|^2_{\dot{H}^1} exp(\gamma T)) + (\|Iu(0)\|^2_{H^1} + \frac{1}{\gamma^2}\|If\|^2_{H^1} exp(\gamma T)),$$

where C_4 is independent of N and T.

Remark 3.4.6. *Without loss of generality, we can replace f with F as F is just a translation of f.*

We can rescale Proposition 3.4.5 by taking $\lambda = N^{\frac{1}{6}(1-\epsilon)}, N' = \frac{N}{\lambda}, T' = \lambda^3 T$. Also, we note that $\|I'v\|^2_{\dot{H}^1} = \lambda^{-3}\|Iu\|^2_{\dot{H}^1}, \|I'g\|^2_{L^\infty_{T'}\dot{H}^1} = \lambda^{-3}\|If\|^2_{\dot{H}^1}$. We rewrite Proposition 3.4.5 as following:

Proposition 3.4.7. *Let $\frac{1}{2} \leqslant s < 1$, $T' > 0$ is given and let v be a solution of IVP (3.2.1)-(3.2.2) on $[0, T']$. Assume that $\lambda^3 \geqslant \gamma$ and that for suitable $C_6, C_3 > 0$, $N'^{-}\lambda^{0-} \geqslant C_6 T' \lambda^2$ and*

$$(\|v(0)\|^2_{L^2} + \frac{1}{\gamma^2}\|g\|^2_{L^2} exp(\gamma\lambda^{-3}T')) \leqslant C_3$$
$$(\|I'v(0)\|^2_{\dot{H}^1} + \frac{1}{\gamma^2}\|I'g\|^2_{L^\infty_{T'}\dot{H}^1} exp(\gamma\lambda^{-3}T')) \leqslant C_3.$$

Then, we have

$$\begin{aligned}
\|I'v(T')\|^2_{L^2} exp(\gamma\lambda^{-3}T') &\leqslant C_4(\|v(0)\|^2_{L^2} + \frac{1}{\gamma^2}\|g\|^2_{L^2} exp(\gamma\lambda^{-3}T')) \\
\|I'v(T')\|^2_{\dot{H}^1} exp(\gamma\lambda^{-3}T') &\leqslant C_4(\|I'v(0)\|^2_{\dot{H}^1} + \frac{1}{\gamma^2}\|I'g\|^2_{L^\infty_{T'}\dot{H}^1} exp(\gamma\lambda^{-3}T') \\
&\quad + \|v(0)\|^6_{L^2} + \frac{1}{\gamma^4}\|g\|^6_{L^\infty_{T'}L^2} exp(\gamma\lambda^{-3}T')) \\
&\quad + \lambda^{-2}(\|I'v(0)\|^2_{H^1} + \frac{1}{\gamma^2}\|I'g\|^2_{H^1} exp(\gamma\lambda^{-3}T')),
\end{aligned}$$

where C_4 is independent of N', T' and λ.

Remark 3.4.8. *Because of non homogeneity of non homogeneous Sobolev space, we can not rescale the Proposition 3.4.5 into Proposition 3.4.7 with the order of rescaling factor as λ^{-3} like the KdV equation. Also, if we consider the homogeneous Sobolev space, the trilinear and multilinear estimates may not follows for counterexample see appendix. Therefore, we consider the non homogeneous Sobolev space with the rescaling estimate $\|I'v\|^2_{H^1} \lesssim \lambda^{-1}\|Iu\|^2_{H^1}$. We estimate L^2 and $\dot{H}^1$ separately to prove Proposition 3.4.7 in H^1. Although, it is not necessary for our problem to have the separate estimates but for the shake of general proof, we estimate it separately.*

Proof of Proposition 3.4.7. Take $\delta > 0$ and $j \in \mathbb{N}$ such that $\delta j = T'$ where $\delta \sim (\|I'v(0)\|_{H^1} + \|I'g\|_{L^\infty_{T'}H^1} + \gamma\lambda^{-3})^{-\alpha}$, $\alpha > 0$. For $0 \leqslant m \leqslant j$, $m \in \mathbb{Z}$, we prove

$$\begin{aligned}
&\|I'v(m\delta)\|^2_{\dot{H}^1} exp(\gamma\lambda^{-3}m\delta)\\
\leqslant& 2C_1(\|I'v(0)\|^2_{\dot{H}^1} + \|v(0)\|^6_{L^2} + \frac{1}{\gamma^2}\|I'g\|^2_{\dot{H}^1} exp(\gamma\lambda^{-3}m\delta)\\
&+ \frac{1}{\gamma^4}\|g\|^6_{L^2} exp(\gamma\lambda^{-3}k\delta)) + \lambda^{-2}(\|I'v(0)\|^2_{H^1} + \frac{1}{\gamma^2}\|I'g\|^2_{H^1} exp(\gamma\lambda^{-3}T'))\\
\leqslant& 4C_1C_3 + \lambda^{-2}(\|I'v(0)\|^2_{H^1} + \frac{1}{\gamma^2}\|I'g\|^2_{H^1} exp(\gamma\lambda^{-3}T'))
\end{aligned} \tag{3.4.5}$$

by induction.

For $m = 0$, (3.4.5) hold trivially. We assume (3.4.5) hold true for $m = l$ where $0 \leqslant l \leqslant j - 1$. From Lemma 3.4.2, we have

$$\begin{aligned}
&\|I'v((l+1)\delta)\|^2_{\dot{H}^1} exp(\gamma\lambda^{-3}(l+1)\delta) \leqslant C_1(\|I'v(0)\|^2_{\dot{H}^1} + \|v(0)\|^6_{L^2}\\
&+ \frac{1}{\gamma^2}\|I'g\|^2_{\dot{H}^1} exp(\gamma\lambda^{-3}(l+1)\delta) \ + \frac{1}{\gamma^4}\|g\|^6_{L^2} exp(\gamma\lambda^{-3}(l+1)\delta) + \left|\int_0^{(l+1)\delta} M(t)dt\right|
\end{aligned}$$

Therefore, it suffices to prove

$$\left|\int_0^{(l+1)\delta} M(t)dt\right| + \lesssim \lambda^{-2}(\|I'v(0)\|_{H^1} + \frac{1}{\gamma^2}\|I'g\|_{L^\infty_{(l+1)\delta}H^1} exp(\gamma\lambda^{-3}(l+1)\delta)).$$

If $\gamma = 0$ and $f = 0$ in Equation (3.4.2), then we have the following estimate:

Lemma 3.4.9.

$$\left|\int_0^{T'} M(t)dt\right| \lesssim \lambda^{0+}N'^{-1+}\|Iu\|^4_{X^{1,\frac{1}{2}}_{T'}} + \lambda^{0+}N'^{-2}\|Iu\|^6_{X^{1,\frac{1}{2}}_{T'}}.$$

We prove Lemma 3.4.9 in last section.

Lemma 3.4.9 implies that

$$\begin{aligned}
\left|\int_0^{(l+1)\delta} M(t)dt\right| &\sim \sum_{k=0}^{l}\left|\int_{k\delta}^{(k+1)\delta} M(x,t)dt\right|,\\
&\lesssim (N')^{-1+}\lambda^{0+}\sum_{k=0}^{l}\|exp(\frac{1}{4}\gamma\lambda^{-3}t)I'v\|^4_{X^{1,\frac{1}{2}}_{([0,\lambda]\times[k\delta,(k+1)\delta])}}
\end{aligned}$$

$$+ (N')^{-2}\lambda^{0+}\sum_{k=0}^{l}\|exp(\frac{1}{6}\gamma\lambda^{-3}t)I'v\|^6_{X^{1,\frac{1}{2}}_{([0,\lambda]\times[k\delta,(k+1)\delta])}}.$$

From Proposition 3.4.4, we obtain

$$\begin{aligned}
&\left|\int_0^{(l+1)\delta} M(t)dt\right| \\
\lesssim&(N')^{-1+}\lambda^{0+}\sum_{k=0}^{l}\|\widehat{exp(\gamma\lambda^{-3}}t)\|_{L^1_{[k\delta,(k+1)\delta]}}\|I'v\|^4_{X^{1,\frac{1}{2}}_{([0,\lambda]\times[k\delta,(k+1)\delta])}} \\
&+ (N')^{-1+}\lambda^{0+}\sum_{k=0}^{l}\|exp(\gamma\lambda^{-3}t)\|_{H^{\frac{1}{2}}_{[k\delta,(k+1)\delta]}}\|\langle k\rangle^s\tilde{I'v}\|^4_{L^2_{[0,\lambda]}L^1_{[k\delta,(k+1)\delta]}} \\
&+ (N')^{-2}\lambda^{0+}\sum_{k=0}^{l}\|\widehat{exp(\gamma\lambda^{-3}}t)\|_{L^1_{[k\delta,(k+1)\delta]}}\|I'v\|^6_{X^{1,\frac{1}{2}}_{([0,\lambda]\times[k\delta,(k+1)\delta])}} \\
&+ (N')^{-2}\lambda^{0+}\sum_{k=0}^{l}\|exp(\gamma\lambda^{-3}t)\|_{H^{\frac{1}{2}}_{[k\delta,(k+1)\delta]}}\|\langle k\rangle^s\tilde{I'v}\|^6_{L^2_{[0,\lambda]}L^1_{[k\delta,(k+1)\delta]}}.
\end{aligned}$$

From simple computations, we can verify that

$$\max_{0\leqslant l\leqslant k}\|\widehat{exp(\gamma\lambda^{-3}}t)\|_{L^1_{[l\delta,(l+1)\delta]}} \lesssim C\,exp(\gamma\lambda^{-3}(l+1)\delta)$$

and

$$\max_{0\leqslant l\leqslant k}\|exp(\gamma\lambda^{-3}t)\|_{H^{\frac{1}{2}}_{[l\delta,(l+1)\delta]}} \lesssim C\,exp(\gamma\lambda^{-3}(l+1)\delta)$$

are bounded. From the first inequality of Proposition 3.2.5, we have

$$\|I'v\|^4_{X^{1,\frac{1}{2}}_{([0,\lambda]\times[k\delta,(k+1)\delta])}} + \|\langle\partial_x\rangle I'v\|^4_{L^2_{[0,\lambda]}L^1_{[k\delta,(k+1)\delta]}} \lesssim \|I'v(k\delta)\|^4_{H^1_{[0,\lambda]}} + (\lambda^{-3}\|I'g\|)^4_{L^\infty_{(l+1)\delta}H^1_{[0,\lambda]}}. \tag{3.4.6}$$

$$\|I'v\|^6_{X^{1,\frac{1}{2}}_{([0,\lambda]\times[k\delta,(k+1)\delta])}} + \|\langle\partial_x\rangle I'v\|^6_{L^2_{[0,\lambda]}L^1_{[k\delta,(k+1)\delta]}} \lesssim \|I'v(k\delta)\|^6_{H^1_{[0,\lambda]}} + (\lambda^{-3}\|I'g\|)^6_{L^\infty_{(l+1)\delta}H^1_{[0,\lambda]}}. \tag{3.4.7}$$

Therefore, we have

$$\left|\int_0^{(l+1)\delta} M(t)dt\right| \lesssim (C_6\lambda^2T')^{-1}\sum_{k=0}^{l}(\|I'v(k\delta)\|^4_{H^1_{[0,\lambda]}} + (\lambda^{-3}\|I'g\|)^4_{L^\infty_{(l+1)\delta}H^1_{[0,\lambda]}}exp(\gamma\lambda^{-3}(l+1)\delta)). \tag{3.4.8}$$

From inequalities (3.4.6),(3.4.7) and the assumption in Proposition 3.4.7, we get

$$\left|\int_0^{(l+1)\delta} M(t)dt\right| \lesssim 2(C_6\lambda^2 T')^{-1}C_3(C_1^2+C_1^3)(l+1)(\|I'v(0)\|_{H^1} + \frac{1}{\gamma^2}\|I'g\|^2_{L^\infty_{T'}H^1_{[0,\lambda]}} exp(2\gamma\lambda^{-3}(l+1)\delta)).$$

We choose C_6 sufficiently large such that $2(C_6T')^{-1}C_3(C_1^2+C_1^3)(l+1) \leqslant 2(C_6\delta)^{-1}C_3(C_1^2+C_1^3) \ll 1$, which leads to Proposition 3.4.7. □

3.5 Proof of Theorem 3.1.1

In this section, we describe the proof of Theorem 3.1.1.

Proof of Theorem 3.1.1. Let $0 < \epsilon \ll 12s - 11$ be fixed. We choose $T_1 > 0$ so that

$$exp(\gamma T_1) > (\|u_0\|^2_{H^s} + \|u_0\|^6_{L^2})(\frac{1}{\gamma^2}\|f\|^2_{H^1} + \frac{1}{\gamma^4}\|f\|^6_{L^2})^{-1} max\left\{\gamma^{\frac{4(1-s)}{1-\epsilon}}, (C_6T_1)^{\frac{2(1-s)}{\epsilon-}}, \left(\frac{C_3}{2}\|u_0\|^{-2}_{H^s}\right)^{\frac{12(s-1)}{(1-\epsilon)+12(s-1)}}, \left(2C_3^{-1}\gamma^{-2}\|f\|^2_{H^1}exp(\gamma T_1)\right)^{\frac{6(-2s+2)}{1-\epsilon}}\right\}, \tag{3.5.1}$$

which is possible as $\frac{6(-2s+2)}{1-\epsilon} < 1$. T_1 depends only on $\|u_0\|_{H^s}$, $\|f\|_{H^1}$ and γ. Set

$$N = max\left\{\gamma^{\frac{2}{1-\epsilon}}, (C_6T_1)^{\frac{1}{\epsilon-}}, \left(\frac{C_3}{2}\|u_0\|^{-2}_{H^s}\right)^{\frac{-6}{12(1-s)+(1-\epsilon)}}, \left(2C_2^{-1}\gamma^{-2}\|f\|^2_{H^1}e^{2\gamma T_1}\right)^{\frac{6}{1-\epsilon}}\right\}. \tag{3.5.2}$$

From the choice of T_1 and N, we know

$$N^{\frac{1-\epsilon}{2}} \geqslant \gamma, \qquad N^{\epsilon-} \geqslant C_6T_1,$$

and

$$\|Iu_0\|^2_{H^1} \leqslant N^{2-2s}\|u_0\|^2_{H^s} \leqslant \frac{C_3}{2}N^{\frac{1-\epsilon}{6}-},$$
$$\gamma^{-2}\|If\|^2_{H^1}e^{2\gamma T_1} \leqslant \frac{C_3}{2}N^{\frac{1-\epsilon}{6}-}.$$

Hence, from Proposition 3.4.5, we gains

$$\|u(T_1)\|^2_{H^s} \leqslant \|Iu(T_1)\|^2_{H^1}$$

$$\leqslant C_3(\|Iu_0\|_{H^1}^2 exp(-\gamma T_1) + \|u_0\|_{L^2}^6 exp(-\gamma T_1) + \frac{1}{\gamma^2}\|If\|_{H^1}^2 + \frac{1}{\gamma^4}\|f\|_{L^2}^6)$$
$$\leqslant C_3(N^{2(1-s)}(\|u_0\|_{H^s}^2 exp(-\gamma T_1) + \|u_0\|_{L^2}^6 exp(-\gamma T_1)) + \frac{1}{\gamma^2}\|f\|_{H^1}^2 + \frac{1}{\gamma^4}\|f\|_{L^2}^6).$$

From (3.5.1) and (3.5.2) , we get

$$N^{2(1-s)} exp(-\gamma T_1)(\|u_0\|_{H^s}^2 + \|u_0\|_{L^2}^6) < \frac{1}{\gamma^2}\|f\|_{H^1}^2 + \frac{1}{\gamma^4}\|f\|_{L^2}^6$$

which helps us give the bound

$$\|u(T_1)\|_{H^s}^2 \leqslant 2C_3(\frac{1}{\gamma^2}\|f\|_{H^1}^2 + \frac{1}{\gamma^4}\|f\|_{L^2}^6) < K_1,$$

where K_1 depends only on $\|f\|_{H^1}$ and γ.

In the next place, one can fix $T_2 > 0$ and solve mKdV equation on time interval $[T_1, T_1 + T_2]$ with initial data replaced by $u(T_1)$. Let $K_2 > 0$ be sufficiently large such that

$$K_2 exp(\gamma t) > (\|u_0\|_{H^s}^2 + \|u_0\|_{L^2}^6)(\frac{1}{\gamma^2}\|f\|_{H^1}^2 + \frac{1}{\gamma^4}\|f\|_{L^2}^6)^{-1} max\Big\{\gamma^{\frac{4(1-s)}{1-\epsilon}}, (C_6 t)^{\frac{2(1-s)}{\epsilon-}},$$
$$\Big((C_3)^{-1}2K_1\Big)^{\frac{12(s-1)}{(1-\epsilon)+12(s-1)}}, \Big(2C_3^{-1}\gamma^{-2}\|f\|_{H^1}^2 exp(\gamma T_1)\Big)^{\frac{6(-2s+2)}{1-\epsilon}}\Big\}, \qquad (3.5.3)$$

for any $t > 0$. Set $N^{2(1-s)} = K_2 exp(\gamma T_2)$, then inequality (3.5.3) verifies the assumptions in Proposition 3.4.5 and hence we obtain

$$\|Iu(T_1 + T_2)\|_{H^1}^2 \leqslant C_4(N^{2(1-s)}\|u(T_1)\|_{H^s}^2 exp(\gamma T_2) + \|u(T_1)\|_{L^2}^6 exp(-\gamma T_2) + \frac{1}{\gamma^2}\|f\|_{H^1}^2 + \frac{1}{\gamma^4}\|f\|_{L^2}^6)$$
$$\leqslant C_4(K_1K_2 + K_1^2 + \frac{1}{\gamma^2}\|f\|_{H^1}^2 + \frac{1}{\gamma^4}\|f\|_{L^2}^6) < K_3.$$

For $t > T_1$, we define the maps $L_1(t)$ and $L_2(t)$ as

$$\widehat{L_1(t)u_0} = \widehat{S(t)u_0}|_{|\zeta|<N_t}, \quad \widehat{L_2(t)u_0} = \widehat{S(t)u_0}|_{|\zeta|>N_t},$$

where $S(t)u_0 = u(t)$ and $N_t = (K_2 exp(\gamma(t - T_1))^{-\frac{1}{2(1-s)}}$.

It's easy to see that for $t > T_1$,

$$\|L_1(t)u_0\|_{H^1}^2 \leqslant \|Iu(t)\|_{H^1}^2 < K_3,$$

$$\|L_2(t)u_0\|_{H^s}^2 \leqslant N^{2s-2}\|Iu(t)\|_{H^1}^2 < K_2^{-1}K_3 ezp(-\gamma(t-T_1)).$$

Hence we obtain Theorem 3.1.1 by taking $K = max\{K_3^{\frac{1}{2}}, K_2^{-\frac{1}{2}}K_3^{\frac{1}{2}}\}$. □

3.6 Multilinear Estimates

In this section, we prove the 4-linear and 6-linear estimates given in Lemma 3.4.9.

Proof of Lemma 3.4.9. For $\gamma = 0$ and $g = 0$ in (3.4.2), we have

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \left[\int(-\partial_x^2 I'v - (I'v)^3)(-\partial_x^3 I'v - \partial_x I'v^3)\right],$$

$$\begin{aligned} E(I'v(T)) - E(I'v(0)) &= \int_0^T\int_0^\lambda \partial_x^3 I'v[(I'v)^3 - I'v^3]dxdt + \int_0^T\int_0^\lambda \partial_x(I'v)^3[(I'v)^3 - I'v^3]dxdt, \\ &= I_1' + I_2', \end{aligned}$$

for any arbitrary $T > 0$. For an $\epsilon > 0$ let $w_j \in X^{s,\frac{1}{2}}$ such that $w|_{[0,\lambda]\times[0,T]} = v_j$ and $\|v_j\|_{X_T^{s,\frac{1}{2}}} \leqslant C\|w_j\|_{X^{s,\frac{1}{2}}} \leqslant C\|v_j\|_{X_T^{s,\frac{1}{2}+\epsilon}}$ for $1 \leqslant j \leqslant 4$. Let $\eta_T(t) = \eta(t/T)$ and let $\tilde{\eta}$ denotes the Fourier transform only in t. From the Plancherel's theorem, it suffices to prove the following:

$$\begin{aligned} I_1' = \int_{\mathbb{R}}\int_0^\lambda \eta(t)\partial_x^3 I'w[(I'w)^3 - I'w^3]dxdt, \lesssim \int_{\substack{k_1+k_2+k_3+k_4=0 \\ (k_1+k_2)(k_2+k_3)(k_3+k_1)\neq 0}} \int \tilde{\eta}(\tau_1+\tau_2+\tau_3+\tau_4) \\ \left|\langle k_1\rangle^3(\widetilde{I'w_1})\left(1 - \frac{m(k_2+k_3+k_4)}{m(k_2)m(k_3)m(k_4)}\right)(\widetilde{I'w_2})(\widetilde{I'w_3})(\widetilde{I'w_4})\right|(dk_i)_\lambda d\tau_i \\ + \int_\Omega\int\left|\langle k_1\rangle^3(\widetilde{I'w_1})\left(1 - \frac{m(k_2+k_3+k_4)}{m(k_2)m(k_3)m(k_4)}\right)(\widetilde{I'w_2})(\widetilde{I'w_3})(\widetilde{I'w_4})\right|(dk_i)_\lambda d\tau_i, = I_{11} + I_{12}, \end{aligned}$$

where $\Omega = \{k_1+k_2+k_3+k_4 = 0 : \ |k_1+k_2| \neq 0, \ (|k_2+k_3||(k_3+k_1|) = 0\}$ and $w_i = w_i(k_i,\tau_i)$. Let $w = w_L + w_H$ where *supp* $\hat{w}_L(k) \subset \{|k| \ll N'\}$ and *supp* $\hat{w}_H(k) \subset \{|k| \gtrsim N'\}$. From dyadic partition of $|k_i|$, we let $|k_i| \sim N_i'$. Let $\sigma_i = \tau_i - 4\pi^2 k_i^3$ for $1 \leqslant i \leqslant 4$. We can assume that $\langle\sigma_4\rangle = \max\{\langle\sigma_i\rangle, \ 1 \leqslant i \leqslant 4\}$ as all other cases can be treated in the same way. Let $*$ be the region of integration for I_{11}. After substituting $w = w_L + w_H$, we can write I_{11} as a sum of the following three integrals:

Integral 1.

$$\int_* \int \tilde{\eta}(\tau_1 + \tau_2 + \tau_3 + \tau_4) \Bigg| \langle k_1 \rangle^3 (\widetilde{I' w_H}) \left(1 - \frac{m(k_2 + k_3 + k_4)}{m(k_2)m(k_3)m(k_4)}\right) (\widetilde{I' w_L})(\widetilde{I' w_L})(\widetilde{I' w_H}) \Bigg| (dk_i)_\lambda d\tau_i. \tag{3.6.1}$$

Integral 2.

$$\int_* \int \tilde{\eta}(\tau_1 + \tau_2 + \tau_3 + \tau_4) \Bigg| \langle k_1 \rangle^3 (\widetilde{I' w_H}) \left(1 - \frac{m(k_2 + k_3 + k_4)}{m(k_2)m(k_3)m(k_4)}\right) (\widetilde{I' w_L})(\widetilde{I' w_H})(\widetilde{I' w_H}) \Bigg| (dk_i)_\lambda d\tau_i. \tag{3.6.2}$$

Integral 3.

$$\int_* \int \tilde{\eta}(\tau_1 + \tau_2 + \tau_3 + \tau_4) \Bigg| \langle k_1 \rangle^3 (\widetilde{I' w_H}) \left(1 - \frac{m(k_2 + k_3 + k_4)}{m(k_2)m(k_3)m(k_4)}\right) (\widetilde{I' w_H})(\widetilde{I' w_H})(\widetilde{I' w_H}) \Bigg| (dk_i)_\lambda d\tau_i. \tag{3.6.3}$$

Remark 3.6.1. *We omit other cases as they follows in the similar manner.*

Integral 1. For this case, we have $|k_1| \sim |k_4| \gtrsim N'$ and $|k_2| \sim |k_3| \ll N'$. Hence, by using mean value theorem, we get

$$\left| \left(1 - \frac{m(k_2 + k_3 + k_4)}{m(k_2)m(k_3)m(k_4)}\right) \right| \lesssim \frac{|k_2| + |k_3|}{|k_4|}.$$

For *Integral 1*, we get

$$\textit{Integral 1} \lesssim N_4^{-1+2\epsilon} \int_* \int \tilde{\eta}(\tau_1 + \tau_2 + \tau_3 + \tau_4)(\langle k_1 \rangle \widetilde{I' w_H} \langle \sigma \rangle^{\frac{1}{2}}) \Big[\langle k_1 \rangle \{ (|k_2| \widetilde{I' w_L})(\widetilde{I' w_L}) + (\widetilde{I' w_L})(|k_3| \widetilde{I' w_L}) \} (\langle k_1 \rangle \widetilde{I' w_H}) \langle \sigma \rangle^{-\frac{1}{2}}) \Big].$$

Plancherel's theorem, Schwarz's inequality and Corollary 3.3.9(1) imply

$$\textit{Integral 1} \lesssim \lambda^{0+} N'^{-1+2\epsilon} \|I' w_H\|_{X^{1,\frac{1}{2}}} \|I' w_L\|_{X^{1,\frac{1}{2}}} (N_3)^{-\frac{1}{2}} \|I' w_L\|_{X^{1,\frac{1}{2}}} \|I' w_H\|_{X^{1,\frac{1}{2}}},$$

$$\lesssim \lambda^{0+} N'^{-1+2\epsilon} \|I'w\|^4_{X^{1,\frac{1}{2}}}.$$

Note that, we neglect $(N_3)^{-\frac{1}{2}}$ as it is not contributing in the decay.

Integral 2. From given conditions, we have $|k_1| \sim |k_4| \gg |k_3| \gtrsim N'$ and $|k_2| \ll N'$. Also, the definition of m implies $m(k_2) \sim 1$. Therefore,

$$\begin{aligned}\left|\left(1 - \frac{m(k_2+k_3+k_4)}{m(k_2)m(k_3)m(k_4)}\right)\right| &\lesssim \frac{m(k_1)}{m(k_2)m(k_3)m(k_4)} \\ &\sim \frac{1}{m(k_3)} \\ &\lesssim N'^{-1+s}|k_3|^{1-s} \\ &\lesssim N'^{-1}|k_3|.\end{aligned}$$

For *Integral 2*, we get

Integral 2

$$\lesssim N'^{-1+2\epsilon} \int_* \int \tilde{\eta}(\tau_1+\tau_2+\tau_3+\tau_4)(\langle k_1\rangle \widetilde{I'w_H}\langle\sigma\rangle^{\frac{1}{2}})\Big[\langle k_1\rangle(\widetilde{I'w_L})(|k_3|\widetilde{I'w_H})(\langle k_1\rangle \widetilde{I'w_H})\langle\sigma\rangle^{-\frac{1}{2}})\Big].$$

From Plancherel's theorem, Schwarz's inequality and Corollary 3.3.9(2), we have

$$\begin{aligned}\textit{Integral 2} &\lesssim N'^{-1+2\epsilon} N_2^{-\frac{1}{2}} \|I'w_L\|_{X^{1,\frac{1}{2}}} \|I'w_H\|_{X^{1,\frac{1}{2}}} \|I'w_H\|_{X^{1,\frac{1}{2}}} \|I'w_L\|_{X^{1,\frac{1}{2}}} \\ &\lesssim N'^{-1+2\epsilon} \|I'w\|^4_{X^{1,\frac{1}{2}}}.\end{aligned}$$

Integral 3. Clearly, we have $|k_1| \sim |k_2| \sim |k_3| \sim |k_4| \gtrsim N'$. Hence, from definition of m, we have

$$\begin{aligned}\left|\left(1 - \frac{m(k_2+k_3+k_4)}{m(k_2)m(k_3)m(k_4)}\right)\right| &\lesssim \frac{m(k_1)}{m(k_2)m(k_3)m(k_4)} \\ &\sim \frac{N'^{-2s+2}|k_1|^{s-1}}{|k_2|^{s-1}|k_3|^{s-1}|k_4|^{s-1}}|k_4||k_4|^{-1} \\ &\lesssim N'^{-2+2s}|k_2|^{1-s}|k_3|^{1-s}|k_4|^{1-s}|k_1|^{s-1}|k_4||k_4|^{-1} \\ &\lesssim N'^{-1}|k_4|,\end{aligned}$$

for $1/2 \leqslant s < 1$. Therefore, *Integral 3* implies

Integral 3

$$\lesssim N'^{-1+2\epsilon} \int_* \int \tilde{\eta}(\tau_1+\tau_2+\tau_3+\tau_4)(\langle k_1\rangle \widetilde{I'w_H}\langle\sigma\rangle^{\frac{1}{2}})\Big[(\langle k_1\rangle \widetilde{I'w_H})(\langle k_1\rangle \widetilde{I'w_H})(|k_4|\widetilde{I'w_H})\langle\sigma\rangle^{-\frac{1}{2}})\Big].$$

From Plancherel's theorem, Schwarz's inequality and Corollary 3.3.9(3), we have

$$\begin{aligned} \textit{Integral 3} &\lesssim \lambda^{0+}N'^{-1+2\epsilon}\|I'w_H\|_{X^{1,\frac{7}{18}+}}\|I'w_H\|_{X^{1,\frac{7}{18}+}}\|I'w_H\|_{X^{1,\frac{7}{18}+}}\|I'w_H\|_{X^{1,\frac{7}{18}+}} \\ &\lesssim \lambda^{0+}N'^{-1+2\epsilon}\|I'w\|^4_{X^{1,\frac{1}{2}}}. \end{aligned}$$

Remark 3.6.2. *Note that*

$$\left[k_1^3\left(1-\frac{m(k_2+k_3+k_4)}{m(k_2)m(k_3)m(k_4)}\right)\right]_{sym} = \sum_{j=1}^4 k_j^3 - \frac{1}{m_1m_2m_3m_4}\sum_{j=1}^4 k_j^3 m_j^2$$

for details (see [8, Section 4]). Although, even after using symmetrization, we are not able to improve the decay for the above 4-linear estimate for nonresonant frequencies. Although, this symmetrization leads to the cancellation in the resonant case.

Hence, for the term I_{11}, the estimate holds. For I_{12}, we use the symmetrization as follow:

Case 1. $k_2+k_3=0$.

Case 2. $k_1+k_3=0$.

Case 1. Clearly, we have $k_2=-k_3$ and $k_1=-k_4$. Therefore, from Remark 3.6.2, we have

$$\left[k_1^3\left(1-\frac{m(k_2+k_3+k_4)}{m(k_2)m(k_3)m(k_4)}\right)\right]_{sym} = \sum_{j=1}^4 k_j^3 - \frac{1}{m_1m_2m_3m_4}\sum_{j=1}^4 k_j^3 m_j^2,$$

which vanishes for $k_1=-k_4$ and $k_2=-k_3$.

Case 2. This case is similar to **Case 1.**

Now, we consider I_2. From the Fourier transformation, we get

$$\begin{aligned} I_2 &= \int_0^T\int_0^\lambda \partial_x(I'v)^3[(I'v)^3 - I'v^3]dxdt, \\ &\lesssim \int_{\sum_{i=1}^6 k_i=0}\int_{\sum_{i=1}^6 \tau_i=0}\Big|\langle k_1+k_2+k_3\rangle(\widetilde{I'v_1})(\widetilde{I'v_2})(\widetilde{I'v_3}) \\ &\qquad \left(1-\frac{m(k_4+k_5+k_6)}{m(k_4)m(k_5)m(k_6)}\right)(\widetilde{I'v_4})(\widetilde{I'v_5})(\widetilde{I'v_6})\Big|(dk_i)_\lambda d\tau_i, \end{aligned}$$

We may suppose $\langle k_1 \rangle = \max\{\langle k_i \rangle, 1 \leqslant i \leqslant 3\}$. Putting $v = v_L + v_H$, we divide the integral I_2 into the following three integrals:

Integral 4.

$$\int_{\sum_{i=1}^{6} k_i=0} \int_{\sum_{i=1}^{6} \tau_i=0} (\langle k_1 \rangle \widetilde{I'v_H})(\widetilde{I'v_L} + \widetilde{I'v_H})(\widetilde{I'v_L} + \widetilde{I'v_H}) \left(1 - \frac{m(k_4+k_5+k_6)}{m(k_4)m(k_5)m(k_6)}\right)$$
$$(\widetilde{I'v_L})(\widetilde{I'v_L})(\widetilde{I'v_H})(dk_i)_\lambda d\tau_i.$$

Integral 5.

$$\int_{\sum_{i=1}^{6} k_i=0} \int_{\sum_{i=1}^{6} \tau_i=0} (\langle k_1 \rangle \widetilde{I'w_H})(\widetilde{I'v_L} + \widetilde{I'v_H})(\widetilde{I'v_L} + \widetilde{I'v_H}) \left(1 - \frac{m(k_4+k_5+k_6)}{m(k_4)m(k_5)m(k_6)}\right)$$
$$(\widetilde{I'v_H})(\widetilde{I'v_H})(\widetilde{I'v_L})(dk_i)_\lambda d\tau_i.$$

Integral 6.

$$\int_{\sum_{i=1}^{6} k_i=0} \int_{\sum_{i=1}^{6} \tau_i=0} (\langle k_1 \rangle \widetilde{I'v_H})(\widetilde{I'v_L} + \widetilde{I'v_H})(\widetilde{I'v_L} + \widetilde{I'v_H}) \left(1 - \frac{m(k_4+k_5+k_6)}{m(k_4)m(k_5)m(k_6)}\right)$$
$$(\widetilde{I'v_H})(\widetilde{I'v_H})(\widetilde{I'v_H})(dk_i)_\lambda d\tau_i.$$

Integral 4. Clearly, we have $|k_4|, |k_5| \ll N'$ and $|k_6| \gtrsim N'$. Hence, the worst condition is $|k_3|, |k_2| \ll N'$ and $|k_1| \gtrsim N'$. The proof is the same as in I_1. From the mean value theorem, we get

$$\left| \left(1 - \frac{m(k_4+k_5+k_6)}{m(k_4)m(k_5)m(k_6)}\right) \right| \lesssim \frac{|k_4| + |k_5|}{|k_6|}. \tag{3.6.4}$$

We may assume $\langle \sigma_1 \rangle = \max\{\langle \sigma_i \rangle \ : \ 1 \leqslant i \leqslant 6\}$ as other cases can be treated in the same way. Therefore,

$$\langle \sigma_1 \rangle^{2\epsilon} = \langle \sigma_1 \rangle^{3\epsilon} \langle \sigma_1 \rangle^{-\epsilon} \lesssim \langle \sigma_1 \rangle^{3\epsilon} \langle \sigma_2 \rangle^{-\frac{\epsilon}{2}} \min\{\langle \sigma_3 \rangle^{-\frac{\epsilon}{2}}, \langle \sigma_6 \rangle^{-\frac{\epsilon}{2}}\}. \tag{3.6.5}$$

From Plancherel's theorem, Hölder's inequality, Proposition 3.2.2, Lemma 3.3.10 and inequalities (3.6.4) and (3.6.5), we get

$$\begin{aligned}\textit{Integral 4} \lesssim N'^{-1} \|\mathcal{F}^{-1}(\langle\sigma\rangle^{3\epsilon}\langle k_1\rangle \widetilde{I'v_H})\|_{L^4_{x,t}} \|\mathcal{F}^{-1}(\langle\sigma_2\rangle^{-\frac{\epsilon}{2}} \widetilde{I'v_L})\|_{L^\infty_{x,t}} \|\mathcal{F}^{-1}(\langle\sigma_3\rangle^{-\frac{\epsilon}{2}} \widetilde{I'v_L})\|_{L^\infty_{x,t}} \\ \|\mathcal{F}^{-1}(\langle k_4\rangle \widetilde{I'v_L})\|_{L^4_{x,t}} \|I'v_L\|_{L^4_{x,t}} \|I'v_H)\|_{L^4_{x,t}} \\ \lesssim N'^{-2} \|I'v_H\|_{X^{1,\frac{1}{3}+4\epsilon}} \|I'v_L\|_{X^{\frac{1}{2}+\epsilon,\frac{1}{2}-\frac{\epsilon}{2}}} \|I'v_L\|_{X^{\frac{1}{2}+\epsilon,\frac{1}{2}-\frac{\epsilon}{2}}} \|I'v_L\|_{X^{1,\frac{1}{3}+\epsilon}} \\ \|I'v_L\|_{X^{0,\frac{1}{3}+\epsilon}} \|I'v_H\|_{X^{1,\frac{1}{3}+\epsilon}}.\end{aligned}$$

We neglect extra derivatives corresponding to N_2, N_3 and N_5 to get

$$\textit{Integral 4} \lesssim N'^{-2} \|I'v\|^6_{X^{1,\frac{1}{2}}}.$$

Integral 5. Clearly, we have $|k_4|, |k_5| \gtrsim N'$ and $|k_6| \ll N'$. Hence, the worst condition is $|k_3| \ll N'$ and $|k_1|, |k_2| \gtrsim N'$ as $|k_1|$ always have high frequency. From definition of m, we get

$$\left|\left(1 - \frac{m(k_4+k_5+k_6)}{m(k_4)m(k_5)m(k_6)}\right)\right| \lesssim \left|\frac{m(k_1)}{m(k_4)m(k_5)}\right| \lesssim N'^{-1}N_5. \tag{3.6.6}$$

From Plancherel's theorem, Hölder's inequality, Proposition 3.2.2, Lemma 3.3.10 and inequalities (3.6.5) and (3.6.6), we get

$$\begin{aligned}\textit{Integral 5} \lesssim N'^{-1} \|\mathcal{F}^{-1}(\langle\sigma\rangle^{3\epsilon}\langle k_1\rangle \widetilde{I'v_H})\|_{L^4_{x,t}} \|I'v_H\|_{L^4_{x,t}} \|\mathcal{F}^{-1}(\langle\sigma_3\rangle^{-\frac{\epsilon}{2}} \widehat{I'v_L})\|_{L^\infty_{x,t}} \|I'v_H\|_{L^4_{x,t}} \\ \|\mathcal{F}^{-1}(\langle k_5\rangle \widetilde{I'v_H})\|_{L^4_{x,t}} \|\mathcal{F}^{-1}(\langle\sigma_6\rangle^{-\frac{\epsilon}{2}} \widetilde{I'v_H})\|_{L^4_{x,t}} \\ \lesssim N'^{-1} \|I'v_H\|_{X^{1,\frac{1}{3}+4\epsilon}} \|I'v_H\|_{X^{0,\frac{1}{3}+\epsilon}} \|I'v_L\|_{X^{\frac{1}{2}+\epsilon,\frac{1}{2}-\frac{\epsilon}{2}}} \|I'v_H\|_{X^{0,\frac{1}{3}+\epsilon}} \\ \|I'v_H\|_{X^{1,\frac{1}{3}+\epsilon}} \|I'v_L\|_{X^{\frac{1}{2}+\epsilon,\frac{1}{2}-\frac{\epsilon}{2}}}.\end{aligned}$$

We neglect extra derivatives corresponding to N_3 and N_6 to get

$$\textit{Integral 4} \lesssim N'^{-3} \|I'v\|^6_{X^{1,\frac{1}{2}}}.$$

Integral 6. Clearly, we have $|k_4|, |k_5|, |k_6| \gtrsim N'$. Hence, the worst condition is $|k_3|, |k_2| \ll N'$ and $|k_1| \gtrsim N'$.. From definition of m, we get

$$\left|\left(1 - \frac{m(k_4+k_5+k_6)}{m(k_4)m(k_5)m(k_6)}\right)\right| \lesssim \left|\frac{m(k_1)}{m(k_4)m(k_5)m(k_6)}\right| \lesssim N'^{-2}|k_5||k_6|. \tag{3.6.7}$$

From Plancherel's theorem, Hölder's inequality, Proposition 3.2.2, Lemma 3.3.10 and inequalities (3.6.5) and (3.6.7), we get

$$\begin{aligned}\textit{Integral 6} \lesssim N'^{-2} \|\mathcal{F}^{-1}(\langle\sigma\rangle^{3\epsilon}\langle k_1\rangle\widetilde{I'v_H})\|_{L^4_{x,t}} \|\mathcal{F}^{-1}(\langle\sigma_2\rangle^{-\frac{\epsilon}{2}}\widetilde{I'v_L})\|_{L^\infty_{x,t}} \|\mathcal{F}^{-1}(\langle\sigma_3\rangle^{-\frac{\epsilon}{2}}\widetilde{I'v_L})\|_{L^\infty_{x,t}} \\ \|\mathcal{F}^{-1}(\langle k_4\rangle\widetilde{I'v_H})\|_{L^4_{x,t}} \|\mathcal{F}^{-1}(\langle k_5\rangle\widetilde{I'v_H})\|_{L^4_{x,t}} \|I'v_H)\|_{L^4_{x,t}} \\ \lesssim N'^{-2} \|I'v_H\|_{X^{1,\frac{1}{3}+4\epsilon}} \|I'v_L\|_{X^{\frac{1}{2}+\epsilon,\frac{1}{2}-\frac{\epsilon}{2}}} \|I'v_L\|_{X^{\frac{1}{2}+\epsilon,\frac{1}{2}-\frac{\epsilon}{2}}} \|I'v_H\|_{X^{1,\frac{1}{3}+\epsilon}} \\ \|I'v_H\|_{X^{1,\frac{1}{3}+\epsilon}} \|I'v_H\|_{X^{0,\frac{1}{3}+\epsilon}}.\end{aligned}$$

We neglect extra derivatives corresponding to N_2 and N_3 to get

$$\textit{Integral 4} \lesssim N'^{-3} \|I'v\|^6_{X^{1,\frac{1}{2}}}.$$

Remark 3.6.3. *Note that the sexalinear term does not depend on the scaler parameter λ.*

Appendix

The following example is given by Prof. Nobu Kishimoto which explain why we need to use the inhomogeneous Soblev norm in place of homogeneous norm. In fact, for homogeneous norm the Proposition 3.3.1 does not hold. Define the space $\dot{X}^{s,\frac{1}{2}}$ via the norm

$$\|u\|_{\dot{X}^{s,\frac{1}{2}}} = \||k|^s\langle\tau - 4\pi^2k^3\rangle^b\tilde{u}(k,\tau)\|_{L^2((dk)_\lambda, d\tau)}.$$

Examples 3.6.4. *Assume $\lambda \geqslant 1$ and $\sqrt{\lambda} \in \mathbb{Z}/\lambda$. Let $\lambda\mathbb{T} = \mathbb{R}/\lambda\mathbb{Z}$. We define the functions v_1, v_2, v_3 on $\lambda\mathbb{T}\times\mathbb{R}$ by*

$$\begin{aligned}\tilde{v}_1(k,\tau) &= 1_{[-1,1]}(\tau - 4\pi^2k^3)\cdot 1_{\{1/\lambda\}}(k), \\ \tilde{v}_2(k,\tau) &= 1_{[-1,1]}(\tau - 4\pi^2k^3)\cdot 1_{\{-2/\lambda\}}(k), \\ \tilde{v}_3(k,\tau) &= 1_{[-1,1]}(\tau - 4\pi^2k^3)\cdot 1_{\{\sqrt{\lambda}\}}(k).\end{aligned}$$

We have

$$\begin{aligned}\|v_1\|_{\dot{X}^{s,\frac{1}{2}}} \sim \|v_2\|_{\dot{X}^{s,\frac{1}{2}}} \sim \left(\frac{1}{\lambda}\right)^s \lambda^{-\frac{1}{2}} = \lambda^{s-\frac{1}{2}}, \\ \|v_3\|_{\dot{X}^{s,\frac{1}{2}}} \sim (\sqrt{\lambda})^s\lambda^{-\frac{1}{2}} = \lambda^{\frac{s}{2}-\frac{1}{2}}.\end{aligned}$$

We see that

$$
\begin{aligned}
&\left|\tilde{J}[v_1, v_2, v_3](\sqrt{\lambda}) - \frac{1}{\lambda}, \tau)\right| \\
&\sim \sqrt{\lambda}\left|\int_{\tau_1+\tau_2+\tau_3=\tau}\int_{\substack{k_1+k_2+k_3=\sqrt{\lambda}-\lambda^{-1} \\ (k_1+k_2)(k_2+k_3)(k_3+k_1)\neq 0}} \prod_{j=1}^{3} \tilde{v}_j(k_j, \tau_j)(dk_1)_\lambda (dk_2)_\lambda d\tau_1 d\tau_2\right| \\
&\gtrsim \lambda^{-3/2} 1_{[-1,1]}(\tau - 4\pi^2(\sqrt{\lambda} - \lambda^{-1})^3 + 4\pi^2 M),
\end{aligned}
$$

where

$$M = 3\left(\frac{1}{\lambda} + \frac{-2}{\lambda}\right)\left(\frac{-2}{\lambda} + \sqrt{\lambda}\right)\left(\sqrt{\lambda} + \frac{1}{\lambda}\right),$$

so that $|M| \sim 1$. *Hence, we have*

$$\|J[v_1, v_2, v_3]\|_{\dot{X}^{s,\frac{1}{2}}} \gtrsim \lambda^{-\frac{3}{2}} \cdot (\sqrt{\lambda})^s \lambda^{-\frac{1}{2}} = \lambda^{\frac{s}{2}-2}.$$

Therefore, if the trilinear estimate

$$\|J[v_1, v_2, v_3]\|_{\dot{X}^{s,\frac{1}{2}}} \lesssim \lambda^{0+} \|v_1\|_{\dot{X}^{s,\frac{1}{2}}} \|v_2\|_{\dot{X}^{s,\frac{1}{2}}} \|v_3\|_{\dot{X}^{s,\frac{1}{2}}}$$

were true, it would imply that

$$\lambda^{\frac{s}{2}-2} \lesssim (\lambda^{-s-\frac{1}{2}})^2 \lambda^{\frac{s}{2}-\frac{1}{2}} \quad \Leftrightarrow \quad \lambda^{2s} \lesssim \lambda^{\frac{1}{2}+} \ (\lambda \geqslant 1).$$

For large λ, *this holds only if* $s \leqslant \frac{1}{4}+$.

□

Appendix A

Proof Of The Uniqueness For KdV Equation

Definition A.0.1. *Let* $\varphi \in C_0^\infty(\mathbb{R})$ *with* $\varphi \equiv 1$ *on* $[-1,1]$ *and* $Supp\varphi \subseteq [-2,2]$. *Let* $T < T^*$. *We define* $\varphi_T = \varphi(\frac{t}{T})$ *and* $\varphi_{T^*} = \varphi(\frac{t}{T^*})$.

Definition A.0.2.

$$||u||_{X_T} = \inf_{\omega}\{||\omega||_{X_{s,b}} : \omega \in X_{s,b}, u(t) = \omega(t), t \in [0,T] \; in \; H^s\}.$$

Lemma A.0.3. *If* $s \leqslant 0$ *and* $b \in (\frac{1}{2}, 1)$ *then for any* $\delta \in (0,1)$, *we have*

$$||\varphi(\delta^{-1}t)F||_{X_{s,b}} \leqslant c\delta^{\frac{1-2b}{2}}||F||_{X_{s,b}},$$
$$||\varphi(\delta^{-1}t)\int_0^t W(t-t^{'})F(t^{'})dt^{'}||_{X_{s,b}} \leqslant c\delta^{\frac{1-2b}{2}}||F||_{X_{s,b-1}}.$$

Proposition A.0.4. *Let* $a, b \in (0, \frac{1}{2})$ *with* $a < b$ *and* $\delta \in (0,1)$, *then for* $f \in X_{s,-a}$ *we have*

$$||\varphi_\delta F||_{X_{s,-b}} \leqslant \delta^{\frac{(b-a)}{4(1-a)}}||F||_{X_{s,-a}}.$$

Lemma A.0.5. *For given* $s \in (-\frac{3}{4}, 0]$ $\exists\, b \in (\frac{1}{2}, 1)$ *and* $a \in (0, \frac{1}{2})$ *with* $a < b$ *and* a, b *are sufficiently close to* $\frac{1}{2}$, *such that*

$$||B(F,F)||_{X_{s,-a}} \leqslant c||F||^2_{X_{s,b}}.$$

Proof. First of all, we will rewrite the estimate. Let $\rho = -s \in]0, \frac{3}{4})$. From the definition of $||.||_{X_{s,b}}$ for $F \in X_{s,b} = X_{-\rho,b}$, we have

$$f(\xi,\tau) = (1 + |\tau - \xi^3|)^b(1 + |\xi|)^{-\rho}\hat{f}(\xi,\tau) \in \mathbb{L}^2(\mathbb{R}^2).$$

and $||f||_{\mathbb{L}^2_\xi \mathbb{L}^2_\tau} = ||F||_{X_{s,b}} = ||F||_{X_{-\rho,b}}$.
As we know that

$$\partial_x \widehat{(F^2)(\xi,\tau)} = c\xi(\hat{F} * \hat{F}).$$

So the bilinear estimate can be written as

$$\begin{aligned} ||B(F,F)||_{X_{s,-a}} &= ||(1+|\tau-\xi^3|)^{-a}(1+|\xi|)^{-\rho}\widehat{\partial_x F^2}||_{\mathbb{L}^2_\xi \mathbb{L}^2_\tau} \\ &= c||(1+|\tau-\xi^3|)^{-a}(1+|\xi|)^{-\rho}\xi(\hat{F}*\hat{F})||_{\mathbb{L}^2_\xi \mathbb{L}^2_\tau} \\ &= \left|\left| \frac{\xi}{(1+|\tau-\xi^3|)^a(1+|\xi|)^\rho} \times \int\int \frac{f(\xi_1,\tau_1)(1+|\xi_1|)^\rho}{(1+|\tau_1-\xi_1^3|)^b} \frac{f(\xi-\xi_1,\tau-\tau_1)(1+|\xi-\xi|)^\rho}{(1+|\tau-\tau_1-(\xi-\xi_1)^3|)^b} d\xi_1 d\tau_1 \right|\right|_{\mathbb{L}^2_\xi \mathbb{L}^2_\tau} \\ &\leqslant c||F||^2_{X_{s,b}}. \end{aligned}$$

Now for $s = 0$, by using Cauchy-Schwartz inequality, we will get

$$\left|\left| \frac{\xi}{(1+|\tau-\xi^3|)^a} \times \int\int \frac{f(\xi_1,\tau_1)}{(1+|\tau_1-\xi_1^3|)^b} \frac{f(\xi-\xi_1,\tau-\tau_1)}{(1+|\tau-\tau_1-(\xi-\xi_1)^3|)^b} d\xi_1 d\tau_1 \right|\right|_{\mathbb{L}^2_\xi \mathbb{L}^2_\tau}$$

$$\leqslant \left|\left| \frac{\xi}{(1+|\tau-\xi^3|)^a} \times \left(\int\int \frac{d\xi_1 d\tau_1}{(1+|\tau-\tau_1-(\xi-\xi_1)^3|)^{2b}(1+|\tau_1-\xi_1^3|)^{2b}} \right)^{\frac{1}{2}} \right|\right|_{\mathbb{L}^\infty_\xi \mathbb{L}^\infty_\tau} \left|\left| \left(\int\int |f(\xi_1,\tau_1)|^2 |f(\xi-\xi_1,\tau-\tau_1)|^2 \right)^{\frac{1}{2}} \right|\right|_{\mathbb{L}^2_\xi \mathbb{L}^2_\tau}$$

So to prove the lemma we just need to show that the first term is finite, which we will prove in the following proposition.

Proposition A.0.6. *If $b \in (\frac{1}{2}, \frac{3}{4}], a \in (0, \frac{1}{2})$ and $b' \in (\frac{1}{2}, b]$, then there exists $c > 0$ such that*

$$\frac{\xi}{(1+|\tau-\xi^3|)^a} \times \left(\int\int \frac{d\xi_1 d\tau_1}{(1+|\tau-\tau_1-(\xi-\xi_1)^3|)^{2b'}(1+|\tau_1-\xi_1^3|)^{2b'}} \right)^{\frac{1}{2}} \leqslant c$$

Proof. We know that, for $l > \frac{1}{2}$ $\exists$ $c > 0$ such that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+|x-a|)^{2l}(1+|x-b|)^{2l}} \leqslant \frac{c}{(1+|a-b|)^{2l}}.$$

Science, $b' > \frac{1}{2}$, the above inequality implies that

$$\int_{-\infty}^{\infty} \frac{d\tau_1}{(1+|\tau-\tau_1-(\xi-\xi_1)^3|)^{2b'}(1+|\tau_1-\xi_1^3|)^{2b'}}$$

$$\leqslant \frac{c}{(1+|\tau-\xi^3+3\xi\xi_1(\xi-\xi_1)|^{2b'})}.$$

To integrate with respect to ξ_1, we will change the variable.

$$\mu = \lambda - \xi^3 + 3\xi\xi_1(\xi - \xi_1), \text{ then } d\mu = 3\xi(\xi - 2\xi_1)d\xi_1.$$

and

$$\xi_1 = \frac{1}{2}\left\{\xi \pm \sqrt{\frac{4\tau - \xi^3 - 4\mu}{3\xi}}\right\}.$$

Therefor,

$$|\xi(\xi - 2\xi_1)| = c\sqrt{|\xi|}|\sqrt{4\tau - 4\xi^3 - 4\mu}|$$

and

$$d\xi_1 = c\frac{d\mu}{\sqrt{|\xi|}|\sqrt{4\tau - 4\xi^3 - 4\mu}|}.$$

Combine these identities, we will get

$$\frac{c}{(1 + |\tau - \xi^3 + 3\xi\xi_1(\xi - \xi_1)|^{2b'})}$$

$$\leqslant \frac{1}{|\sqrt{\xi}|}\int_{-\infty}^{\infty}\frac{d\mu}{(1 + |\mu|)^{2b'}|\sqrt{4\tau - 4\xi^3 - 4\mu}|} \leqslant \frac{c}{|\xi|^{\frac{1}{2}}(1 + |4\tau - \xi^3|)^{\frac{1}{2}}}.$$

Hence by using the identity. For $l > \frac{1}{2}$ $\exists\, c > 0$ such that

$$\int_{-\infty}^{\infty}\frac{dx}{(1 + |x - a|)^{2(1-l)}(1 + |x - b|)^{2l}} \leqslant \frac{c}{(1 + |a - b|)^{2(1-l)}},$$

we will get

$$\frac{|\xi|^{\frac{3}{4}}}{(1 + |4\tau - \xi^3|)^{\frac{1}{2}}(1 + |\tau - \xi^3|)^{a}}.$$

The above term is finite for $b \leqslant \frac{3}{4}$ and $a \in (0, \frac{1}{2})$.

□

Hence, by using the above proposition, we are done. □

Theorem A.0.7. *Let $s \in (-\frac{3}{4}, 0]$. Then $\exists\, b \in (\frac{1}{2}, 1)$ such that for any $u_0 \in H^s(\mathbb{R})$ $\exists\, T = T(||u_0||_{H^s} > 0)$ with $T(\delta) \to \infty$ as $\delta \to \infty$ and a unique solution of the KdV equation on the time interval $[-T, T]$.*

Proof. Let u_1 and $\varphi_{T^*}u_2$ be two solutions of the KdV equation where $u_1$is the solution we obtained and $\varphi_{T^*}u_2$ be the solution of the integral equation associated to the KdV equation. Let u_0 be the initial data. For $M > 0$, let

$$||u_1||_{X_{s,b}}, ||\varphi_{T^*}u_2||_{X_{s,b}} \leqslant M.$$

We can assume that $M > 1$ and $T < 1$. Also assume that $T^* < T$. Now consider,

$$u_1 - \varphi_{T^*}u_2 = \varphi_T(t)W(t)u_0 - \frac{\varphi_T(t)}{2}\int_0^t W(t-t^{'})\varphi_T^2(t^{'})\partial_x u_1^2(t^{'})dt^{'}$$

$$-\varphi_{T^*}(t)W(t)u_0 - \frac{\varphi_{T^*}(t)}{2}\int_0^t W(t-t^{'})\varphi_{T^*}^2(t^{'})\partial_x\varphi_{T^*}u_2^2(t^{'})dt^{'},$$

For $t \in [0, T^*]$

$$= -\frac{\varphi_{T^*}(t)}{2}\int_0^t W(t-t^{'})\varphi_{T^*}^2(t^{'})\partial_x(u_1^2(t^{'}) - \varphi_{T^*}u_2^2(t^{'}))dt^{'}. \qquad (1)$$

Now for any $\epsilon > 0$, $\exists\, \omega \in X_{s.b}$, such that for $t \in [0, T^*]$

$$\omega(t) = u_1 - \varphi_{T^*}u_2$$

and by the definition of $||.||_{X_{T^*}}$

$$||\omega(t)||_{X_{s,b}} \leqslant ||u_1 - \varphi_{T^*}u_2||_{X_{T^*}} + \epsilon.$$

Now (1) can be restated as

$$\omega^{'} = -\frac{\varphi_{T^*}(t)}{2}\int_0^t W(t-t^{'})\varphi_{T^*}^2(t^{'})\partial_x(\omega(t^{'}))(u_1(t^{'}) + \varphi_{T^*}u_2(t^{'}))dt^{'},$$

As we can see for $t \in [0, T^*]$

$$\omega^{'} = \omega = u_1 - \varphi_{T^*}u_2.$$

Now.

$$||u_1 - \varphi_{T^*}u_2||_{X_{T^*}} \leqslant ||\omega^{'}||_{X_{s.b}} = ||\frac{\varphi_{T^*}(t)}{2}\int_0^t W(t-t^{'})\varphi_{T^*}^2(t^{'})\partial_x(\omega(t^{'}))(u_1(t^{'}) + \varphi_{T^*}u_2(t^{'}))dt^{'}||_{X_{s,b}}.$$

Now we can use the second part of lemma (3) for $F = \varphi_{T^*}^2\partial_x(\omega)(u_1 + \varphi_{T^*}u_2)$. So we will get

$$||\omega^{'}||_{X_{s.b}} \leqslant CT^{*(\frac{(1-2b)}{2})}||\varphi_{T^*}^2\partial_x(\omega)(u_1 + \varphi_{T^*}u_2)||_{X_{s,b-1}}.$$

Now use proposition (4) for $F = \partial_x(\omega)(u_1 + \varphi_{T^*}u_2)$. While using proposition 4 for the space $||.||_{X_{s,b-i}}$, we will replace $-b$ by $b-1$. For $b \in (0, \frac{1}{2})$.

$$\begin{aligned} ||\omega^{'}||_{X_{s.b}} &\leqslant CT^{*(\frac{(1-2b)}{2})+(\frac{(1-b-a)}{4(1-a)})}||\partial_x(\omega)(u_1 + \varphi_{T^*}u_2)||_{X_{s,-a}} \\ &\leqslant CT^{*(\frac{(1-2b)}{2})+(\frac{(1-b-a)}{4(1-a)})}(||\partial_x(\omega u_1)||_{X_{s,-a}} + ||\partial_x(\omega\varphi_{T^*}u_2)||_{X_{s,-a}}). \end{aligned}$$

Let $p = (\frac{(1-2b)}{2}) + (\frac{(1-b-a)}{4(1-a)})$. Now by using lemma (5), we will get

$$\begin{aligned} ||\omega^{'}||_{X_{s.b}} &\leqslant CT^{*p}(||\omega|||_{X_{s,b}}||u_1||_{X_{s,b}} + ||\omega|||_{X_{s,b}}||\varphi_{T^*}u_2||_{X_{s,b}}) \\ &\leqslant CT^{*p}M(||u_1 - \varphi_{T^*}u_2||_{X_{T^*}} + \epsilon) \\ ||u_1 - \varphi_{T^*}u_2||_{X_{T^*}} &\leqslant \frac{\epsilon}{1 - CT^{*p}M} \end{aligned}$$

We have, $p = (\frac{(1-2b)}{2}) + (\frac{(1-b-a)}{4(1-a)})$.
For $a \in (0, \frac{1}{2}), b \in (\frac{1}{2}, 1), p$ is positive and hence we are done. □

Appendix B

A Counterexample

The following example is given by Prof. Nobu Kishimoto which explain why we need to use the inhomogeneous Soblev norm in place of homogeneous norm. In fact, for homogeneous norm the Proposition 3.3.1 does not hold. Define the space $\dot{X}^{s,\frac{1}{2}}$ via the norm

$$\|u\|_{\dot{X}^{s,\frac{1}{2}}} = \||k|^s \langle \tau - 4\pi^2 k^3 \rangle^b \tilde{u}(k,\tau)\|_{L^2((dk)_\lambda, d\tau)}.$$

Examples B.0.1. *Assume $\lambda \geqslant 1$ and $\sqrt{\lambda} \in \mathbb{Z}/\lambda$. Let $\lambda\mathbb{T} = \mathbb{R}/\lambda\mathbb{Z}$. We define the functions v_1, v_2, v_3 on $\lambda\mathbb{T} \times \mathbb{R}$ by*

$$\begin{aligned}
\tilde{v}_1(k,\tau) &= 1_{[-1,1]}(\tau - 4\pi^2 k^3) \cdot 1_{\{1/\lambda\}}(k), \\
\tilde{v}_2(k,\tau) &= 1_{[-1,1]}(\tau - 4\pi^2 k^3) \cdot 1_{\{-2/\lambda\}}(k), \\
\tilde{v}_3(k,\tau) &= 1_{[-1,1]}(\tau - 4\pi^2 k^3) \cdot 1_{\{\sqrt{\lambda}\}}(k).
\end{aligned}$$

We have

$$\begin{aligned}
\|v_1\|_{\dot{X}^{s,\frac{1}{2}}} \sim \|v_2\|_{\dot{X}^{s,\frac{1}{2}}} \sim \left(\frac{1}{\lambda}\right)^s \lambda^{-\frac{1}{2}} &= \lambda^{s-\frac{1}{2}}, \\
\|v_3\|_{\dot{X}^{s,\frac{1}{2}}} \sim (\sqrt{\lambda})^s \lambda^{-\frac{1}{2}} &= \lambda^{\frac{s}{2}-\frac{1}{2}}.
\end{aligned}$$

We see that

$$\begin{aligned}
&\left|\tilde{J}[v_1, v_2, v_3](\sqrt{\lambda}) - \frac{1}{\lambda}, \tau)\right| \\
&\sim \sqrt{\lambda} \left| \int_{\tau_1+\tau_2+\tau_3=\tau} \int_{\substack{k_1+k_2+k_3=\sqrt{\lambda}-\lambda^{-1} \\ (k_1+k_2)(k_2+k_3)(k_3+k_1)\neq 0}} \prod_{j=1}^{3} \tilde{v}_j(k_j,\tau_j)(dk_1)_\lambda (dk_2)_\lambda d\tau_1 d\tau_2 \right| \\
&\gtrsim \lambda^{-3/2} 1_{[-1,1]}(\tau - 4\pi^2(\sqrt{\lambda} - \lambda^{-1})^3 + 4\pi^2 M),
\end{aligned}$$

where

$$M = 3\left(\frac{1}{\lambda} + \frac{-2}{\lambda}\right)\left(\frac{-2}{\lambda} + \sqrt{\lambda}\right)\left(\sqrt{\lambda} + \frac{1}{\lambda}\right),$$

so that $|M| \sim 1$. *Hence, we have*

$$\|J[v_1, v_2, v_3]\|_{\dot{X}^{s,\frac{1}{2}}} \gtrsim \lambda^{-\frac{3}{2}} \cdot (\sqrt{\lambda})^s \lambda^{-\frac{1}{2}} = \lambda^{\frac{s}{2}-2}.$$

Therefore, if the trilinear estimate

$$\|J[v_1, v_2, v_3]\|_{\dot{X}^{s,\frac{1}{2}}} \lesssim \lambda^{0+} \|v_1\|_{\dot{X}^{s,\frac{1}{2}}} \|v_2\|_{\dot{X}^{s,\frac{1}{2}}} \|v_3\|_{\dot{X}^{s,\frac{1}{2}}}$$

were true, it would imply that

$$\lambda^{\frac{s}{2}-2} \lesssim (\lambda^{-s-\frac{1}{2}})^2 \lambda^{\frac{s}{2}-\frac{1}{2}} \quad \Leftrightarrow \quad \lambda^{2s} \lesssim \lambda^{\frac{1}{2}+} \ (\lambda \geqslant 1).$$

For large λ, *this holds only if* $s \leqslant \frac{1}{4} + .$

Appendix C

A Counterexample

The following example is given by Prof. Nobu Kishimoto which explain why we need to use the inhomogeneous Soblev norm in place of homogeneous norm. In fact, for homogeneous norm the Proposition 3.3.1 does not hold. Define the space $\dot{X}^{s,\frac{1}{2}}$ via the norm

$$\|u\|_{\dot{X}^{s,\frac{1}{2}}} = \||k|^s\langle\tau - 4\pi^2k^3\rangle^b\tilde{u}(k,\tau)\|_{L^2((dk)_\lambda,d\tau)}.$$

Examples C.0.1. *Assume $\lambda \geqslant 1$ and $\sqrt{\lambda} \in \mathbb{Z}/\lambda$. Let $\lambda\mathbb{T} = \mathbb{R}/\lambda\mathbb{Z}$. We define the functions v_1, v_2, v_3 on $\lambda\mathbb{T} \times \mathbb{R}$ by*

$$\begin{aligned}
\tilde{v}_1(k,\tau) &= 1_{[-1,1]}(\tau - 4\pi^2k^3) \cdot 1_{\{1/\lambda\}}(k),\\
\tilde{v}_2(k,\tau) &= 1_{[-1,1]}(\tau - 4\pi^2k^3) \cdot 1_{\{-2/\lambda\}}(k),\\
\tilde{v}_3(k,\tau) &= 1_{[-1,1]}(\tau - 4\pi^2k^3) \cdot 1_{\{\sqrt{\lambda}\}}(k).
\end{aligned}$$

We have

$$\begin{aligned}
\|v_1\|_{\dot{X}^{s,\frac{1}{2}}} \sim \|v_2\|_{\dot{X}^{s,\frac{1}{2}}} \sim \left(\frac{1}{\lambda}\right)^s \lambda^{-\frac{1}{2}} &= \lambda^{s-\frac{1}{2}},\\
\|v_3\|_{\dot{X}^{s,\frac{1}{2}}} \sim (\sqrt{\lambda})^s\lambda^{-\frac{1}{2}} &= \lambda^{\frac{s}{2}-\frac{1}{2}}.
\end{aligned}$$

We see that

$$\begin{aligned}
&\left|\tilde{J}[v_1,v_2,v_3](\sqrt{\lambda}) - \frac{1}{\lambda},\tau)\right|\\
&\sim \sqrt{\lambda}\left|\int_{\tau_1+\tau_2+\tau_3=\tau}\int_{\substack{k_1+k_2+k_3=\sqrt{\lambda}-\lambda^{-1}\\(k_1+k_2)(k_2+k_3)(k_3+k_1)\neq 0}} \prod_{j=1}^{3}\tilde{v}_j(k_j,\tau_j)(dk_1)_\lambda(dk_2)_\lambda d\tau_1 d\tau_2\right|\\
&\gtrsim \lambda^{-3/2}1_{[-1,1]}(\tau - 4\pi^2(\sqrt{\lambda} - \lambda^{-1})^3 + 4\pi^2M),
\end{aligned}$$

where

$$M = 3\left(\frac{1}{\lambda} + \frac{-2}{\lambda}\right)\left(\frac{-2}{\lambda} + \sqrt{\lambda}\right)\left(\sqrt{\lambda} + \frac{1}{\lambda}\right),$$

so that $|M| \sim 1$. *Hence, we have*

$$\|J[v_1, v_2, v_3]\|_{\dot{X}^{s,\frac{1}{2}}} \gtrsim \lambda^{-\frac{3}{2}} \cdot (\sqrt{\lambda})^s \lambda^{-\frac{1}{2}} = \lambda^{\frac{s}{2}-2}.$$

Therefore, if the trilinear estimate

$$\|J[v_1, v_2, v_3]\|_{\dot{X}^{s,\frac{1}{2}}} \lesssim \lambda^{0+} \|v_1\|_{\dot{X}^{s,\frac{1}{2}}} \|v_2\|_{\dot{X}^{s,\frac{1}{2}}} \|v_3\|_{\dot{X}^{s,\frac{1}{2}}}$$

were true, it would imply that

$$\lambda^{\frac{s}{2}-2} \lesssim (\lambda^{-s-\frac{1}{2}})^2 \lambda^{\frac{s}{2}-\frac{1}{2}} \quad \Leftrightarrow \quad \lambda^{2s} \lesssim \lambda^{\frac{1}{2}+} \; (\lambda \geqslant 1).$$

For large λ, *this holds only if* $s \leqslant \frac{1}{4}+$.

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