Title	On minimal model program in positive characteristic
Author(s)	Tanaka, Hiromu
Citation	代数幾何学シンポジウム記録 (2018), 2018: 3-8
Issue Date	2018
URL	http://hdl.handle.net/2433/236399
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

ON MINIMAL MODEL PROGRAM IN POSITIVE CHARACTERISTIC

HIROMU TANAKA

Abstract. In this note, we first overview the current status of minimal model program in positive characteristic. We then discuss difference between the situations in characteristic 0 and p. In particular, we will exhibit some pathological examples that appear only in positive characteristic. This article is based on my talk given in Kinosaki symposium 2018.

1. Introduction

We work over an algebraically closed field k, unless otherwise specified. One of the goal of minimal model program is as follows.

Conjecture 1.1 (MMP conjecture)**.** Let X be a projective $\mathbb{Q}$-factorial terminal variety. Then there exists a sequence

$$X =: X_0 \overset{f_0}{\dashrightarrow} X_1 \overset{f_1}{\dashrightarrow} \cdots \overset{f_{N-1}}{\dashrightarrow} X_N$$

that satisfies the following properties.

(1) Each X_i is a projective $\mathbb{Q}$-factorial terminal variety.
(2) Each f_i is either a divisorial contraction or a flip.
(3) X_N is either a minimal model or a Mori fibre space $g : X_N \to Z$.

So far, the conjecture has been solved in the following cases.

Theorem 1.2. *Assume that* $\mathrm{char} k = 0$*. Then MMP conjecture holds if*

(1) $\dim \leq 4$*, or*
(2) X *is of genera type.*

Theorem 1.3. *Assume that* $p := \mathrm{char} k > 0$*. Then MMP conjecture holds if*

(3) $\dim = 2$*, or*
(4) $\dim = 3$ *and* $p > 5$*.*

2010 *Mathematics Subject Classification.* 14E30.
Key words and phrases. minimal model program, positive characteristic.

Remark 1.4. Both Theorem 1.2 and Theorem 1.3 hold even for klt varieties:

$$\text{smooth} \Rightarrow \text{terminal} \Rightarrow \text{klt}.$$

Remark 1.5. For the above results, we refer to the following articles and references therein.

(1) [Mor].
(2) [BCHM10].
(3) [Tan14].
(4) [BW17].

2. Base point free theorem

In this section, we summarise results on the Kawamata–Shokurov base point free theorem in positive characteristic. In particular, we give an example of positive characteristic that violates the original statement of the relative Kawamata–Shokurov base point free theorem.

2.1. **The current status.** Let us recall the statement of the Kawamata–Shokurov base point free theorem.

Theorem 2.1 (Kawamata–Shokurov base point free theorem, [KM98])**.** *Assume* $\operatorname{char} k = 0$. *Let* X *be a projective klt variety. Let* L *be a nef Cartier divisor such that* $L - K_X$ *is ample. Then there exists* $b_0 \in \mathbb{Z}_{>0}$ *such that* $|bL|$ *is base point free for any integer* $b \geq b_0$.

Remark 2.2. If $L \equiv 0$ and $-K_X$ is ample, then $L \sim 0$. Indeed, we have $b_0 L \sim 0$ and $(b_0+1)L \sim 0$, hence take the difference: $L = (b_0+1)L - b_0 L \sim 0$.

It is natural to ask whether the same statement holds also in positive characteristic. The following results are known to hold.

Theorem 2.3 (cf. [Tan15])**.** *Assume* $\operatorname{char} k > 0$. *If* $\dim X = 2$, *then the same statement as in Theorem 2.1 holds.*

Theorem 2.4 ([BW17])**.** *Assume* $\operatorname{char} k > 5$. *Let* X *be a projective klt threefold. Let* L *be a nef Cartier divisor such that* $L - K_X$ *is ample. Then there exists* $c_0 \in \mathbb{Z}_{>0}$ *such that* $|cc_0 L|$ *is base point free for any* $c \in \mathbb{Z}_{>0}$.

Remark 2.5. We use notation as in Theorem 2.4. If $L \equiv 0$ and $-K_X$ is ample, then $c_0 L \sim 0$. Indeed, we have $c_0 L \sim 0$ and $2c_0 L \sim 0$, hence take the difference: $c_0 L = 2c_0 L - c_0 L \sim 0$.

The consequences of Theorem 2.1 and Theorem 2.4 are similar, but slightly different. The latter one is weaker than the former. In many applications, Theorem 2.4 is enough, however it is a fundamental problem whether such difference occurs. It has not been known yet whether such examples actually exist, whilst such phenomena occurs at least in the relative situation. Indeed, the following theorem shows that, in positive characteristic, there exists an example which violates the relative version of the Kawamata–Shokurov base point free theorem.

Theorem 2.6 ([Tan]). *The relative version of Theorem 2.1 does not hold in positive characteristic. More explicitly, the following hold.*

Assume that the base field k is an algebraically closed field of characteristic $p \in \{2, 3\}$. Then there exists a projective morphism $g : \mathcal{Y} \to B$ with $g_*\mathcal{O}_{\mathcal{Y}} = \mathcal{O}_B$ such that

1. *$\mathcal{Y}$ is a $\mathbb{Q}$-factorial klt threefold,*
2. *B is a smooth curve,*
3. *$-K_{\mathcal{Y}}$ is g-ample, and*
4. *there is a Cartier divisor L on $\mathcal{Y}$ such that $L \equiv_g 0$ and $L \not\sim_g 0$.*

Then it is reasonable to expect that the following conjecture is true:

Conjecture 2.7. There exist a smooth Fano variety X and a Cartier divisor L on X such that $L \equiv 0$ and $L \not\sim 0$. In particular, Theorem 2.1 does not hold in positive characteristic.

2.2. **Construction of the example.** We now overview the construction of the example in Theorem 2.6. Note that B will be defined as a non-empty open subset of $\mathbb{A}^1_k = \operatorname{Spec} k[t]$. The construction consists of the following two steps.

(i) We first construct a pathological two-dimensional klt del Pezzo surface Y over the imperfect field $k(t)$.

(ii) Then we get $\mathcal{Y} \to B$ by killing denominators, i.e. there exists a non-empty open subset B of $\mathbb{A}^1_k = \operatorname{Spec} k[t]$ with the following cartesian square:

$$\begin{array}{ccc} \mathcal{Y} & \longleftarrow & Y \\ \downarrow & & \downarrow \\ B & \longleftarrow & \operatorname{Spec} k(t). \end{array}$$

The second step (ii) is just a routine, hence let us focus on the first step (i). The surface Y, appearing in (i), is obtained by contracting the curve Γ of X, where X and Γ are given as follows.

Theorem 2.8. *In this theorem, we work over the imperfect field* $k(t)$ *(recall that* k *is an algebraically closed field of characteristic* $p \in \{2,3\}$*). Then there exists* $f : X \to Z$ *such that*

(1) Z *is a regular projective curve with* $K_Z \sim 0$*,*
(2) f *is a* $\mathbb{P}^1$*-bundle (in particular,* f *is a smooth projective morphism and* X *is a regular surface),*
(3) *there exists an effective* $\mathbb{Q}$*-divisor* Δ *on* X *such that* (X, Δ) *is klt and* $-(K_X + \Delta)$*,*
(4) *there is a Cartier divisor* M *on* Z *such that* $pM \sim 0$ *and* $M \not\sim 0$*, and*
(5) *there is a curve* Γ *on* X *with* $\Gamma^2 < 0$*.*

Remark 2.9. For $N := f^*M$, we have that $pN \sim 0$ and $N \not\sim 0$. This is because f is a $\mathbb{P}^1$-bundle.

Construction. We only treat the case when $p = 3$. Set $F := k(t)$.

- $Z := \mathrm{Proj} F[x,y,z]/(-y^2z + x^3 + t^2z^3)$. Note that this is nothing but the generic fibre of a quasi-elliptic fibration. That is why we need to assume that $p \in \{2,3\}$.
- $Q := [0:1:0] \in Z(F)$.
- We have $H^1(Z, \mathcal{O}_Z(-Q)) \neq 0$ by $K_Z \sim 0$ and Serre duality.
- Fix a nonzero element $\alpha \in H^1(Z, \mathcal{O}_Z(-Q))$. Then α corresponds to a locally free sheaf E of rank two, which completes the following non-splitting exact sequence:

$$0 \to \mathcal{O}_Z(-Q) \to E \to \mathcal{O}_Z \to 0.$$

- $X := \mathbb{P}_Z(E)$.
- Set $M := Q_1 - Q_2$ for $Q_1, Q_2 \in Z(F)$ with $Q_1 \neq Q_2$, e.g. $[0 : \pm t : 1] \in Z(F)$.

For more details, see [Tan]. □

3. Flips

From now on, we again work over an algebraically closed field k.

Theorem 3.1 ([BCHM10], [HX15])**.** *Assume that either*

- $\mathrm{char} k = 0$*, or*
- $\mathrm{char} k > 5$ *and* $\dim = 3$*.*

Then flips exist.

Mori's original proof of the existence of flips for terminal threefolds in characteristic zero is given by thorough study of three-dimensional terminal singularities. On the other hand, both the proofs of [BCHM10] and [HX15] follow the strategy initiated by Shokurov:

(1) We first reduce the existence of flips to the existence of pl-flips. The advantage of pl-filps is easier to apply induction on dimension.
(2) Given a plt pair (X, Δ), it suffices to show that, under suitable assumptions, the ring

$$\mathrm{Im}\left(\bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X(\llcorner m(K_X+\Delta)\lrcorner)) \to \bigoplus_{m=0}^{\infty} H^0(\llcorner \Delta \lrcorner, \mathcal{O}_X(\llcorner m(K_X+\Delta)\lrcorner)|_{\llcorner \Delta \lrcorner})\right)$$

is a finitely generated k-algebra, where $\llcorner \Delta \lrcorner$ is a prime divisor on X.

To get such a finite generation, the first step is to establish the normality of $\llcorner \Delta \lrcorner$.

Theorem 3.2. *Let (X, Δ) be a plt pair. Assume that one of the following conditions holds.*

- $\mathrm{char} k = 0$.
- $\mathrm{char} k > 0$ *and* $\dim X = 2$.
- $\mathrm{char} k > 5$ *and* $\dim X = 3$.

Then $\llcorner \Delta \lrcorner$ is normal.

Then it is natural to hope that the same statement holds in general. Unfortunately, this is not the case.

Theorem 3.3 ([CT], [Ber])**.** *The following hold.*

(1) *Assume that* $\mathrm{char} k = 2$*. Then there is a 3-dimensional plt pair (X, Δ) such that $\llcorner \Delta \lrcorner$ is not normal.*
(2) *Assume that* $\mathrm{char} k > 2$*. Then there is a $(2p+2)$-dimensional plt pair (X, Δ) such that $\llcorner \Delta \lrcorner$ is not normal.*

Remark 3.4. The example in (1) is obtained by applying cone-like construction to surfaces obtained by Keel–M^{c}Kernan. The example in (2) is obtained by taking cones of Totaro's smooth Fano varieties. Such Fano varieties is defined as $\mathrm{SL}(n, k)/P$, where P is a non-reduced parabolic subgroup of $\mathrm{SL}(n, k)$ (cf. [Tot]).

Acknowledgement: The author would like to thank the organisers H. Uehara, K. Watanabe, and A. Kanazawa of Kinosaki Symposium 2018 for the invitation and their generous hospitality. The author was funded by the Grant-in-Aid for Scientific Research (KAKENHI No. 18K13386).

References

[Ber] F. Bernasconi, Non-normal purely log terminal centres in characteristic $p \geq 3$, preprint available at arXiv:1802.04896.

[BCHM10] C. Birkar, P. Cascini, C. D. Hacon, J. McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc., Vol. **23**, no. 2 (2010), 405–468.

[BW17] C. Birkar, J. Waldron, *Existence of Mori fibre spaces for 3-folds in char p*, Adv. Math., **313**, 2017, 62–101.

[CT] P. Cascini, H. Tanaka, *Purely log terminal threefolds with non-normal centres in characteristic two*, preprint available at arXiv:1607.08590. to appear in Amer. J. Math.

[HX15] C. D. Hacon, C. Xu, *On the three dimensional minimal model program in positive characteristic*, J. Amer. Math. Soc. **28** (2015), 711-744 .

[KM98] J. Kollár, S. Mori, *Birational geometry of algebraic varieties*, Cambrigde Tracts in Mathematics, Vol. **134**, 1998.

[Mor] J. Moraga, *Termination of pseudo-effective 4-fold flips*, preprint available at arXiv:1802.10202.

[Tan14] H. Tanaka, *Minimal models and abundance for positive characteristic log surfaces*, Nagoya Math. J. **216** (2014), 1–70.

[Tan15] H. Tanaka, *The X-method for klt surfaces in positive characteristic*, J. Algebraic Geom. **24** (2015), no. 4, 605–628.

[Tan] H. Tanaka, *Pathologies on Mori fibre spaces in positive characteristic*, preprint available at arXiv:1609.00574. to appear in Annali della Scuola Normale Superiore di Pisa.

[Tot] B. Totaro, *The failure of Kodaira vanishing for Fano varieties, and terminal singularities that are not Cohen-Macaulay*, preprint available at arXiv:1710.04364.

Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, JAPAN

E-mail address: `tanaka@ms.u-tokyo.ac.jp`