STABILITY CONDITIONS ON THREEFOLDS WITH NEF TANGENT BUNDLES

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1. Introduction

The notion of stability conditions on a triangulated category was introduced by Bridgeland in his papaer [13]. Although the original motivation to study Bridgeland stability conditions came from string theory, it has found many applications to the classical problems in algebraic geometry. We will survey a part of them in Section 3.

For such applications, the starting problem is to prove the existence of Bridgeland stability conditions on (the derived category of coherent sheaves on) a given smooth projective variety X. When the dimension of a variety X is less than or equal to two, the construction problem is solved by Bridgeland [14] and Arcara-Bertram [1]. In the proof, the classical Bogomolov-Gieseker inequality for torsion free slope stable sheaves is crucial.

However, in dimension three, the existence of Bridgeland stability conditions is an open problem in general. By the work [10] of Bayer-Macrì-Toda, the problem was reduced to proving the so-called *Bogomolov-Gieseker (BG) type inequality conjecture*. It is a conjectural inequality for the Chern characters of certain stable objects in the derived category, called *tilt stable objects*. The BG type inequality conjecture is known to be true in the following cases.

- Abelian threefolds ([29, 9]).
- Fano threefolds of Picard number one ([10, 31, 35, 25]).
- some toric threefolds ([11]).
- product threefolds of projective spaces and Abelian varieties ([23]).
- quintic threefolds ([26]).

In this article, we will explain the following result:

Theorem 1.1 ([24])**.** *For any smooth projective threefold with nef tangent bundle, the BG type inequality conjecture holds.*

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Notation and Convention. In this paper we always work over $\mathbb{C}$. We use the following notations:

- $\mathrm{ch}^B = (\mathrm{ch}^B_0, \cdots, \mathrm{ch}^B_n) := e^{-B}.\,\mathrm{ch}$, where ch denotes the Chern character and $B \in \mathrm{NS}(X)_{\mathbb{R}}$.
- $v^B := \omega.\,\mathrm{ch}^B := (\omega^n.\,\mathrm{ch}^B_0, \cdots, \omega.\,\mathrm{ch}^B_{n-1}, \mathrm{ch}^B_n)$, where $B, \omega \in \mathrm{NS}(X)_{\mathbb{R}}$.
- $K(\mathcal{A})$: the Grothendieck group of an abelian category $\mathcal{A}$.

- $\hom(E,F) := \dim \operatorname{Hom}(E,F)$.
- $\operatorname{ext}^i(E,F) := \dim \operatorname{Ext}^i(E,F)$.
- $D^b(X) := D^b(\operatorname{Coh}(X))$: the bounded derived category of coherent sheaves on a smooth projective variety X.

2. Bridgeland stability conditions

2.1. **Definitions.** In this subsection, we recall the notion of Bridgeland stability conditions on a triangulated category. The reference for this subsection is Bridgeland's original paper [13]. First, we define the notion of stability functions:

Definition 2.1. Let $\mathcal{A}$ be an Abelian category.

(1) A *stability function* on $\mathcal{A}$ is a group homomorphism $Z\colon K(\mathcal{A}) \to \mathbb{C}$ satisfying the condition

$$Z(\mathcal{A} \setminus \{0\}) \subset \mathcal{H} \cup \mathbb{R}_{<0},$$

where $\mathcal{H}$ is the upper half plane.

(2) Let Z be a stability function on $\mathcal{A}$. An object $E \in \mathcal{A}$ is called *Z-stable (resp. semistable)* if for every non zero proper subobject $0 \neq F \subset E$, we have an inequality

$$-\frac{\Re Z(F)}{\Im Z(F)} < (\text{resp. } \leq) - \frac{\Re Z(E)}{\Im Z(E)}.$$

Here, we define $-\frac{\Re Z(E)}{\Im Z(E)} := +\infty$ if $\Im Z(E) = 0$.

(3) A stability function Z on $\mathcal{A}$ satisfies the *Harder-Narasimhan (HN) property* if the following holds: for every object $E \in \mathcal{A}$, there exists a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{m-1} \subset E_m = E$$

such that $F_i := E_i/E_{i-1}$ are Z-semistable and

$$-\frac{\Re Z(F_1)}{\Im Z(F_1)} > \cdots > -\frac{\Re Z(F_m)}{\Im Z(F_m)}.$$

We now define the notion of stability conditions on a triangulated category:

Definition 2.2. Let $\mathcal{D}$ be a triangulated category. A *stability condition* on $\mathcal{D}$ is a pair consisting of the heart $\mathcal{A}$ of a bounded t-structure on $\mathcal{D}$ and a stability function Z on $\mathcal{A}$ satisfying the HN property. A stability function Z is called a *central charge*.

2.2. **Bogomolov-Gieseker type inequality conjecture.** In this subsection, we recall the conjectural approach for the construction of stability conditions on threefolds. Let X be a smooth projective threefold. Fix a class $B + i\omega \in \operatorname{NS}(X)_{\mathbb{C}}$ with ω ample. Conjecturally, there exists a stability condition on $D^b(X)$ with its central charge given as follows (cf. [10, Conjecture 2.1.2]):

$$Z_{\omega,B} := -\int_X e^{-i\omega}.\operatorname{ch}^B.$$

It is easy to see that the pair $(Z_{\omega,B}, \mathrm{Coh}(X))$ does not define a stability condition when X is a threefold. Hence we need to introduce new hearts. Our hearts are obtained by the double-tilting construction [10] which we explain below, see the paper [22] for the general theory of torsion pairs and tilting. In the following, we assume that $B \in \mathrm{NS}(X)_{\mathbb{Q}}$ and $\omega = mH$ for some ample divisor H and $m \in \mathbb{R}_{>0}$ with $m^2 \in \mathbb{Q}$. We use the following notation:

$$v^B = (v_0^B, v_1^B, v_2^B, v_3^B) := (\omega^3.\,\mathrm{ch}_0^B, \omega^2.\,\mathrm{ch}_1^B, \omega.\,\mathrm{ch}_2^B, \mathrm{ch}_3^B).$$

First tilting: We define the slope function on $\mathrm{Coh}(X)$ as follows:

$$\mu_{\omega,B} := \frac{v_1^B}{v_0^B} \colon \mathrm{Coh}(X) \to (-\infty, +\infty].$$

Then define the full subcategories $\mathcal{T}_{\omega,B}, \mathcal{F}_{\omega,B} \subset \mathrm{Coh}(X)$ as follows:

$$\mathcal{T}_{\omega,B} := \langle T \in \mathrm{Coh}(X) : T \text{ is } \mu_{\omega,B}\text{-semistable with } \mu_{\omega,B}(T) > 0 \rangle\,,$$
$$\mathcal{F}_{\omega,B} := \langle F \in \mathrm{Coh}(X) : F \text{ is } \mu_{\omega,B}\text{-semistable with } \mu_{\omega,B}(F) \leq 0 \rangle\,.$$

Here, $\mu_{\omega,B}$-stability for coherent sheaves is defined in a standard manner, and we denote by $\langle S \rangle$ the extension closure of a set of objects $S \subset \mathrm{Coh}(X)$. Now we define a new heart, called tilted heart by

$$\mathrm{Coh}^{\omega,B}(X) := \langle \mathcal{F}_{\omega,B}[1], \mathcal{T}_{\omega,B} \rangle\,.$$

Second tilting: As in the first tilting, we introduce a new slope function and tilting of $\mathrm{Coh}^{\omega,B}(X)$: A slope function $\nu_{\omega,B}$ on $\mathrm{Coh}^{\omega,B}(X)$ is defined to be

$$\nu_{\omega,B} := \frac{v_2^B - \frac{1}{6}v_0^B}{v_1^B} \colon \mathrm{Coh}^{\omega,B}(X) \to (-\infty, +\infty],$$

and the notion of $\nu_{\omega,B}$-stability for objects in $\mathrm{Coh}^{\omega,B}(X)$ is defined similarly as $\mu_{\omega,B}$-stability for coherent sheaves. We also refer to $\nu_{\omega,B}$-stability as *tilt stability*. Note that the existence of Harder-Narasimhan filtration with respect to $\nu_{\omega,B}$-stability is shown in the paper [10]. We define full subcategories of $\mathrm{Coh}^{\omega,B}(X)$ as

$$\mathcal{T}'_{\omega,B} := \left\langle T \in \mathrm{Coh}^{\omega,B}(X) : T \text{ is } \nu_{\omega,B}\text{-semistable with } \nu_{\omega,B}(T) > 0 \right\rangle,$$
$$\mathcal{F}'_{\omega,B} := \left\langle F \in \mathrm{Coh}^{\omega,B}(X) : F \text{ is } \nu_{\omega,B}\text{-semistable with } \nu_{\omega,B}(F) \leq 0 \right\rangle.$$

Now we reach the definition of the double-tilted heart:

$$\mathcal{A}_{\omega,B} := \left\langle \mathcal{F}'_{\omega,B}[1], \mathcal{T}'_{\omega,B} \right\rangle.$$

In the paper [10], Bayer, Macrì, and Toda conjectured the following:

Conjecture 2.3 ([10])**.** The pair $(Z_{\omega,B}, \mathcal{A}_{\omega,B})$ is a stability condition on $D^b(X)$.

Let us denote

$$\overline{\Delta}_{\omega,B}(E) := v_1^B(E)^2 - 2v_0^B(E)v_2^B(E)$$

and

$$\overline{\nabla}_{\omega,B}(E) := 2(v_2^B(E))^2 - 3v_1^B(E)v_3^B(E).$$

The following is the so-called Bogomolov-Gieseker (BG) type inequality conjecture ([10, 9, 34]).

Conjecture 2.4 ([34, Conjecture 3.8])**.** For any $\nu_{\omega,B}$-stable object E, we have the inequality

$$\overline{\Delta}_{\omega,B}(E) + 6\overline{\nabla}_{\omega,B}(E) \geq 0.$$

The BG type inequality conjecture implies the existence of a stability condition:

Proposition 2.5 ([34])**.** *Assume that Conjecture 2.4 holds. Then Conjecture 2.3 also holds.*

2.3. **Counter-examples.** Counter-examples to Conjecture 2.3 are constructed in the papers [23, 32, 36]. In particular, we have the following result:

Lemma 2.6 ([32, Lemma 3.1])**.** *Let H be an ample divisor. Assume that there exists an effective divisor D such that*

$$D^3 > \frac{(H^2.D)^3}{4(H^3)^2} + \frac{3}{4}\frac{(H.D^2)^2}{H^2.D}. \tag{2.1}$$

Then there exists a pair (α, β) of real numbers with $\alpha > 0$, such that the pair $(Z_{\alpha H, \beta H}, \mathcal{A}_{\alpha H, \beta H})$ does not define a stability condition.

Remark 2.7. Let D be a nef divisor. Then we can show that D does not satisfy the inequality (2.1) by using the Hodge index theorem for nef divisors. In particular, there are no known counter-examples to Conjecture 2.3 when the pseudo-effective cone agrees with the nef cone.

In Section 4, we will explain how to prove Conjecture 2.4 for threefolds with nef tangent bundles.

3. Applications

In this section, we will review various applications of the theory of Bridgeland stability to the problems in algebraic geometry.

3.1. **Birational geometry.** Let S be a smooth projective surface. Then we know the existence of stability conditions on S. Fix a cohomology class $v \in H^{2*}(S, \mathbb{Q})$. For any choice of a stability condition σ on S, we can consider the moduli space $M_\sigma(v)$ of σ-semistable objects with Chern character v. When we vary a stability condition σ, the moduli space $M_\sigma(v)$ may change. Such a phenomenon is called *wall crossing*. Using wall crossing, birational geometry of the moduli spaces of Gieseker stable sheaves is studied in several cases.

- When $S = \mathbb{P}^2$, many people studied the wall crossings (see e.g. [2, 12, 16, 17, 18, 19, 27, 28]). For example, we have the following result:

  Theorem 3.1 ([27, Theorem 0.2])**.** *We can run the whole minimal model program for the moduli space of Gieseker stable sheaves on $\mathbb{P}^2$ via wall crossing in the space of stability conditions. Moreover, the minimal model is smooth.*

- When S is a K3 surface, Bayer and Macrì [7, 8] studied wall crossings in detail. Note that the moduli spaces of Gieseker stable sheaves on a K3 surface are important examples of irreducible holomorphic symplectic manifolds. For example, we have the following result:

Theorem 3.2 ([7, Theorem 1.2])**.** *Every smooth K-tirvial birational model of the moduli space of Gieseker stable sheaves on S appears as the moduli space of Bridgeland stable objects.*

3.2. **Clifford type theorem.** The classical Clifford index theorem states the following. Let C be a smooth projective curve of genus g, F a slope semistable vector bundle on C with rank r and slope $\mu \in [0, g]$. Then we have an inequality

$$h^0(F)/r \leq 1 + \frac{\mu}{2}.$$

When our curve C is embedded into a surface S in a special way, we can obtain a stronger Clifford type theorem via wall crossing in the space of Bridgeland stability on S. The basic idea is to regard a vector bundle F on C as a torsion sheaf on S and analyze the length of the Harder-Narasimhan filtration of F with respect to certain Bridgeland stability conditions on S. Such a study was started by [3], and developed in the papers [6, 20, 21]. For example, we have the following result:

Theorem 3.3 ([21, Theorem 1.1])**.** *Let (S, H) be a polarized K3 surface with $\mathrm{Pic}(S) = \mathbb{Z}[H]$. Let $C \in |H|$ be a smooth member of genus g. Let E be a slope semistable vector bundle on C of rank r, degree d. Assume that $d \leq r(g-1)$. Then we have the inequality*

$$h^0(C, E) < r + \frac{g}{4r(g-1)^2}d^2 + \frac{r}{g}.$$

To obtain the above theorem, one of the key point is a stronger BG inequality on a surface S. In fact, we can prove the similar result for curves in del Pezzo surfaces etc. In particular, the Clifford type theorem for curves on degree four del Pezzo surfaces was proved, and used to construct Bridgeland stability conditions on quintic threefolds in [26].

3.3. **Donaldson-Thomas invariants.** Let A be an Abelian threefold. For a fixed polarization H and a cohomology class $v \in H^{2*}(A, \mathbb{Q})$, we can consider the reduced Donaldson-Thomas (DT) invariant $\mathrm{DT}_H(v)$. In the paper [33], the autors studied the invariance of DT invariants under autoequivalences on the derived category $D^b(A)$ via wall crossing.

Theorem 3.4 ([33, Theorem 1.1])**.** *Under a certain condition on the class v, $\mathrm{DT}_H(v)$ is independent of H and we have*

$$\mathrm{DT}_H(g_* v) = \mathrm{DT}_H(v)$$

for every autoequivalence $g \in \mathrm{Aut}\, D^b(A)$.

It is the first example where the wall crossing of Bridgeland stability conditions is applied to the study of DT invariants on Calabi-Yau (CY) threefolds. One of the technical difficulty is to construct Bridgeland stability conditions on a given CY threefold, and the another difficulty is to prove the so-called support property of a stability condition which require the stronger BG type inequality. Abelian threefolds are the first class of CY threefolds for which the above technical difficulties were solved. It is now possible to treat the case of quintic CY threefolds due to the work [26].

3.4. **Fujita's conjecture.** One of the motivation to study the BG type inequality conjecture is the following theorem, which states the relation between Conjecture 2.4 and Fujita's conjecture:

Theorem 3.5 ([4, Corollary 1.1]). *Let L be an ample line bundle on a smooth projective threefold X. Assume that Conjecture 2.4 holds for a certain choice of a pair (ω, B). Then the following hold:*

(1) *$K_X \otimes L^{\otimes m}$ is globally generated for $m \geq 4$. Moreover, if $L^3 \geq 2$, then $K_X \otimes L^{\otimes 3}$ is also globally generated.*
(2) *$K_X \otimes L^{\otimes m}$ is very ample for $m \geq 6$.*

The proof uses Reider type method: Assume for a contradiction that there exists a zero-dimensional subscheme $Z \subset X$ such that $H^1(X, K_X \otimes L^{\otimes m} \otimes I_Z) \neq 0$. Then by the Serre duality, we obtain a non-trivial extension class

$$\mathcal{O}_X[1] \to E \to L^{\otimes m} \otimes I_Z$$

which is an short exact sequence in the tilted category $\mathcal{B}_{\omega,B}$. By using Conjecture 2.4, we can show that E is not tilt-semistable for ω sufficiently small. Then analyzing the Chern characters of destabilizing objects in the heart $\mathcal{B}_{\omega,B}$, we will get the contradiction.

4. Idea of proof

In this section, we give an outline of the proof of Theorem 1.1. There are three steps.

Step 1. (classification). The first step is the classification theorem of such threefolds due to [15].

Theorem 4.1 ([15]). *Let X be a smooth projective threefold with nef tangent bundle. Then up to taking finite étale coverings, X is one of the following:*

(1) $\mathbb{P}^3$.
(2) *a three dimensional smooth quadric.*
(3) $\mathbb{P}^1 \times \mathbb{P}^2$.
(4) $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.
(5) $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$.
(6) *$\mathbb{P}_A(\mathcal{E})$, where A is an Abelian surface and $\mathcal{E}$ is a rank two vector bundle obtained as an extension of two line bundles in* $\mathrm{Pic}^0(A)$.
(7) *$\mathbb{P}_C(\mathcal{E})$, where C is an elliptic curve and $\mathcal{E}$ is a rank three vector bundle obtained as extensions of three line bundles of degree zero.*
(8) *$\mathbb{P}_C(\mathcal{E}_1) \times_C \mathbb{P}_C(\mathcal{E}_2)$, where C is an elliptic curve and $\mathcal{E}_i$ are rank two vector bundles obtained as extensions of degree zero line bundles.*
(9) *an Abelian threefold.*

Among the above threefolds, the existence of Bridgeland stability conditions is known in the following cases:

- $\mathbb{P}^3$ by [10, 31].
- a three dimensional smooth quadric by [35].
- (3) – (5) in Theorem 4.1 by [11].
- an Abelian threefold by [9, 29, 30].

Hence it is enough to consider threefolds in (6)–(8) of Theorem 4.1. For simplicity, we only consider the case of $X = \mathbb{P}_A(\mathcal{E})$, where A is an Abelian surface, and $\mathcal{E}$ is a rank two vector bundle fitting into the exact sequence

$$0 \to \mathcal{O}_A \to \mathcal{E} \to \mathcal{O}_A \to 0.$$

STEP 2. (degeneration technique). The next step is to reduce to the case when $\mathcal{E} = \mathcal{O}_A \oplus \mathcal{O}_A$. Assume that $\mathcal{E}$ is non-split. The following theorem is crucial:

Theorem 4.2 ([5])**.** *Let $f\colon \mathcal{X} \to D$ be a smooth projective family of threefolds over a smooth curve D and fix a point $0 \in D$. Suppose that f is a trivial family over $U := D \setminus \{0\}$, i.e. $f^{-1}(U) \cong X \times U$ for some threefold X. Take an f-ample $\mathbb{Q}$-divisor $\mathcal{H}$ and an arbitrary $\mathbb{Q}$-divisor $\mathcal{B}$ on $\mathcal{X}$. Let $\mathcal{H}_0$, $\mathcal{B}_0$ (resp. H, B) be restriction of $\mathcal{H}$, $\mathcal{B}$ to the special fiber $f^{-1}(0)$ (resp. the general fiber X). If Conjecture 2.4 is true for $(f^{-1}(0), \mathcal{H}_0, \mathcal{B}_0)$, then it also holds for (X, H, B).*

The above result follows from the existence of the relative moduli spaces of tilt-stable objects over the base D, satisfying the valuative criterion for universal closedness.

In our case, take an affine line in $\mathrm{Ext}^1(\mathcal{O}_A, \mathcal{O}_A)$ passing through the class $[\mathcal{E}]$ and the origin. Over this affine line $\mathbb{A}^1$, we have a family $f\colon \mathcal{X} \to \mathbb{A}^1$ such that

- $f^{-1}(U) \cong X \times U$, where $U := \mathbb{A}^1 \setminus \{0\}$.
- $f^{-1}(0) \cong X_0 := \mathbb{P}^1 \times A$.

Hence it is enough to consider the case of $X_0 = \mathbb{P}^1 \times A$.

STEP 3. (endomorphism technique). The good point of the variety $\mathbb{P}^1 \times A$ is that it has many endomorphisms, i.e., the product

$$F_m := \underline{m}^2_{\mathbb{P}^1} \times \underline{m}_A \colon X_0 \to X_0$$

of the toric Frobenius morphism $\underline{m}^2_{\mathbb{P}^1}$ on $\mathbb{P}^1$ and the multiplication map $\underline{m}_A$ on A for any non negative integer $m \in \mathbb{Z}_{\geq 0}$.

Now let $E \in \mathcal{B}_{\omega,B}$ be a tilt-stable object. For simplicity, assume that $B = 0$, ω is sufficiently small, and $\nu_{\omega,B}(E) = 0$. Then the desired inequality becomes $\mathrm{ch}_3(E) \leq 0$. By the Riemann-Roch theorem, we have

$$\begin{aligned} m^6 \,\mathrm{ch}_3(E) + \mathcal{O}(m^4) &= \chi(\mathcal{O}_{X_0}, F_m^* E) \\ &\leq \hom(\mathcal{O}_{X_0}, F_m^* E) + \mathrm{ext}^2(\mathcal{O}_{X_0}, F_m^* E). \end{aligned}$$

We can estimate the dimensions $\hom(\mathcal{O}_{X_0}, F_m^* E)$ and $\mathrm{ext}^2(\mathcal{O}_{X_0}, F_m^* E)$ by order m^4 using the following facts:

- the toric Frobenius splitting on $\mathbb{P}^1$,
- the morphism $\underline{m}_A$ is étale,
- every line bundle on X_0 is tilt-stable.

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