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Author(s)	Nakajima, Hiraku
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# Geometric Satake correspondence for affine Kac-Moody Lie algebras of type $A$

$A$ 型のアファイン・カツ・ムーディー・リー環における幾何学的佐武対応

Hiraku Nakajima (中島 啓)

ABSTRACT. This is an informal expository article on geometric Satake correspondence for affine Kac-Moody Lie algebras of type  $A$  given in [Nak18b]. We emphasize formal analogies between this result and the author's earlier results on geometric approaches to the representation via quiver varieties.

At Kinosaki Algebraic Geometry Symposium 2018, the author gave a talk “Coulomb branches of  $3d$  SUSY gauge theories” explaining

- (1) the provisional mathematical definition of Coulomb branches of 3-dimensional  $\mathcal{N} = 4$  SUSY gauge theories in [Nak16a, BFN16a], and
- (2) geometric Satake correspondence for affine Kac-Moody Lie algebras of type  $A$  in [Nak18b].

Since we already have an expository article for the first part [Nak18a] (see also an earlier Japanese version [Nak16b]), we here review only the second part. We make this article as a continuation of [Nak18a], as we need to presuppose the first part. When we refer a section numbered between 1 and 8, it means the corresponding section in [Nak18a].

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## 9. Geometric Satake correspondence for complex reductive groups — disclaimer

Our main result Theorem 12.4 can be regarded as an affine Lie algebra  $\mathfrak{sl}(n)_{\text{aff}}$  version of the geometric Satake correspondence for a complex reductive group  $\mathbf{G}$  [Lus83, Gin95, BD00, MV07]. Our approach is close to one due to Mirković-Vilonen [MV07]. Fortunately we have many good expository articles on [MV07], and there is no reason to try to produce a worse one. There is another reason to

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decide not to explain geometric Satake correspondence: We avoid to use sheaf theoretic language as much as possible in the spirit of the talk in Algebraic Geometry Symposium. If we would explain geometric Satake, this decision makes no sense.

We also omit historical accounts. Interested readers should read [BFN16b, §3(viii)] and [Fin18].

### 10. Quiver gauge theories of affine type $A$

Let  $n$  be an integer greater than 1. We have a similar story for the case  $n = 1$ , but we omit it for brevity.

Let  $V = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} V_i$ ,  $W = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} W_i$  be finite dimensional  $\mathbb{Z}/n\mathbb{Z}$  graded complex vector spaces. We define

$$G \stackrel{\text{def.}}{=} \prod_i \text{GL}(V_i), \quad \mathbf{N} \stackrel{\text{def.}}{=} \prod_i \text{Hom}(V_i, V_{i+1}) \oplus \text{Hom}(W_i, V_i).$$

We regard  $\mathbf{N}$  as a representation of  $G$  in a natural way. Therefore we can apply the definition of the Coulomb branch in §3 for  $G, \mathbf{N}$ . We denote it by  $\mathcal{M}(\lambda, \mu)$ , where  $\lambda, \mu$  are weights of the affine Lie algebra  $\mathfrak{sl}(n)_{\text{aff}}$  given by

$$\lambda = \sum_i \dim W_i \cdot \Lambda_i, \quad \mu = \lambda - \sum_i \dim V_i \cdot \alpha_i,$$

where  $\Lambda_i, \alpha_i$  are fundamental weights and simple roots respectively. These notation are *not* important at this stage, as one can identify them with dimension vectors  $\mathbf{w} = (\dim W_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ ,  $\mathbf{v} = (\dim V_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$  respectively. Nevertheless it will be convenient when we relate Coulomb branches to representation theory of  $\mathfrak{sl}(n)_{\text{aff}}$ . The corresponding Higgs branch, which is the symplectic reduction  $\mathbf{N} \oplus \mathbf{N}^* // G = \mu^{-1}(0) // G$  (see §1), is nothing but a quiver variety of affine type  $A$ . We denote it by  $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ . In standard notation for a quiver variety, it is usually denoted by  $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ . Taking a character  $\chi: G \rightarrow \mathbb{C}^\times$  given by the product of determinants of factors  $\text{GL}(V_i)$ , we can consider a GIT quotient  $\mu^{-1}(0) //_\chi G$ . Let us denote it by  $\mathfrak{M}_\chi(\lambda, \mu)$ . It is equipped with a projective morphism  $\pi: \mathfrak{M}_\chi(\lambda, \mu) \rightarrow \mathfrak{M}_0(\lambda, \mu)$ . One see that  $\chi$ -stability and  $\chi$ -semistability are equivalent and  $\mathfrak{M}_\chi(\lambda, \mu)$  is smooth.

### 11. Quiver varieties and affine Lie algebras

**11(i). Integrable highest weight representations.** Recall the following, which was proved for more general quiver:

**Theorem 11.1** ([Nak94, Nak98]). (1) *Let  $0$  be the point in  $\mathfrak{M}_0(\lambda, \mu)$  corresponding to  $0$  in  $\mathbf{N} \oplus \mathbf{N}^*$ . Then  $\mathfrak{L}_\chi(\lambda, \mu) \stackrel{\text{def.}}{=} \pi^{-1}(0)$  is a lagrangian subvariety in  $\mathfrak{M}_\chi(\lambda, \mu)$ .*

(2) *The direct sum*

$$\bigoplus_{\mu} H_{\text{top}}(\mathfrak{L}_\chi(\lambda, \mu))$$

*of top degree homology groups of  $\mathfrak{L}_\chi(\lambda, \mu)$  over  $\mu$  has a structure of an integrable highest weight representation of an affine Lie algebra  $\mathfrak{sl}(n)_{\text{aff}}$  of type  $A_{n-1}^{(1)}$ .*

Recall  $\mathfrak{sl}(n)_{\text{aff}}$  is an infinite dimensional Lie algebra, defined as a central extension of the finite dimensional complex simple Lie algebra  $\mathfrak{sl}(n): \mathfrak{sl}(n) \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$ . (Here  $d$  is the so-called degree operator.) Integrable highest weight representations are natural analog of finite dimensional irreducible representations

of  $\mathfrak{sl}(n)$ . They are classified by dominant integral weights (i.e. highest weights),  $n$ -tuple of nonnegative integers in this case. For the representation in (2), the highest weight is given by  $\lambda$ , which was essentially  $(\dim W_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$  as we mentioned above. Let us denote the corresponding integrable highest weight representation by  $V(\lambda)$ . Its weight  $\mu$  subspace is denoted by  $V_\mu(\lambda)$ .

Let us note that  $\bigoplus_\mu H_{\text{top}}(\mathfrak{L}_\chi(\lambda, \mu))$  has a distinguished base given by fundamental classes of irreducible components of  $\mathfrak{L}_\chi(\lambda, \mu)$ . It is known that the set  $\bigsqcup_\mu \text{Irr } \mathfrak{L}_\chi(\lambda, \mu)$  of irreducible components has a structure of Kashiwara crystal isomorphic to one for crystal base of  $U_q(\mathfrak{sl}(n)_{\text{aff}})$ -version for  $V(\lambda)$ . See [KS97, Sai02, Nak01b]. We do not give a precise explain of its meaning (which requires to recall the definition of Kashiwara crystal): it vaguely means that there is a recursive way to parametrize irreducible components of  $\mathfrak{L}_\chi(\lambda, \mu)$  with varying  $\mu$  starting from the highest weight vector  $\mathfrak{L}_\chi(\lambda, 0) = \{0\}$ . In fact, this recursive structure was used to show that the above representation is of highest weight.

Our goal is to relate the Coulomb branch  $\mathcal{M}(\lambda, \mu)$  to  $\mathfrak{sl}(n)_{\text{aff}}$ . In fact, we conjecture that we have a similar relation for Coulomb branches for general quiver gauge theories. An affine type  $A$  quiver is special, as Coulomb branches have description as Cherkis bow varieties [NT17], which can be defined by symplectic reduction of certain finite dimensional symplectic manifolds (products of vector spaces and Slodowy slices of type  $A$ ). This description is crucially used in the proof of the main result Theorem 12.4 below, although we hide it from the statement in order to make the length of this article reasonable. We lack tools to establish various expected properties of Coulomb branches of general quiver gauge theories, such as finiteness of symplectic leaves, description of leaves, etc.

In order to illustrate a formal similarity between two constructions, one in Theorem 11.1 and another in Theorem 12.4, let us briefly review the definition of  $\mathfrak{sl}(n)_{\text{aff}}$  representation, formulated in a slightly different manner from the original.

We use a presentation of  $\mathfrak{sl}(n)_{\text{aff}}$  as a Kac-Moody Lie algebra: it is generated by elements  $e_i, f_i, h_i$  ( $i \in \mathbb{Z}/n\mathbb{Z}$ ),  $d$  with certain defining relations which we omit. Operators  $h_i, d$  are given by dimensions of  $V_i, W_i$  on the component  $H_{\text{top}}(\mathfrak{L}_\chi(\lambda, \mu))$ , hence not so interesting. More precisely they are defined so that  $H_{\text{top}}(\mathfrak{L}_\chi(\lambda, \mu))$  is identified with the weight space  $V_\mu(\lambda)$ .

Operators  $e_i, f_i$  are defined via a correspondence in  $\mathfrak{M}_\chi(\lambda, \mu) \times \mathfrak{M}_\chi(\lambda, \mu - \alpha_i)$ . Note that  $\mu - \alpha_i$  corresponds to  $V \oplus S_i$ , where  $S_i$  is a one dimensional vector space with grading  $i \in \mathbb{Z}/n\mathbb{Z}$ .

Let  $\chi_i: G \rightarrow \mathbb{C}^\times$  be a character, which is the product of determinants of  $\text{GL}(V_j)$  for  $j \neq i$ . We consider the corresponding GIT quotients  $\mathfrak{M}_{\chi_i}(\lambda, \mu), \mathfrak{M}_{\chi_i}(\lambda, \mu - \alpha_i)$ . The morphism  $\pi$  factors as  $\mathfrak{M}_\chi(\lambda, \mu) \rightarrow \mathfrak{M}_{\chi_i}(\lambda, \mu) \rightarrow \mathfrak{M}_0(\lambda, \mu)$ . The same is true for  $\mathfrak{M}_{\chi_i}(\lambda, \mu)$ . Moreover we have a closed embedding  $\mathfrak{M}_{\chi_i}(\lambda, \mu) \rightarrow \mathfrak{M}_{\chi_i}(\lambda, \mu - \alpha_i)$ . This is induced from the corresponding morphism for  $\mu^{-1}(0)$  given by extending the data for  $V, W$  to  $V \oplus S_i, W$  by zero on the  $S_i$  component. This extension does not preserve the stability condition for  $\chi$ , hence we do not have a morphism  $\mathfrak{M}_\chi(\lambda, \mu) \rightarrow \mathfrak{M}_\chi(\lambda, \mu - \alpha_i)$ . But it is well-defined on  $\mathfrak{M}_{\chi_i}(\lambda, \mu)$ , as we take categorical quotients with respect to  $\text{GL}(V_i)$  at  $i$ .

We consider the fiber product

$$\mathfrak{M}_\chi(\lambda, \mu) \times_{\mathfrak{M}_{\chi_i}(\lambda, \mu - \alpha_i)} \mathfrak{M}_\chi(\lambda, \mu - \alpha_i),$$

where  $\mathfrak{M}_\chi(\lambda, \mu) \rightarrow \mathfrak{M}_{\chi_i}(\lambda, \mu - \alpha_i)$  is the composition  $\mathfrak{M}_\chi(\lambda, \mu) \rightarrow \mathfrak{M}_{\chi_i}(\lambda, \mu) \rightarrow \mathfrak{M}_{\chi_i}(\lambda, \mu - \alpha_i)$ . It is known that the fiber product is lagrangian in  $\mathfrak{M}_\chi(\lambda, \mu) \times \mathfrak{M}_\chi(\lambda, \mu - \alpha_i)$ .

There is a distinguished irreducible component  $\mathfrak{P}_i(\lambda, \mu)$  in the fiber product: it consists of pairs  $x, x'$  such that  $x$  is a restriction of  $x'$  to  $V$  (considered as  $\subset V \oplus S_i$ ). Since  $x'$  is a  $\mathrm{GL}(V \oplus S_i)$ -orbit of linear maps, not a linear map, its restriction to  $V$  needs to be carefully defined. But we omit this technical detail.

One can check that projections  $p_1, p_2: \mathfrak{P}_i(\lambda, \mu) \rightarrow \mathfrak{M}_\chi(\lambda, \mu), \mathfrak{M}_\chi(\lambda, \mu - \alpha_i)$  are proper. Hence operators

$$H_{\mathrm{top}}(\mathfrak{L}_\chi(\lambda, \mu)) \simeq H_{\mathrm{top}}(\mathfrak{L}_\chi(\lambda, \mu - \alpha_i))$$

are defined by convolution  $p_{2*}p_1^*$  and  $p_{1*}p_2^*$ . These are definition of elements  $e_i, f_i$  up to a suitable sign convention.

**11(ii). Tensor product.** In Theorem 11.1 irreducible integrable representations of  $\mathfrak{sl}(n)_{\mathrm{aff}}$  are constructed. Let us review the construction of tensor product representations in [Nak01b], which was motivated by earlier works by Lusztig [Lus99] and Varagnolo-Vasserot [VV02].

Recall that  $(\dim W_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$  gives the highest weight in Theorem 11.1. We consider a decomposition  $W = W^1 \oplus W^2$  of  $\mathbb{Z}/n\mathbb{Z}$ -graded vector spaces, which will be related to the tensor product of representations with highest weights  $\lambda^1, \lambda^2$ , which correspond to  $(\dim W_i^1)$  and  $(\dim W_i^2)$ . The following construction depends on the ordering of  $W^1, W^2$ , though the resulted tensor product does *not*. It is because the construction can be upgraded to representations of the quantum loop algebra  $U_q(\mathbf{L}\mathfrak{sl}(n)_{\mathrm{aff}})$  (toroidal algebra) whose tensor products depend on the order.

Let us take a one parameter subgroup in  $\nu^\bullet: \mathbb{C}^\times \rightarrow \mathrm{GL}(W)$  given by  $\nu^\bullet(t) = \mathrm{id}_{W^1} \oplus t \mathrm{id}_{W^2}$ . It acts on  $\mathfrak{M}_0(\lambda, \mu), \mathfrak{M}_\chi(\lambda, \mu)$  naturally. We define attracting subsets in  $\mathfrak{M}_0(\lambda, \mu)$ :

$$(11.2) \quad \begin{aligned} \mathfrak{T}_0^{\nu^\bullet}(\lambda, \mu) &\stackrel{\mathrm{def.}}{=} \left\{ x \in \mathfrak{M}_0(\lambda, \mu) \mid \lim_{t \rightarrow 0} \nu^\bullet(t)x \text{ exists} \right\}, \\ \tilde{\mathfrak{T}}_0^{\nu^\bullet}(\lambda, \mu) &\stackrel{\mathrm{def.}}{=} \left\{ x \in \mathfrak{M}_0(\lambda, \mu) \mid \lim_{t \rightarrow 0} \nu^\bullet(t)x = 0 \right\}. \end{aligned}$$

Their inverse images under  $\pi$  are denoted by  $\mathfrak{T}_\chi^{\nu^\bullet}(\lambda, \mu), \tilde{\mathfrak{T}}_\chi^{\nu^\bullet}(\lambda, \mu)$  respectively.

**Theorem 11.3** ([Nak01b]). (1) *The subvariety  $\tilde{\mathfrak{T}}_\chi^{\nu^\bullet}(\lambda, \mu)$  is lagrangian in  $\mathfrak{M}_\chi(\lambda, \mu)$ .*

(2) *The direct sum*

$$\bigoplus_{\mu} H_{\mathrm{top}}(\tilde{\mathfrak{T}}_\chi^{\nu^\bullet}(\lambda, \mu))$$

*of top degree homology groups of  $\tilde{\mathfrak{T}}_\chi^{\nu^\bullet}(\lambda, \mu)$  over  $\mu$  has a structure of an integrable representation of  $\mathfrak{sl}(n)_{\mathrm{aff}}$ , which is isomorphic to the tensor product  $V(\lambda^1) \otimes V(\lambda^2)$  of two integrable highest weights representations with highest weights  $\lambda^1$  and  $\lambda^2$ .*

The  $\mathfrak{sl}(n)_{\mathrm{aff}}$ -module structure is given by the convolution product as in Theorem 11.1. In order to see that the representation has a correct ‘size’, let us describe irreducible components of  $\tilde{\mathfrak{T}}_\chi^{\nu^\bullet}(\lambda, \mu)$ . We first observe that the  $\nu^\bullet$ -fixed point set  $\mathfrak{M}_\chi(\lambda, \mu)^{\nu^\bullet}$  decomposes as

$$\mathfrak{M}_\chi(\lambda, \mu)^{\nu^\bullet} \cong \bigsqcup_{\mu = \mu^1 + \mu^2} \mathfrak{M}_\chi(\lambda^1, \mu^1) \times \mathfrak{M}_\chi(\lambda^2, \mu^2).$$

The morphism from the right hand side to the left is given by the direct sum. To show that it is an isomorphism, one uses that  $\mathfrak{M}_\chi(\lambda, \mu)$  is smooth and a fine moduli space. See [Nak01b, Lemma 3.2] for detail. For a point  $x \in \tilde{\mathfrak{X}}_\chi^{\nu^\bullet}(\lambda, \mu)$ , the limit  $\lim_{t \rightarrow 0} tx$  is a  $\nu^\bullet$ -fixed point. According to the above decomposition, we have the induced decomposition

$$\tilde{\mathfrak{X}}_\chi^{\nu^\bullet}(\lambda, \mu) = \bigsqcup_{\mu=\mu^1+\mu^2} \tilde{\mathfrak{X}}_\chi^{\nu^\bullet}(\lambda^1, \mu^1; \lambda^2, \mu^2).$$

Then  $\tilde{\mathfrak{X}}_\chi^{\nu^\bullet}(\lambda^1, \mu^1; \lambda^2, \mu^2)$  is a vector bundle over  $\mathfrak{L}_\chi(\lambda^1, \mu^1) \times \mathfrak{L}_\chi(\lambda^2, \mu^2)$ , where the projection to the base is identified with the map given by  $\lim_{t \rightarrow 0} tx$ . It consists of points  $x$  which are given by ‘exact sequence’  $0 \rightarrow x^2 \rightarrow x \rightarrow x^1 \rightarrow 0$  with  $x^1 \in \mathfrak{L}_\chi(\lambda^1, \mu^1)$ ,  $x^2 \in \mathfrak{L}_\chi(\lambda^2, \mu^2)$ . See [Nak01b, Remark 3.16]. Now irreducible components of  $\tilde{\mathfrak{X}}_\chi^{\nu^\bullet}(\lambda, \mu)$  is in bijection to  $\bigsqcup_{\mu=\mu^1+\mu^2} \text{Irr } \mathfrak{L}_\chi(\lambda^1, \mu^1) \times \text{Irr } \mathfrak{L}_\chi(\lambda^2, \mu^2)$ .

Let us give an example. Let  $V = \bigoplus V_i$ ,  $V_i = \mathbb{C}$  ( $1 \leq i \leq n-1$ ),  $V_0 = 0$ ,  $W = \bigoplus W_i$ ,  $W_i = \mathbb{C}$  ( $i = 1, n-1$ ),  $W_i = 0$  ( $i = 0$  or  $1 < i < n-1$ ). (We understand  $W_1 = \mathbb{C}^2$ ,  $W_0 = 0$  when  $n = 2$ .) The corresponding quiver variety  $\mathfrak{M}_0(\lambda, \mu)$  is  $\mathbb{C}^2/(\mathbb{Z}/n\mathbb{Z})$ , i.e. the simple singularity of type  $A_{n-1}$ , and  $\mathfrak{M}_\chi(\lambda, \mu)$  is its minimal resolution. The lagrangian subvariety  $\mathfrak{L}_\chi(\lambda, \mu)$  consists of a chain of  $n-1$  complex projective lines. We take a decomposition  $W = W^1 \oplus W^2$  with  $\dim W^1 = \dim W^2 = 1$ . The fixed point set  $\mathfrak{M}_\chi(\lambda, \mu)^{\nu^\bullet}$  consists of  $n$  isolated points, which are south and north poles of projective lines. There are  $(n-2)$  intersection points among them, and the remaining two are extremal points of the chain. Thus we have one additional irreducible component in  $\tilde{\mathfrak{X}}_\chi^{\nu^\bullet}(\lambda, \mu)$ . Depending on a choice of the decomposition  $W = W^1 \oplus W^2$ , we choose an either of lines through two extremal points. See Figure 1. The additional irreducible component is drawn by a dotted line.

In representation theoretic term this corresponds to the tensor product representation  $\mathbb{C}^n \times (\mathbb{C}^n)^*$  of  $\mathfrak{sl}(n)$ , where  $\mathbb{C}^n$  is the vector representation and  $(\mathbb{C}^n)^*$  is its dual. It decomposes into the sum of  $\mathfrak{sl}(n)$  (adjoint representation) and  $\mathbb{C} \text{ id}$  (trivial representation). The additional irreducible component gives the trivial representation summand.

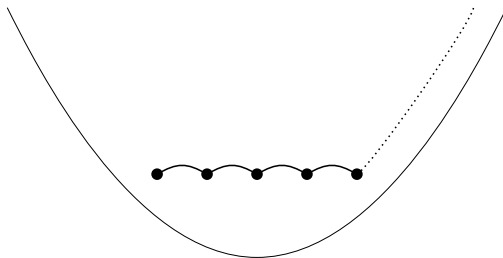


FIGURE 1.  $\tilde{\mathfrak{X}}_\chi^{\nu^\bullet}(\lambda, \mu)$

**Remark 11.4.** Combining Theorem 11.1 with Theorem 11.3, we see that there is an isomorphism

$$(11.5) \quad H_{\text{top}}(\widetilde{\mathfrak{X}}_{\chi}^{\nu^*}(\lambda, \mu)) \cong \bigoplus_{\mu=\mu^1+\mu^2} H_{\text{top}}(\mathfrak{L}_{\chi}(\lambda^1, \mu^1)) \otimes H_{\text{top}}(\mathfrak{L}_{\chi}(\lambda^2, \mu^2)).$$

But the above argument only gives an existence of an isomorphism, and does not give a particular isomorphism. Since the tensor product  $V(\lambda^1) \otimes V(\lambda^2)$  is not irreducible in general, such an isomorphism is not unique. For quiver varieties of finite type, this ambiguity was fixed by regarding one of tensor factors as a lowest weight module. See [Nak01b, Th. 5.9]. The stable envelope introduced by Maulik–Okounkov [MO12] gives a canonical isomorphism in a geometric way for quiver varieties of general types.

**Remark 11.6.** The construction can be easily generalized to the case when we decompose  $W$  into more factors, say  $W = W^1 \oplus W^2 \oplus W^3$ . It corresponds to the triple tensor product  $V(\lambda^1) \otimes V(\lambda^2) \otimes V(\lambda^3)$ .

For quiver varieties of affine type  $A$  (and more generally when the underlying Dynkin diagram contains a loop) there is another type of one parameter subgroup acting on  $\mathfrak{M}_0(\lambda, \mu)$ ,  $\mathfrak{M}_{\chi}(\lambda, \mu)$ : we consider the dilatation action on the factor  $\text{Hom}(V_{n-1}, V_0)$  and the induced action on  $\mathfrak{M}_0(\lambda, \mu)$ ,  $\mathfrak{M}_{\chi}(\lambda, \mu)$ . The fixed point set in  $\mathfrak{M}_{\chi}(\lambda, \mu)$  with respect to this action is union of quiver varieties of type  $A_{\infty}$ , i.e. those defined by replacing  $\mathbb{Z}/n\mathbb{Z}$ -grading above by  $\mathbb{Z}$ -grading. The embedding is giving by regarding  $\mathbb{Z}$ -grading modulo  $n$ .

The corresponding attracting set realizes a representation which is induced by the homomorphism  $\mathfrak{sl}(n)_{\text{aff}} \rightarrow \mathfrak{gl}(\infty)$ , where the latter is the central extension of the Lie algebra of matrices  $(a_{ij})_{i,j \in \mathbb{Z}}$  with  $a_{ij} = 0$  for  $|i - j| \gg 0$ . (See [KRR13].)

**11(iii). Sheaf theoretic formulation.** In order to make analogy between Theorem 11.1 and Theorem 12.4 closer, we further reformulate the above definition using perverse sheaves. A reader who is unfamiliar with sheaf theoretical formulation of convolution products should skip this and next subsections.

It is known [Nak98, Cor. 10.11] that  $\pi: \mathfrak{M}_{\chi}(\lambda, \mu) \rightarrow \mathfrak{M}_0(\lambda, \mu)$  is semismall (when we replace the target by the image of  $\pi$ ). Therefore the direct image  $\pi_!(\mathcal{C}_{\mathfrak{M}_{\chi}(\lambda, \mu)})$  over  $\mathfrak{M}_{\chi}(\lambda, \mu)$  is a semisimple perverse sheaf. Here  $\mathcal{C}_X$  denote the constant sheaf shifted by  $\dim X$ , i.e.  $\mathcal{C}_X[\dim X]$ . The point  $0$  is a stratum and we have

$$H_{\text{top}}(\mathfrak{L}_{\chi}(\lambda, \mu)) \cong \text{Hom}(\mathcal{C}_0, \pi_!(\mathcal{C}_{\mathfrak{M}_{\chi}(\lambda, \mu)})),$$

where the  $\text{Hom}$  in the right hand side is taken in the abelian category of perverse sheaves on  $\mathfrak{M}_0(\lambda, \mu)$  (locally constant along a certain natural stratification).

As we remarked above,  $\pi$  factors through  $\mathfrak{M}_{\chi_i}(\lambda, \mu)$ . Let us denote maps by  $\pi'$ ,  $\pi''$  so that  $\pi = \pi' \circ \pi''$ . Then  $\pi''$  is also semismall, and  $\pi''_!(\mathcal{C}_{\mathfrak{M}_{\chi}(\lambda, \mu)})$  is a semisimple perverse sheaf on  $\mathfrak{M}_{\chi_i}(\lambda, \mu)$ . It decomposes a direct sum of intersection cohomology complexes of various strata. (One can show that no intersection cohomology complexes associated with nontrivial local system appear. cf. [Nak01a, Prop. 15.3.2].) Let us write the decomposition as

$$\pi''_!(\mathcal{C}_{\mathfrak{M}_{\chi}(\lambda, \mu)}) \cong \bigoplus_{\alpha} L_{\alpha} \otimes \text{IC}(\mathfrak{M}_{\chi_i}^{\alpha}),$$

where  $\mathfrak{M}_{\chi_i}^{\alpha}$  are relevant strata, and  $L_{\alpha}$  is the top degree homology of the fiber over a point in a stratum  $\mathfrak{M}_{\chi_i}^{\alpha}$ . It is also known that a stratum is of a form  $\mathfrak{M}_{\chi_i}^s(\lambda, \mu + k\alpha_i)$

with a nonnegative integer  $k$ . Here the superscript ‘s’ means the locus of stable points. See [Nak09, Prop. 2.25].

Since  $\pi = \pi' \circ \pi''$ , we have

$$H_{\text{top}}(\mathfrak{L}_\chi(\lambda, \mu)) \cong \bigoplus_{\alpha} L_{\alpha} \otimes \text{Hom}(\mathbb{C}_0, \pi'_! \text{IC}(\mathfrak{M}_{\chi_i}^{\alpha})).$$

The stratification  $\mathfrak{M}_{\chi_i}(\lambda, \mu) = \bigsqcup \mathfrak{M}_{\chi_i}^{\alpha}$  is compatible with the closed embedding  $\mathfrak{M}_{\chi_i}(\lambda, \mu) \hookrightarrow \mathfrak{M}_{\chi_i}(\lambda, \mu - \alpha_i)$ , hence we may write also

$$H_{\text{top}}(\mathfrak{L}_\chi(\lambda, \mu - \alpha_i)) \cong \bigoplus_{\alpha} L'_{\alpha} \otimes \text{Hom}(\mathbb{C}_0, \pi'_! \text{IC}(\mathfrak{M}_{\chi_i}^{\alpha}))$$

by setting  $L_{\alpha} = 0$  if  $\mathfrak{M}_{\chi_i}^{\alpha}$  is a new stratum appearing in  $\mathfrak{M}_{\chi_i}(\lambda, \mu - \alpha_i)$ .

By the definition of  $e_i, f_i$  above, they respect the decomposition, i.e. they are tensor products of operators  $L_{\alpha} \rightleftharpoons L'_{\alpha}$  and the identity operator of  $\text{Hom}(\mathbb{C}_0, \pi'_! \text{IC}(\mathfrak{M}_{\chi_i}^{\alpha}))$ .

By a local description of a quiver variety (see [Nak01a, §3]), the fiber over a point a stratum  $\mathfrak{M}_{\chi_i}^{\alpha}$  is isomorphic to the central fiber  $\mathfrak{L}_\chi$  of another quiver variety, which is of finite type  $A_1$  in the present case. The construction of  $e_i, f_i$  is compatible with this description, namely operators  $L_{\alpha} \rightleftharpoons L'_{\alpha}$  are given by applying the construction in the previous subsection for quiver varieties of finite type  $A_1$ . Quiver varieties of finite type  $A_1$  are cotangent bundles of Grassmanian [Nak94, §7], and the construction coincides with one done earlier by Ginzburg [Gin91] in this case.

From this explanation it is clear that dimension of  $\text{Hom}(\mathbb{C}_0, \pi'_! \text{IC}(\mathfrak{M}_{\chi_i}^{\alpha}))$  is equal to the multiplicity of an irreducible representation in the restriction

$$V(\lambda) \downarrow_{\mathfrak{l}_i}^{\mathfrak{sl}(n)_{\text{aff}}}$$

where  $\mathfrak{l}_i$  is the Levi factor corresponding to  $i$ . See [Nak09]. In the current situation  $\mathfrak{l}_i$  is the direct sum of  $\mathfrak{sl}_2$  and an abelian Lie algebra. The index  $\alpha$  corresponds to the highest weight of an irreducible representation of  $\mathfrak{l}_i$ . The nonemptiness of  $\mathfrak{M}_{\chi_i}^{\alpha}(V^0, W)$  implies that  $\dim W_i + \dim V_{i-1} + \dim V_{i+1} \geq 2 \dim V_i$ , hence it is dominant for  $\mathfrak{l}_i$ .

**11(iv). Sheaf theoretic formulation for tensor products.** Let us continue sheaf theoretic formulation of results above. We turn to Theorem 11.3. We review [Nak13].

We consider the diagram

$$\mathfrak{M}_0(\lambda, \mu)^{\nu^{\bullet}} \xleftarrow{p} \mathfrak{F}_0^{\bullet}(\lambda, \mu) \xrightarrow{j} \mathfrak{M}_0(\lambda, \mu),$$

where  $j$  is the inclusion and  $p$  is given by  $\lim_{t \rightarrow 0} \nu^{\bullet}(t)$ . We define the hyperbolic restriction functor  $\Phi$  by  $p_* j^!$ . Its fundamental properties are discussed in [Bra03, DG14]. In particular,  $\Phi$  sends  $\pi_!(\mathcal{C}_{\mathfrak{M}_\chi(\lambda, \mu)})$  to a direct sum of simple perverse sheaves with shifts.

Moreover, in the current situation,  $\Phi \pi_!(\mathcal{C}_{\mathfrak{M}_\chi(\lambda, \mu)})$  is a semisimple perverse sheaf, i.e. no shifts are necessary. This is because  $\Phi$  is hyperbolic semi-small in the sense of [BFN16c]. This is, in turn, a consequence of Theorem 11.3(1). Now the top degree homology and the hyperbolic restriction are related by

$$H_{\text{top}}(\tilde{\mathfrak{F}}_{\chi}^{\nu^{\bullet}}(\lambda, \mu)) \cong \text{Hom}(\mathbb{C}_0, \Phi \pi_!(\mathcal{C}_{\mathfrak{M}_\chi(\lambda, \mu)})).$$

See [Nak13, the paragraph after Lemma 4].



The statement (2) in Theorem 11.3 could be regarded as a computation of the hyperbolic restriction  $\Phi\pi_1(\mathcal{C}_{\mathfrak{M}_\chi(\lambda,\mu)})$ . Let

$$\sigma: \bigsqcup_{\mu=\mu^1+\mu^2} \mathfrak{M}_0(\lambda^1, \mu^1) \times \mathfrak{M}_0(\lambda^2, \mu^2) \rightarrow \mathfrak{M}_0(\lambda, \mu)^{\nu^\bullet}$$

denote a morphism given by sum, which is finite and surjective [Nak13, Lemma 1].

**Lemma 11.7** ([Nak13, Lemma 3]). *We have an isomorphism*

$$\sigma_1 \bigoplus_{\mu=\mu^1+\mu^2} (\pi \times \pi)_! \mathcal{C}_{\mathfrak{M}_\chi(\lambda^1, \mu^1) \times \mathfrak{M}_\chi(\lambda^2, \mu^2)} \cong \Phi\pi_1(\mathcal{C}_{\mathfrak{M}_\chi(\lambda, \mu)}).$$

This is a consequence of description of  $\tilde{\mathfrak{T}}_\chi^{\nu^\bullet}(\lambda, \mu)$  in §11(ii).

Taking a fiber at 0, we obtain the isomorphism (11.5). As we mentioned in Remark 11.4, (11.5) is not unique. The same is true in the above situation. We can also show that the stable envelope gives a canonical isomorphism also in above. It is a consequence of [Nak13, Lemma 4].

## 12. Coulomb branches and affine Lie algebras

**12(i). Integrable highest weight representations.** We return back to the Coulomb branch  $\mathcal{M}(\lambda, \mu)$  of a quiver gauge theory of affine type  $A$ . As we explained in §3, the Coulomb branch has a quantization, and hence a Poisson bracket. It is shown that it gives a symplectic form on the smooth locus as a statement for general Coulomb branches. In our case  $\mathcal{M}(\lambda, \mu)$ , a finer statement is known: The decomposition into symplectic leaves are given by

$$(12.1) \quad \mathcal{M}(\lambda, \mu) = \bigsqcup_{\kappa, \underline{k}} \mathcal{M}^s(\kappa, \mu) \times S^k(\mathbb{C}^2 \setminus \{0\}/(\mathbb{Z}/n\mathbb{Z})),$$

where  $\underline{k} = [k_1, k_2, \dots]$  is a partition, and  $\kappa$  is a dominant weight with  $\lambda - |\underline{k}|\delta \geq \kappa \geq \mu$ . And  $\mathcal{M}^s(\kappa, \mu)$  is the regular locus of  $\mathcal{M}(\kappa, \mu)$  if  $\dim W \neq 1$  or  $\kappa = \mu$  and  $\emptyset$  otherwise. See [NT17, Th. 7.26].

Recall that we have chosen a character  $\chi: G \rightarrow \mathbb{C}^\times$  given by the product of determinants. We consider the induced homomorphism  $\pi_1(\chi): \pi_1(G) \rightarrow \pi_1(\mathbb{C}^\times)$  and its Pontryagin dual  $\pi_1(\chi)^\wedge: \pi_1(\mathbb{C}^\times)^\wedge \rightarrow \pi_1(G)^\wedge$ . By §6(iii)  $\pi_1(G)^\wedge$  acts on  $\mathcal{M}(\lambda, \mu)$ . In our case we have  $\pi_1(G)^\wedge = (\mathbb{C}^\times)^n$ . (We assume  $V_i \neq 0$  for all  $i$  for brevity.) Via  $\pi_1(\mathbb{C}^\times)^\wedge = \mathbb{C}^\times$ , we consider  $\pi_1(\chi)^\wedge$  as a one parameter subgroup in  $\pi_1(G)^\wedge = (\mathbb{C}^\times)^n$ . Let us write  $\pi_1(\chi)^\wedge$  by  $\chi$  for brevity.

The following was proved in [NT17, Prop. 7.30] (see also [Nak18b, Prop. 4.1]):

**Lemma 12.2.** *The fixed point set  $\mathcal{M}(\lambda, \mu)^\chi$  is either empty or a single point.*

In analogy with (11.2) we introduce the attracting set

$$\mathfrak{A}_\chi(\lambda, \mu) \stackrel{\text{def.}}{=} \left\{ x \in \mathcal{M}(\lambda, \mu) \mid \lim_{t \rightarrow 0} \chi(t)x \text{ exists} \right\}.$$

Note that a tilde version is same by the above lemma.

**Remark 12.3.** For Higgs branch,  $\chi$  gave a resolution  $\mathfrak{M}_\chi(\lambda, \mu) \rightarrow \mathfrak{M}_0(\lambda, \mu)$ . The same  $\chi$  give a one parameter subgroup acting on the Coulomb branch  $\mathcal{M}(\lambda, \mu)$ . In the next subsection we see the opposite:  $\nu^\bullet$  acts on  $\mathfrak{M}_0(\lambda, \mu)$ , and gives a partial resolution (we will consider the deformation instead) of  $\mathcal{M}(\lambda, \mu)$ .

In physics it is said that the role of FI and mass parameters on Coulomb and Higgs branches are exchanged. Our definition of Coulomb branches is given so that this is established rigorously.

Now we state the main result

**Theorem 12.4** ([Nak18b]). (1) *Every intersection with  $\mathfrak{A}_\chi(\lambda, \mu)$  and a symplectic leaf of  $\mathcal{M}(\lambda, \mu)$  is either empty or a lagrangian subvariety in the leaf.*

(2) *The direct sum*

$$\bigoplus_{\mu} H_{\text{top}}(\mathfrak{A}_\chi(\lambda, \mu))$$

*of top degree homology groups of  $\mathfrak{A}_\chi(\lambda, \mu)$  over  $\mu$  has a structure of an integrable highest weight representation  $V(\lambda)$  of an affine Lie algebra  $\mathfrak{sl}(n)_{\text{aff}}$  with highest weight  $\lambda$ .*

As we already mentioned, the construction in Theorem 12.4 is *formally* similar to one in Theorem 11.1. We define operators  $d$  and  $e_i, f_i, h_i$  ( $i \in \mathbb{Z}/n\mathbb{Z}$ ). The operators  $d$  and  $h_i$  are determined by dimension vectors, namely they are defined so that  $H_{\text{top}}(\mathfrak{A}_\chi(\lambda, \mu))$  is identified with  $V_\mu(\lambda)$ . Operators  $e_i, f_i$  are defined via study of varieties associated with another character  $\chi_i$ . Namely we consider the  $\chi_i$ -fixed point set  $\mathcal{M}(\lambda, \mu)^{\chi_i}$  and the attracting set  $\mathfrak{A}_{\chi_i}(\lambda, \mu) \stackrel{\text{def.}}{=} \{x \in \mathcal{M}(\lambda, \mu) \mid \lim_{t \rightarrow 0} \chi_i(t)x \text{ exists}\}$ . We also consider the  $\chi_i$ -fixed point set  $\mathfrak{A}_\chi(\lambda, \mu)^{\chi_i}$  in the attracting set  $\mathfrak{A}_\chi(\lambda, \mu)$ . Then the original attracting set is the fiber product:

$$(12.5) \quad \mathfrak{A}_\chi(\lambda, \mu) = \mathfrak{A}_\chi(\lambda, \mu)^{\chi_i} \times_{\mathcal{M}(\lambda, \mu)^{\chi_i}} \mathfrak{A}_{\chi_i}(\lambda, \mu),$$

where the morphism  $\mathfrak{A}_{\chi_i}(\lambda, \mu) \rightarrow \mathcal{M}(\lambda, \mu)^{\chi_i}$  is given by  $\lim_{t \rightarrow 0} \chi(t)$ .

We then show that  $\mathcal{M}(\lambda, \mu)^{\chi_i}$  is either empty or isomorphic to the Coulomb branch  $\mathcal{M}_{A_1}(\lambda', \mu')$  of type  $A_1$  quiver gauge theory with some  $\lambda', \mu'$ . Thus  $\mathfrak{A}_\chi(\lambda, \mu)^{\chi_i}$  is the attracting set in the Coulomb branch of type  $A_1$  quiver gauge theory.

This picture enables us to define operators  $e_i, f_i$  by a reduction to the  $\mathfrak{sl}(2)$  case. This is formally similar to the method used in Theorem 11.1. A further detail of the definition of  $e_i, f_i$  requires hyperbolic restriction functors, and hence is postponed to §12(iii).

Let us also note that  $\bigoplus_{\mu} H_{\text{top}}(\mathfrak{A}_\chi(\lambda, \mu))$  has a distinguished base given by fundamental classes of irreducible components of  $\mathfrak{A}_\chi(\lambda, \mu)$ . It can be shown that the set  $\bigsqcup \text{Irr } \mathfrak{A}_\chi(\lambda, \mu)$  of irreducible components has a structure of Kashiwara crystal isomorphic to one for crystal base of  $\mathbf{U}_q(\mathfrak{sl}(n)_{\text{aff}})$ -version of  $V(\lambda)$ . See [Nak18b, §5(vi)].

**12(ii). Tensor product.** Recall that we take a one parameter subgroup  $\nu^\bullet: \mathbb{C}^\times \rightarrow \text{GL}(W)$  in §11(ii) associated with a decomposition  $W = W^1 \oplus W^2$ . It can be considered as a flavor symmetry of the gauge theory. As is explained in §6(iv), we have the deformation of  $\mathcal{M}(\lambda, \mu)$  parametrized by  $\mathbb{C} = \text{Spec } H_{\mathbb{C}^\times}^*(\text{pt})$ , as well as a partial resolution. Let us use the former and denote its fiber over 1 by  $\mathcal{M}^{\nu^\bullet, \mathbb{C}}(\lambda, \mu)$ . (The superscript ‘ $\mathbb{C}$ ’ indicates that we choose the former. The notation is the same as in [Nak18b].)

Let  $\lambda^1, \lambda^2$  denote weights corresponding to  $(\dim W^1), (\dim W^2)$  as before. As in Lemma 12.2 we have

**Lemma 12.6.** *The fixed point set  $\mathcal{M}^{\nu^\bullet, \mathbb{C}}(\lambda, \mu)^{\chi}$  is finite, and corresponds naturally to  $\bigsqcup_{\mu=\mu^1+\mu^2} \mathcal{M}(\lambda^1, \mu^1)^{\chi} \times \mathcal{M}(\lambda^2, \mu^2)^{\chi}$ .*

Recall  $\mathcal{M}(\lambda, \mu)^{\times}$  is either empty or a single point by Lemma 12.2. And in view of Theorem 12.4, it is a single point if and only if  $V_{\mu}(\lambda) \neq 0$ . Therefore the above lemma says that  $\mathcal{M}^{\nu^{\bullet, c}}(\lambda, \mu)^{\times}$  is nonempty if and only if  $\mu$  is a weight of  $V(\lambda^1) \otimes V(\lambda^2)$ . Let us introduce the attracting set:

$$\mathfrak{A}_{\chi}^{\nu^{\bullet, c}}(\lambda, \mu) \stackrel{\text{def.}}{=} \left\{ x \in \mathcal{M}^{\nu^{\bullet, c}}(\lambda, \mu) \mid \lim_{t \rightarrow 0} \chi(t)x \text{ exists} \right\}.$$

**Theorem 12.7** (cf. [Nak18b, Cor. 4.9]). *The direct sum*

$$\bigoplus_{\mu} H_{\text{top}}(\mathfrak{A}_{\chi}^{\nu^{\bullet, c}}(\lambda, \mu))$$

of top degree homology groups of  $\mathfrak{A}_{\chi}^{\nu^{\bullet, c}}(\lambda, \mu)$  over  $\mu$  has a structure of an integrable representation of  $\mathfrak{sl}(n)_{\text{aff}}$ , isomorphic to the tensor product  $V(\lambda^1) \otimes V(\lambda^2)$ .

**12(iii). Sheaf theoretic formulation.** As in §11(iv), we consider the diagram

$$\mathcal{M}(\lambda, \mu)^{\times} \xleftarrow{p} \mathfrak{A}_{\chi}(\lambda, \mu) \xrightarrow{j} \mathcal{M}(\lambda, \mu),$$

and define the hyperbolic restriction functor  $\Phi = p_* j^!$ . We understand  $\Phi = 0$  when  $\mathcal{M}(\lambda, \mu)^{\times} = \emptyset$ . Then  $\Phi(\text{IC}(\mathcal{M}(\lambda, \mu)))$ , if it is nonzero, is a semisimple perverse sheaf over a point  $\mathcal{M}(\lambda, \mu)^{\times}$  as in §11(iv). We have

$$H_{\text{top}}(\mathfrak{A}_{\chi}(\lambda, \mu)) \cong \Phi(\text{IC}(\mathcal{M}(\lambda, \mu))).$$

Let us consider  $\chi_i$  as in §12(i). Then we have hyperbolic restriction functors  $\Phi_i$  and  $\Phi^i$  associated with  $\mathfrak{A}_{\chi}(\lambda, \mu)$  and  $\mathfrak{A}_{\chi}(\lambda, \mu)^{\chi_i}$  respectively. The fiber product property (12.5) implies  $\Phi = \Phi^i \circ \Phi_i$ . Thus  $\Phi(\text{IC}(\mathcal{M}(\lambda, \mu))) = \Phi^i(\Phi_i(\text{IC}(\mathcal{M}(\lambda, \mu))))$ .

Recall that  $\mathcal{M}(\lambda, \mu)^{\chi_i}$  is  $\mathcal{M}_{A_1}(\lambda', \mu')$ . The symplectic leaves (12.1) for type  $A_1$  are just  $\mathcal{M}_{A_1}(\kappa', \mu')$  where positive integers  $\kappa'$  with  $\lambda' \geq \kappa' \geq \mu'$ . The morphisms  $p, j$  are compatible with strata, hence we only get perverse sheaves which are locally constant along strata. By a little more argument one can show that no nonconstant local systems appear. Hence

$$\Phi_i(\text{IC}(\mathcal{M}(\lambda, \mu))) \cong \bigoplus_{\kappa'} M_{\kappa', \mu'}^{\lambda, \mu} \otimes \text{IC}(\mathcal{M}_{A_1}(\kappa', \mu'))$$

for vector spaces  $M_{\kappa', \mu'}^{\lambda, \mu}$ .

For  $\mu - \alpha_i$ , we have

$$\Phi_i(\text{IC}(\mathcal{M}(\lambda, \mu - \alpha_i))) \cong \bigoplus_{\kappa'} M_{\kappa', \mu' - 2}^{\lambda, \mu - \alpha_i} \otimes \text{IC}(\mathcal{M}_{A_1}(\kappa', \mu' - 2)).$$

We then show that there is a natural isomorphism  $M_{\kappa', \mu'}^{\lambda, \mu} \cong M_{\kappa', \mu' - 2}^{\lambda, \mu - \alpha_i}$ . This is a consequence of the factorization property of  $\varpi$  introduced in §6(ii). See [Nak18b, Prop. 5.11].

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KAVLI INSTITUTE FOR THE PHYSICS AND MATHEMATICS OF THE UNIVERSE (WPI), THE UNIVERSITY OF TOKYO, 5-1-5 KASHIWANOHA, KASHIWA, CHIBA, 277-8583, JAPAN  
*E-mail address:* [hiraku.nakajima@ipmu.jp](mailto:hiraku.nakajima@ipmu.jp)