Relative Gromov-Witten theory in Symplectic geometry

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a research in progress
with
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The title of my Kinosaki talk is Relative Gromov-Witten theory in Symplectic geometry.

This document is a slide of my talks at China and France on a related topics, which focus associativity of quantum cohomology.

My Kinosaki talk was more on the side of survey talks.

I feel this slide is more suitable to be public in a proceeding, since it is more focused and contain more mathematical contents than Kinosaki talk.

The contents is related to Section 6 of my joint paper MONOTONE LAGRANGIAN FLOER THEORY IN SMOOTH DIVISOR COMPLEMENTS: I with A. Daemi. (arXiv:1808.089151v).

(X, ω) Symplectic manifold

$D \subset X$ codimension 2 submanifold

J almost complex structure

$U \supset D$ a neighborhood

J is integrable on U

D is a complex submanifold of (U, J)

Λ_0 **Novikov ring**

$$\ni \quad c = \sum_i c_i T^{\lambda_i} \qquad c_i \in \mathbb{R} \qquad \lambda_i \geq 0 \qquad \lim_{i \to \infty} \lambda_i = +\infty$$

$$\mathfrak{v}_T(c) = \inf\{\lambda_i\}$$

$\Lambda_0\{q, q^{-1}\}$

$$\ni \quad a = \sum_{n \in \mathbb{Z}} a_n q^n \qquad a_n \in \Lambda_0 \qquad \lim_{|n| \to \infty} \mathfrak{v}_T(a_n) = +\infty$$

$$H = H(X \setminus D; \Lambda_0) \oplus H(D; \Lambda_0\{q, q^{-1}\})$$

Problem

Some variant of H has a structure of graded commutative ring.

something related to the coefficient of q^0 is to be understood.

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Open closed map

$$H = H(X \setminus D; \Lambda_0) \oplus H(D; \Lambda_-\{q, q^{-1}\})$$

$Fuk(X \setminus D)$ filtered (curved) A infinity category whose object is a compact Lagrangian submanifold $L \subset X \setminus D$

Problem

There is a ring homomorphism

$$H \to HH(Fuk(X \setminus D))\{q, q^{-1}\}$$

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$H(X \setminus D; \Lambda_0) \subset H$ is not a subring in general.

Note if $X \setminus D$ is convex

↓

Usual Gromov-Witten theory defines a ring structure on $H(X \setminus D; \Lambda_0) \subset H$

What's wrong in our case ?

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What's wrong in our case ?

Let me start with reviewing the proof of associativity in usual Gromov-Witten theory.

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$\mathcal{M}_\ell(X;\alpha) =$

$$\left\{ [(\Sigma,\vec{z}),u] \;\middle|\; \begin{array}{l} \Sigma \text{ is genus } 0 \\ \vec{z} = (z_1,\dots,z_\ell) \;\; \ell \text{ marked points} \\ u : \Sigma \to X \;\; \text{holomorphic} \\ u_*([S^2]) = \alpha. \quad \text{stable} \end{array} \right\} \Big/ \sim$$

$\alpha \in H_2(X;\mathbb{Z})$

moduli space of stable map

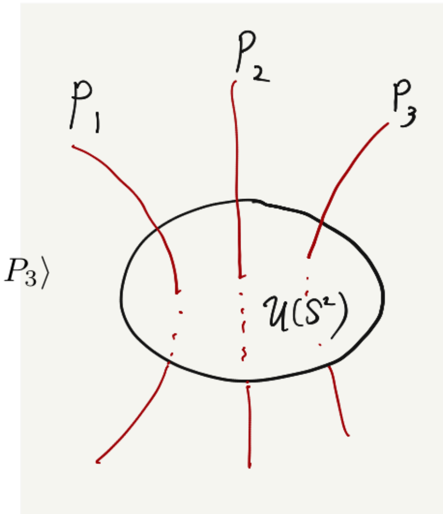
$$\mathrm{ev} : \mathcal{M}_\ell(X;\alpha) \to X^\ell$$

$$[(\Sigma,\vec{z}),u] \mapsto (u(z_1),\dots,u(z_\ell))$$

P_1, P_2, P_3 cycles in $X \setminus D$ (in D)

$$\langle P_1 \cup^Q P_2, P_3 \rangle$$
$$= \sum_\alpha T^{\alpha\cap\omega} \# \left(\mathcal{M}_3(X;\alpha)_{\mathrm{ev}} \times (P_1 \times P_2 \times P_3) \right)$$

$\langle P_1 \cup^Q P_2, P_3 \rangle$

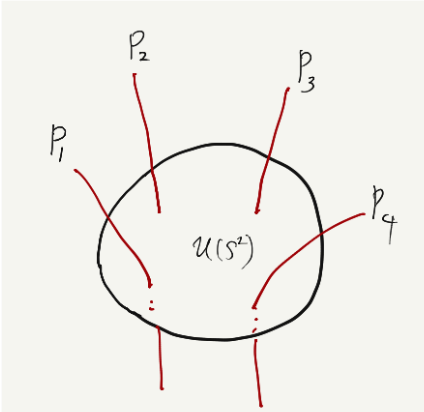


The proof of associativity uses

$$\mathcal{M}_4(X;\alpha) \times_{X^4} (P_1 \times P_2 \times P_3 \times P_4)$$
$$\downarrow$$
$$\mathcal{M}_4$$

Deligne-Mumford moduli of sphere with 4 points

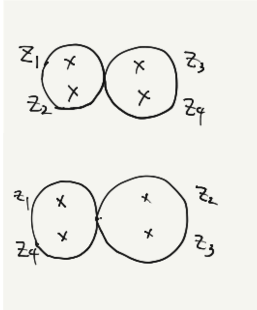
$\mathcal{M}_4 \cong S^2$



$$\mathcal{M}_4(X;\alpha) \times_{X^4} (P_1 \times P_2 \times P_3 \times P_4)$$

$\mathcal{M}_4$ Deligne-Mumford moduli of sphere with 4 points

contains 3 special points.



$x_{12} \in \mathcal{M}_4$

$x_{14} \in \mathcal{M}_4$

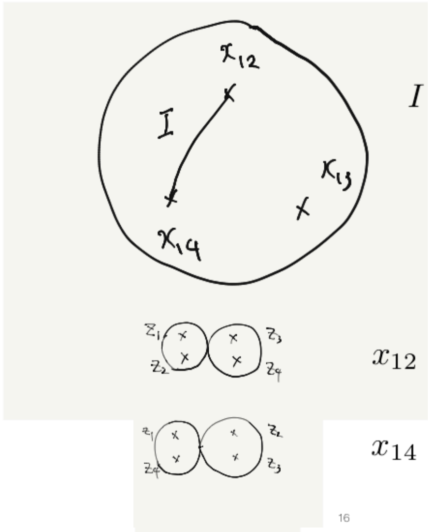
$$\mathcal{M}_4(X;\alpha) \times_{X^4} (P_1 \times P_2 \times P_3 \times P_4)$$

$$\pi^{-1}(x_{12}) \subset \quad \downarrow \pi \quad \supset \pi^{-1}(x_{14})$$

$$x_{12} \in \mathcal{M}_4 \ni x_{14}$$

$\pi^{-1}(x_{12})$ ↔ $\langle (P_1 \cup^Q P_2) \cup^Q P_3, P_4 \rangle$

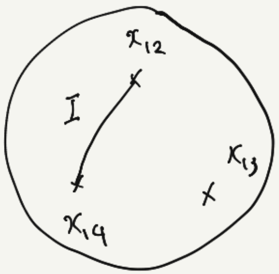
$\pi^{-1}(x_{14})$ ↔ $\langle P_1 \cup^Q (P_2 \cup^Q P_3), P_4 \rangle$



I is a path joining x_{12} and x_{14}

$x_{12} \in \mathcal{M}_4$

$x_{14} \in \mathcal{M}_4$



I is a path joining x_{12} and x_{14}

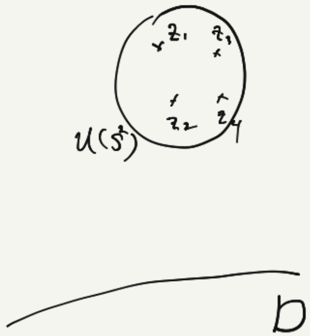
$\pi^{-1}(I)$ is a cobordism between $\pi^{-1}(x_{12})$ and $\pi^{-1}(x_{14})$

$$\langle (P_1 \cup^Q P_2) \cup^Q P_3, P_4 \rangle \quad = \quad \langle P_1 \cup^Q (P_2 \cup^Q P_3), P_4 \rangle$$

Back to the case of divisor complement: $X \setminus D$

$\mathcal{M}_4^\circ(X \setminus D; \alpha)$

$u(S^2) \cap D^2 = \emptyset$



$\mathcal{M}_4^\circ(X \setminus D; \alpha)$ is compact if $X \setminus D$ is convex.

If $X \setminus D$ is not convex

$\mathcal{M}_4^\circ(X \setminus D; \alpha)$ is not compact.

So the cobordism argument breaks down.

If $X \setminus D$ is not convex

$\mathcal{M}_4^\circ(X \setminus D; \alpha)$ is not compact.

We need a different compactification from the usual stable map compactification.

We call it RGW compactification. $\mathcal{M}_4^{\mathrm{RGW}}(X \setminus D, \alpha)$

RGW = relative Gromov Witten theory

RGW compactification $\mathcal{M}_4^{\mathrm{RGW}}(X \setminus D, \alpha)$

There are many related works

J. Li, Gross-Siebert (algebraic case)

Ionel-Paker, A.M. Li - Y. Ruan
B.Parker
Tehurani, Zinger
(symplectic case)

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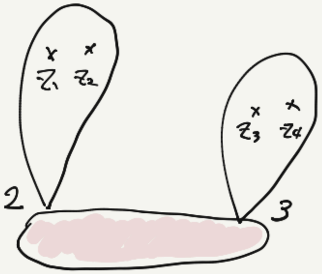
A typical element of $\mathcal{M}_4^{\mathrm{RGW}}(X \setminus D, \alpha)$



this component goes to D

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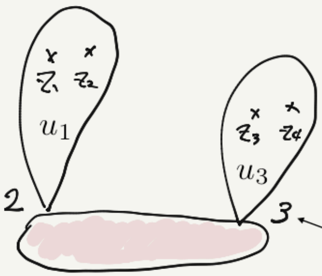

order of tangency

order of tangency

$2 = (u_1)_*([S^2]) \cap D$

$3 = (u_3)_*([S^2]) \cap D$

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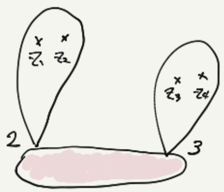
Important point

In RGW compactification
$\mathcal{M}_4^{\mathrm{RGW}}(X \setminus D, \alpha)$
this configuration
appears in codimension 2



In the stable map compactification
this configuration
appears in codimension 4

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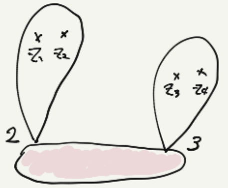
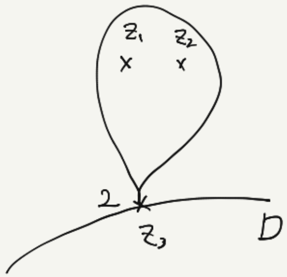


is described by a fiber product

$$\mathcal{M}_{(0)(0)(2)}(X, D; \alpha_1) \ \times_D \ \mathcal{M}_2(D; \alpha_2)$$
$$\times_D \ \mathcal{M}_{(0)(0)(3)}(X, D; \alpha_3)$$

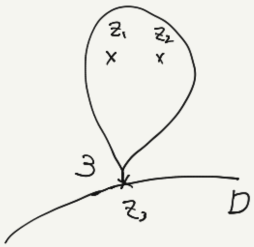
$\mathcal{M}_{(0)(0)(2)}(X, D; \alpha_1)$



sphere with 3 marked points $u_1 : S^2 \to X$

u_1 intersect with D with order 0,0,2 at z_1, z_2, z_3

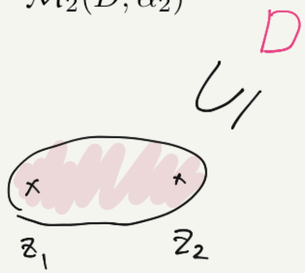
$\mathcal{M}_{(0)(0)(3)}(X, D; \alpha_3)$



sphere with 3 marked points $u_3 : S^2 \to X$

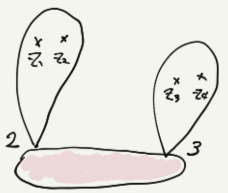
u_3 intersect with D with order 0,0,3 at z_1, z_2, z_3

$\mathcal{M}_2(D; \alpha_2)$



sphere with 2 marked points $u_2 : S^2 \to D$

in *D*



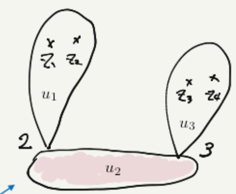
is described by a fiber product

$$\mathcal{M}_{(0)(0)(2)}(X,D;\alpha_1) \times_D \mathcal{M}_2(D;\alpha_2) \times_D \mathcal{M}_{(0)(0)(3)}(X,D;\alpha_3)$$

Put

$$\mathcal{M}_0 = \mathcal{M}_{(0)(0)(2)}(X,D;\alpha_1) \times_D \mathcal{M}_2(D;\alpha_2) \times_D \mathcal{M}_{(0)(0)(3)}(X,D;\alpha_3)$$

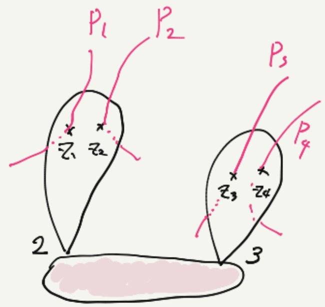
$$\mathrm{ev} : \mathcal{M}_0 \to (X \setminus D)^4$$



$$\mapsto (u_1(z_1), u_1(z_2), u_3(z_3), u_3(z_4))$$

P_1, P_2, P_3, P_4 Cycles in X.

$$\mathcal{M}_0(P_1,P_2,P_3,P_4) = \mathcal{M}_0 \times_{X^4} (P_1 \times P_2 \times P_3 \times P_4)$$

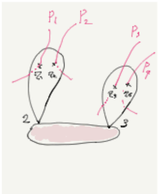


We assume:

$\mathcal{M}_0(P_1, P_2, P_3, P_4)$

is transversal and consists of one point.

In other words



is transversal in its moduli space and is rigid

$$\mathcal{M}_0(P_1, P_2, P_3, P_4)$$

$$\cap\!\!\mid$$

$$\mathcal{M}_4^{\rm RGW}(X;\alpha) \times_{X^4} (P_1 \times P_2 \times P_3 \times P_4)$$

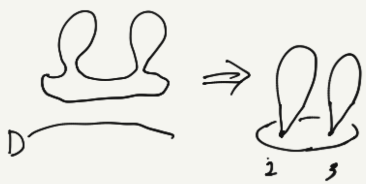
$$\parallel$$

$$\mathcal{M}_4^{\rm RGW}(X;\alpha;\vec{P})$$

$\alpha = \alpha_1 + \alpha_2 + \alpha_3$

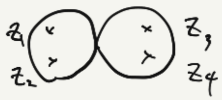
$\alpha \cap D = 0$

Elements of $\mathcal{M}_4^{\circ}(X \setminus D;\alpha)$ converge to an element of $\mathcal{M}_0$



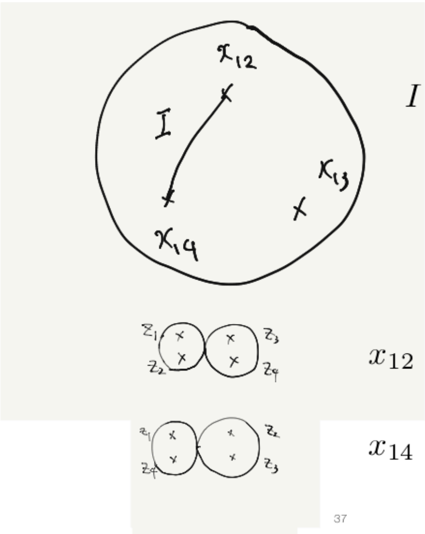
$\alpha \cap D = 0$

$$\begin{array}{ccc} \mathcal{M}_0(P_1, P_2, P_3, P_4) & \subseteq & \mathcal{M}_4^{\rm RGW}(X;\alpha;\vec{P}) \\ \downarrow & & \downarrow \pi \\ x_{12} & \in & \mathcal{M}_4 \end{array}$$



Deligne-Mumford moduli of sphere with 4 points

$\mathcal{M}_4 \cong S^2$

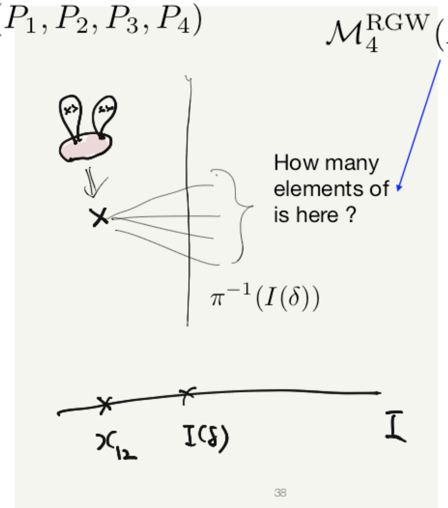


I is a path joining x_{12} and x_{14}

$x_{12} \in \mathcal{M}_4$

$x_{14} \in$

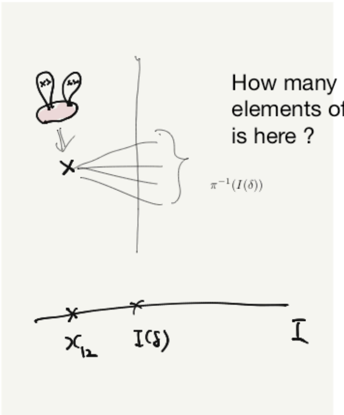
$\mathcal{M}_0(P_1, P_2, P_3, P_4)$ $\qquad$ $\mathcal{M}_4^{\mathrm{RGW}}(X; \alpha; \vec{P})$



How many elements of is here ?

π

$\mathcal{M}_4$



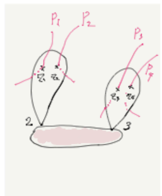
How many elements of $\mathcal{M}_4^{\mathrm{RGW}}(X; \alpha; \vec{P})$ is here ?

The answer is 5

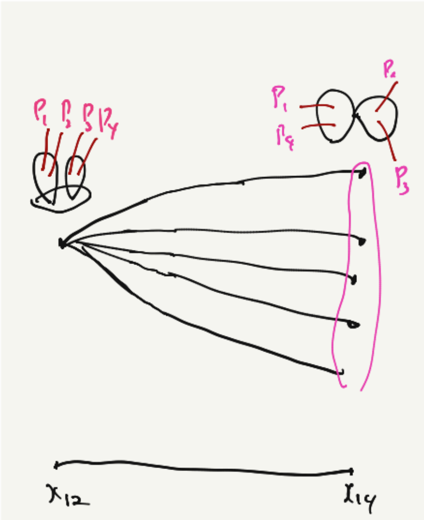
The answer is 5

$$5 \neq 0$$

If we simply forget



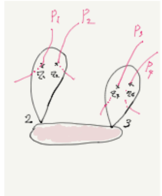
then the quantum cup product is NOT associative.



cobordism
argument
fails
if we forget

Why 5 ?

The fiber product etc. to describe

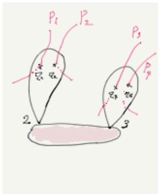


is assumed to be transversal.
However this point is still a singular point
in the moduli space $\mathcal{M}_4^{\mathrm{RGW}}(X;\alpha;\vec{P})$

This is the point very much different from
stable map compactification.

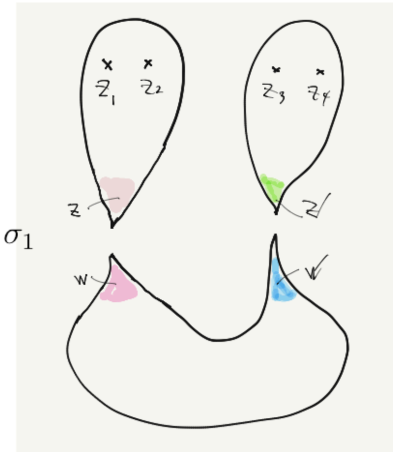
Local Kuranishi model of $\mathcal{M}_4^{\mathrm{RGW}}(X;\alpha;\vec{P})$

in a neighborhood of



is $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$ $\quad (\sigma_1, \sigma_2) \mapsto \sigma_1^2 - \sigma_2^3$

$zw = \sigma_1$

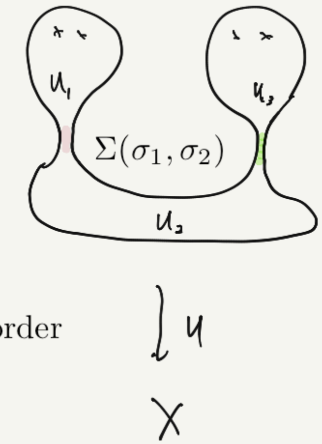


$z'w' = \sigma_2$

Glue the 3 irreducible components via the parameter σ_1, σ_2
$\Sigma(\sigma_1, \sigma_2)$

to obtain a well defined global map

$u : \Sigma(\sigma_1, \sigma_2) \to X$

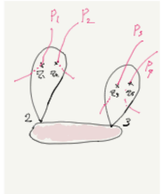


we need a condition:

$\sigma_1^2 = \sigma_2^3 + \text{higher order}$

Local Kuranishi model of $\mathcal{M}_4^{\mathrm{RGW}}(X; \alpha; \vec{P})$

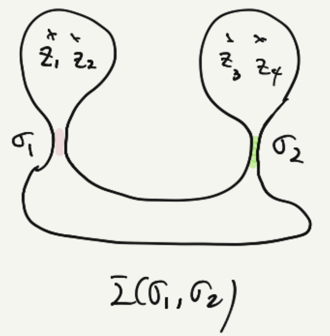
in a neighborhood of

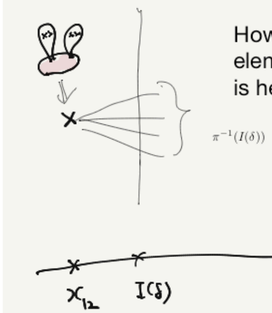


is $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$ $\quad (\sigma_1, \sigma_2) \mapsto \sigma_1^2 - \sigma_2^3$

The moduli parameter of $(\Sigma(\sigma_1, \sigma_2), z_1, z_2, z_3, z_4)$

is $\sigma = \sigma_1 \sigma_2$



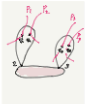


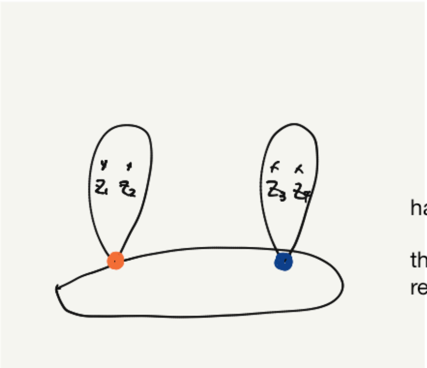
How many elements of $\mathcal{M}_4^{\mathrm{RGW}}(X; \alpha; \vec{P})$ is here ?

The equation we solve is: $\quad \sigma_1 \sigma_2 = \delta \quad \text{given}$

$\sigma_1^2 = \sigma_2^3$

We have 5 solutions !

How  contribute 5 in $\langle (P_1 \cup^Q P_2) \cup^Q P_3, P_4 \rangle$



has two nodes

they contribute 3 and 2 respectively

Let R be a cycle in D.

I will define $\langle P_1 \cup^Q P_2, q^{-2} R \rangle$

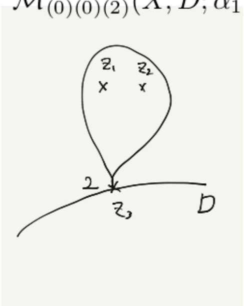
$\langle P_1 \cup^Q P_2, q^{3} R \rangle$

$$H = H(X;\Lambda_0) \oplus H(D;\Lambda_0\{q, q^{-1}\})$$

$$\underset{P_i}{\Cup} \qquad \underset{q^k R}{\Cup}$$

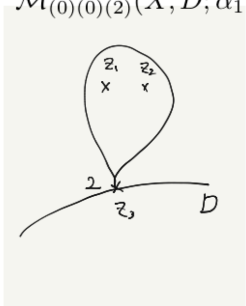
$$\langle q^k R, q^\ell R' \rangle = \delta_{k+\ell} \langle R, R' \rangle$$

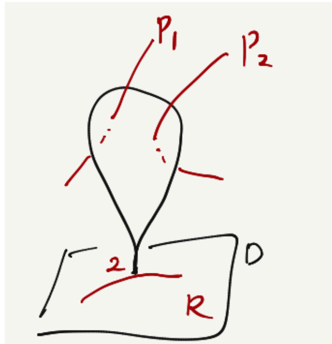
$\mathcal{M}_{(0)(0)(2)}(X, D; \alpha_1)$



$$\langle P_1 \cup^Q P_2, q^{-2} R \rangle$$
$$= \#(\mathcal{M}_{(0)(0)(2)}(X, D; \alpha_1) \times_{X^2 \times D} (P_1 \times P_2 \times R)$$

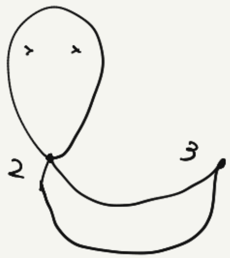
$\mathcal{M}_{(0)(0)(2)}(X, D; \alpha_1)$





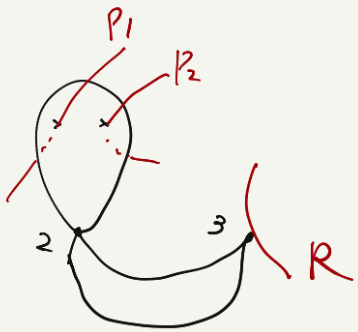
$$\langle P_1 \cup^Q P_2, q^{-2} R \rangle$$
$$= \#(\mathcal{M}_{(0)(0)(2)}(X, D; \alpha_1) \times_{X^2 \times D} (P_1 \times P_2 \times R)$$

$$\mathcal{M}_{(0)(0)(2)}(X, D; \alpha_1) \times_D \mathcal{M}_2(D; \alpha_2)$$



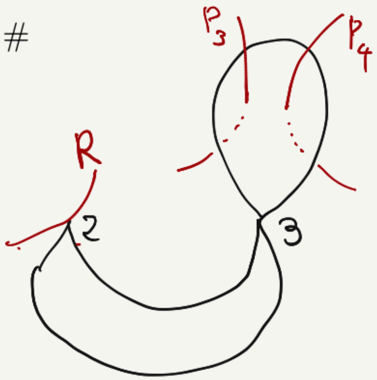
$$\langle P_1 \cup^Q P_2, q^3 R \rangle$$

$$= 2 \times \# \left(\mathcal{M}_{(0)(0)(2)}(X, D; \alpha_1) \times_D \mathcal{M}_2(D; \alpha_2)\right) \times_{X^2 \times D} (P_1 \times P_2 \times R)$$



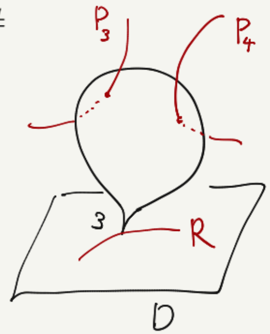
$$\langle q^2 R \cup P_3, P_4 \rangle$$

$$= 3 \times \#$$



$$\langle q^{-3} R \cup^Q P_3, P_4 \rangle$$

$$= \#$$



$$\langle (P_1 \cup^Q P_2) \cup P_3, P_4 \rangle$$

$$= \langle P_1 \cup^Q P_2, q^{-2} R \rangle \langle q^2 R^* \cup^Q P_3, P_4 \rangle$$
$$+ \langle P_1 \cup^Q P_2, q^3 R \rangle \langle q^{-3} R^* \cup^Q P_3, P_4 \rangle$$
$$+ \ldots$$
$$= 1 \times 3 + 2 \times 1 + \ldots$$
$$= 5 + \ldots$$

this is a part of the
`proof' of associativity.