ARITHMETIC APPROACH TO NEWTON-OKOUNKOV BODIES

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Newton-Okounkov body is an efficient tool to study the volume function on the group of Cartier divisors over a projective variety. Initialised by Okounkov [14, 15], this theory is then developed by Lazarsfeld and Mustaţă [12] and Kaveh and Khovanskii [10, 11]. We remind briefly its construction. Let k be a field, X be an integral scheme over $\operatorname{Spec} k$, and K be the field of rational functions on X. Let d be the Krull dimension of the scheme X. We equip $\mathbb{Z}^d$ with a monomial order (for example the lexicographic order) and fix a $\mathbb{Z}^d$-valuation $\operatorname{val}(\cdot)$ on K such that

(i) for any $a \in k \setminus \{0\}$, $\operatorname{val}(a) = 0$,
(ii) for any $\alpha \in \mathbb{Z}^d$, the quotient k-vector space

$$\{x \in K \mid \operatorname{val}(x) \geqslant \alpha\}/\{x \in K \mid \operatorname{val}(x) > \alpha\}$$

has dimension 0 or 1.

Any $\mathbb{Z}^d$-valuation satisfying these properties is said to be *of one-dimensional leaves*. For example, such $\mathbb{Z}^d$-valuation can be obtained by choosing a regular rational point P of X and a regular sequence in the local ring $\mathcal{O}_{X,P}$ (see [10, §2.2]). Given a graded sub-k-algebra $V_\bullet = \bigoplus_{n\in\mathbb{N}} V_n T^n$ of the polynomial ring $K[T]$, the *Newton-Okounkov body* of $V_\bullet$ is defined as

$$\Delta(V_\bullet) := \text{Convex hull of} \bigcup_{n\in\mathbb{N},\, n\geqslant 1} \Big\{\frac{1}{n}\operatorname{val}(x) \,\Big|\, x \in V_n,\ x \neq 0\Big\}.$$

In the case where the Newton-Okounkov semigroup

$$\Gamma(V_\bullet) := \bigcup_{n\in\mathbb{N},\, n\geqslant 1} \{(n, \operatorname{val}(x)) \mid x \in V_n,\ x \neq 0\}$$

generates $\mathbb{Z}^{d+1}$ as a group, the sequence

$$\frac{\dim_k(V_n)}{n^d}, \quad n \in \mathbb{N},\ n \geqslant 1$$

converges to the Lebesgue measure of the convex set $\Delta(V_\bullet)$. Thus we associate graded linear series with convex bodies in $\mathbb{R}^d$ in order to understand the asymptotic behaviour of these graded linear series, generalising the classic combinatoric study of toric varieties.

From the point of view of arithmetic geometry, where the base field is often not algebraically closed, the existence of a $\mathbb{Z}^d$-valuation of one-dimensional leaves on K is not guaranteed in general. The existence of such a $\mathbb{Z}^d$-valuation $\operatorname{val}(\cdot)$ implies actually that the extension K/k is geometrically integral. In fact, any non-zero

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element α of $K \otimes_k \overline{k}$ can be written in the form $x_1 \otimes a_1 + \cdots + x_n \otimes a_n$ with $(x_1, \ldots, x_n) \in (K \setminus \{0\})^n$ and $(a_1, \ldots, a_n) \in (\overline{k} \setminus \{0\})^n$ satisfying

$$\mathrm{val}(x_1) > \mathrm{val}(x_2) \geqslant \ldots \geqslant \mathrm{val}(x_n).$$

Moreover, the value of $\mathrm{val}(x_1)$ does not depend on the choice of the decomposition. These facts can be shown by an induction argument on the tensorial rank of α (the minimal positive integer n such that α can be written as the sum of n split tensors). Hence we can extend the valuation function $\mathrm{val}(\cdot)$ to $K \otimes_k \overline{k}$ by associating $\mathrm{val}(x_1)$ to α. The extended map is additive with respect to the multiplication law. This implies that $K \otimes_k \overline{k}$ is an integral domain. Therefore it is interesting to have an alternative approach of Newton-Okounkov bodies for the case where K does not admit a $\mathbb{Z}^d$-valuation of one-dimensional leaves.

Before explaining the main idea for the alternative approach of Newton-Okounkov bodies, let me briefly recall the construction of arithmetic Newton-Okounkov bodies. Let $\mathscr{X}$ be an *arithmetic projective variety*, that is, an integral, flat, and projective scheme over $\operatorname{Spec} \mathbb{Z}$. As *Hermitian line bundle* over $\mathscr{X}$, we refer to the data $\overline{\mathscr{L}}$ consisting of an invertible sheaf $\mathscr{L}$ on $\mathscr{X}$ together with metric $\varphi = (|\cdot|_\varphi(x))_{x \in \mathscr{X}(\mathbb{C})}$ on $\mathscr{L}(\mathbb{C})$, which is assumed to be continuous with respect to the analytic topology, and invariant by the complex conjugation. We are interested in the asymptotic behaviour of $\ln(\operatorname{card} \widehat{H}^0(\overline{\mathscr{L}}^{\otimes n}))$ when $n \to +\infty$, where

$$\widehat{H}^0(\overline{\mathscr{L}}^{\otimes n}) := \Big\{ s \in \Gamma(\mathscr{X}, \mathscr{L}^{\otimes n}) \ \Big| \ \sup_{x \in \mathscr{X}(\mathbb{C})} |s|_{\varphi^{\otimes n}}(x) \leqslant 1 \Big\}.$$

In the philosophy of Arakelov geometry, Hermitian line bundles are analogous to invertible sheaves in the geometry of a projective scheme, and the family $(\widehat{H}^0(\overline{\mathscr{L}}^{\otimes n}))_{n \in \mathbb{N}}$ should play the role of a graded linear series in the arithmetic setting. However, it turns out that the family $(\widehat{H}^0(\overline{\mathscr{L}}^{\otimes n}))_{n \in \mathbb{N}}$ does not have the structure of graded algebra over a field (or over the ring of integers), and the arithmetic replication of geometric constructions and arguments is often difficult. For example, in [18], an analogue of the approach of Lazarsfeld and Mustaţă has been developed in the arithmetic setting by much more arduous combinatoric arguments.

In [2], a new idea has been proposed to transform the study of "arithmetic graded linear series" $(\widehat{H}^0(\overline{\mathscr{L}}^{\otimes n}))_{n \in \mathbb{N}}$ into that of a family (indexed by $\mathbb{R}$) of graded linear series of the generic fibre $\mathscr{L}_{\mathbb{Q}}$. For any $t \in \mathbb{R}$, we consider the graded linear series

$$V_\bullet^t(\overline{\mathscr{L}}) = \bigoplus_{n \in \mathbb{N}} \mathrm{Vect}_{\mathbb{Q}} \Big\{ s \in \Gamma(\mathscr{X}, \mathscr{L}^{\otimes n}) \ \Big| \ \sup_{x \in \mathscr{X}(\mathbb{C})} |s|_{\varphi^{\otimes n}}(x) \leqslant \mathrm{e}^{-nt} \Big\}.$$

From the point of view of the geometry of numbers, the family of graded linear series $(V_\bullet^t(\overline{\mathscr{L}}))_{t \in \mathbb{R}}$ encodes the *successive minima* of the free $\mathbb{Z}$-modules $\Gamma(\mathscr{X}, \mathscr{L}^{\otimes n})$ equipped with the norms $\|\cdot\|_{\varphi^{\otimes n}} := \sup_{x \in \mathscr{X}(\mathbb{C})} |\cdot|_{\varphi^{\otimes n}}(x)$. Thus by the second theorem of Minkowski one can naturally associate the asymptotic behaviour of $(\widehat{H}^0(\overline{\mathscr{L}}^{\otimes n}))_{n \in \mathbb{N}}$ with the value of the volume function on the family of graded linear series $(V_\bullet^t(\overline{\mathscr{L}}))_{t \in \mathbb{R}}$. This new approach also permits to obtain the arithmetic analogue of the main results of Lazarsfeld and Mustaţă (see [3]), and a new construction of arithmetic Newton-Okounkov bodies (see [1]).

The purpose of this lecture is to explain how the above idea leads to an alternative construction of Newton-Okounkov bodies in the geometric setting. The key argument relies on the analogy between number fields and function fields, where instead of the geometry of numbers we use the geometry of vector bundles on a curve, or function field arithmetic. In what follows, we fix a base field k. For simplicity, we assume that the characteristic of k is zero (we will discuss the general case in the end of the lecture). Let K be a finitely generated extension of k and d be the transcendence degree of K over k. By *graded linear series* of K/k, we refer to a graded sub-k-algebra $V_\bullet = \bigoplus_{n\in\mathbb{N}} V_n T^n$ of the polynomial ring $K[T] = \bigoplus_{n\in\mathbb{N}} KT^n$, such that each V_n is a finite-dimensional vector space over k. We say that $V_\bullet$ is *birational* if one has

$$k(V_\bullet) := k\bigg(\bigcup_{n\in\mathbb{N},\, n\geqslant 1} \Big\{\frac{f}{g} \;\Big|\; (f,g) \in (V_n \setminus \{0\})^2\Big\}\bigg) = K.$$

In order to determine the family of graded linear series on which we develop a theory of Newton-Okounkov bodies, we make the following observations. First of all, given a graded linear series $V_\bullet$ of K/k, the set

$$\mathbb{N}(V_\bullet) := \{n \in \mathbb{N} \mid V_n \neq \{0\}\}$$

is a subsemigroup of $\mathbb{N}$. Moreover, if we denote by $\mathbb{Z}(V_\bullet)$ the subgroup of $\mathbb{Z}$ generated by $\mathbb{N}(V_\bullet)$ then $\mathbb{N}(V_\bullet)\setminus\mathbb{Z}(V_\bullet)$ is a finite set. Therefore, by changing the grading we may assume without loss of generality that $V_n \neq \{0\}$ for sufficiently large n. Secondly, a graded linear series $V_\bullet$ is always birational viewed as a graded linear series of the extension $k(V_\bullet)/k$. Therefore, by changing the underlying field extension K/k we may assume without loss of generality that $V_\bullet$ is birational. Thirdly, we are mainly interested in graded linear series of a Cartier divisor, which is necessarily a sub-k-algebra of a graded linear series of finite type. Such graded linear series is said to be *of subfinite type*. A priori this condition depends on the choice of the extension K/k with respect to which we consider the graded linear series $V_\bullet$. However, as shown by [8, Theorem 1.2], if $V_\bullet$ is a graded linear series of subfinite type of K/k, then it is also a graded linear series of subfinite type of $k(V_\bullet)/k$.

In the following, we denote by $\mathcal{A}(K/k)$ the set of all birational graded linear series $V_\bullet$ of K/k which are of subfinite type and such that $V_n \neq \{0\}$ for sufficiently large n. Our purpose is to construct a map Δ from $\mathcal{A}(K/k)$ to the set of convex bodies in $\mathbb{R}^d$ (namely convex and compact subset with non-empty interior) which satisfies the following properties:

(a) for graded linear series $V_\bullet$ and $W_\bullet$ in $\mathcal{A}(K/k)$ such that $V_n \subset W_n$ for sufficiently large n, one has $\Delta(V_\bullet) \subset \Delta(W_\bullet)$;

(b) for any $V_\bullet$ in $\mathcal{A}(K/k)$ and any integer $m \in \mathbb{N}_{\geqslant 1}$, one has $\Delta(V_\bullet^{(m)}) = m\Delta(V_\bullet)$, where $V_\bullet^{(m)}$ denotes the graded linear series $\bigoplus_{n\in\mathbb{N}} V_{mn}T^n$;

(c) for graded linear series $V_\bullet$ and $W_\bullet$ in $\mathcal{A}(K/k)$ one has

$$\Delta(V_\bullet) + \Delta(W_\bullet) \subset \Delta(V_\bullet \cdot W_\bullet),$$

where "+" denotes the Minkowski sum of convex bodies, and $V_\bullet \cdot W_\bullet$ denotes the graded linear series

$$\bigoplus_{n\in\mathbb{N}} \mathrm{Vect}_k\big(\{fg \,|\, (f,g)\in V_n\times W_n\}\big)T^n;$$

(d) for any graded linear series $V_\bullet$, the *volume* of $V_\bullet$, which is defined as

$$\mathrm{vol}(V_\bullet) := \limsup_{n\to+\infty} \frac{\dim_k(V_n)}{n^d/d!},$$

is equal to the Lebesgue measure of $\Delta(V_\bullet)$ times $d!$.

We will see from the construction of the map Δ that, for any graded linear series $V_\bullet \in \mathcal{A}(K/k)$, the sequence defining the volume of $V_\bullet$ actually converges. Moreover, the graded linear series $V_\bullet$ satisfies the Fujita approximation property, namely $\mathrm{vol}(V_\bullet)$ is equal to the supremum of volumes of graded linear series of finite type which are contained in $V_\bullet$. This generalises [10, Corollary 3.11] to the case where the extension K/k is not geometrically integral.

Similarly to the approach of [3, 10], the construction of the map $\Delta(\cdot)$ is not intrinsic. However, instead of choosing a $\mathbb{Z}^d$-valuation of the field K over k, our construction depends on the choice of a flag of intermediate extensions of K/k. In the following we fix a sequence of field extensions

$$k = K_0 \subsetneq K_1 \subsetneq \ldots \subsetneq K_d = K$$

such that each extension K_i/K_{i-1} is transcendental of transcendence degree 1. Note that the extension K/K_i is of transcendence degree $d-i$. We will construct, by backward induction on i, a map $\Delta^{(i)}$ from $\mathcal{A}(K/K_i)$ to the set of convex bodies in $\mathbb{R}^{d-i}$, which satisfies the conditions (a)–(d) above, and such that, for any graded linear series $V_\bullet$ in $\mathcal{A}(K/K_{i-1})$, the projection of $\Delta^{(i-1)}(V_\bullet)$ on its first $d-i$ coordinates gives a convex body which is contained in $\Delta^{(i)}(V_{\bullet,K_i})$, where $V_{\bullet,K_i}$ denotes the graded sub-K_i-algebra of $K[T]$ generated by $V_\bullet$. Then the map $\Delta^{(0)}$ from $\mathcal{A}(K/k)$ to the set of convex bodies in $\mathbb{R}^d$ is just what we need.

We first consider the case where $i = d$. If $V_\bullet$ is a graded linear series in $\mathcal{A}(K/K)$, then $V_n = K$ for sufficiently large n, and one has $\mathrm{vol}(V_\bullet) = 1$. We let $\Delta^{(d)}(V_\bullet) = \mathbb{R}^0$. It is easy to see that the conditions (a)–(d) are satisfied by the map $\Delta^{(d)}$. Moreover, any graded linear series $V_\bullet$ in $\mathcal{A}(K/K)$ is necessarily of finite type, and $\dim_k(V_n) = 1$ for sufficiently large n, and hence the sequence defining $\mathrm{vol}(V_\bullet)$ converges.

It is in the induction procedure that we applies the approach of concave transform and arithmetic Newton-Okounkov body in the function field setting. We assume that the map $\Delta^{(i)}$ has been constructed. The main idea is to identifie K_i with the field of rational functions of a regular projective curve C_i over $\mathrm{Spec}\, K_{i-1}$. We denote by $\eta_i : \mathrm{Spec}\, K_i \to C_i$ the morphism associated with the generic point of C_i. Let W be a vector space over K_i and M be a finite-dimensional K_{i-1}-vector subspace of W. The sub-$\mathcal{O}_{C_i}$-module of $\eta_{i,*}(W)$ generated by M is a vector bundle (namely, locally free sheaf of finite rank) on C_i, called the *vector bundle generated by* (M,W)*, denoted by* $E(M,W)$. Note that M identifies with a K_{i-1}-vector subspace of $H^0(C_i, E(M,W))$, and the generic fibre of E identifies with the K_i-vector subspace of W generated by M. If $V_\bullet$ is a graded linear series in $\mathcal{A}(K/K_{i-1})$, then

$$E(V_\bullet) := \bigoplus_{n\in\mathbb{N}} E(V_n, V_{n,K_i})$$

is a graded $\mathcal{O}_{C_i}$-algebra whose generic fibre identifies with $V_{\bullet,K_i}$. Note that the volume of $V_\bullet$ can be described by the asymptotic behaviour of $E(V_\bullet)$: one has

$$\mathrm{vol}(V_\bullet) = \limsup_{n\to+\infty} \frac{\dim_k(H^0(C_i, E(V_n, V_{n,K_i})))}{n^{d-i+1}/(d-i+1)!}.$$

We refer to [7, Lemma 4.5] for a proof.

The above construction allows to apply the method of $\mathbb{R}$-filtration to the graded $\mathcal{O}_{C_i}$-algebra $E(V_\bullet)$ to construct the convex body associated with $V_\bullet$ (similarly to the arithmetic Newton-Okounkov body mentioned above). It turns out that Harder-Narasimhan $\mathbb{R}$-filtration is a suitable choice. Let F be a non-zero vector bundle on C_i. Recall that the *slope* of F is defined as the quotient of the degree of F by the rank of F, denoted by $\mu(F)$. If any non-zero vector subbundle of F has a slope $\leqslant \mu(F)$, we say that F is *semistable*. Harder and Narasimhan have shown that, for any non-zero vector bundle F, there exists a unique flag of vector subbundles

$$0 = F_0 \subsetneq F_1 \subsetneq \ldots \subsetneq F_m = F,$$

which is called *Harder-Narasimhan flag of* F, such that each subquotient F_j/F_{j-1} is semistable, and that

$$\mu(F_1/F_0) > \ldots > \mu(F_m/F_{m-1}).$$

We refer to [9, §1.3] for more details. Note that the last slope $\mu(F_m/F_{m-1})$ is the smallest one among the slopes of non-zero quotient vector bundles of F. It is called the *minimal slope* of F and denoted by $\mu_{\min}(F)$. The first slope $\mu(F_1/F_0)$ is the largest one among the slopes of non-zero vector subbundle of F, which is called the *maximal slope* of F and denoted by $\mu_{\max}(F)$. We can encode the Harder-Narasimhan flag and the successive slopes into an $\mathbb{R}$-filtration of the generic fibre: for any $t \in \mathbb{R}$, we let

$$\mathcal{F}^t_{\mathrm{HN}}(F) = \sum_{\substack{0\neq G\subset F\\ \mu_{\min}(G)\geqslant t}} G_{K_i}$$

and call it the *Harder-Narasimhan* $\mathbb{R}$*-filtration* of F. It can be shown that

$$\mathcal{F}^t_{\mathrm{HN}}(F) = \begin{cases} 0, & \text{if } \mu(F_1/F_0) < t,\\ F_{j,K_i}, & \text{if } \mu(F_{j+1}/F_j) < t \leqslant \mu(F_j/F_{j-1}),\\ F_{K_i}, & \text{if } t \leqslant \mu(F_m/F_{m-1}). \end{cases}$$

We refer to [4, §§2.2-2.3] for more details.

Note that we have assumed that the base field k is of characteristic zero. Under this condition it has been shown by Narasimhan and Seshadri [13] that the tensor product of two semistable vector bundles is still semistable (see also the algebraic proof of Ramanan and Ramanathan [16]). As a consequence, the minimal slope of the tensor product of two (non-necessarily semistable) vector bundles is equal to the sum of the minimal slopes of these vector bundles. Therefore, if $V_\bullet$ is a graded linear series in $\mathcal{A}(K/K_{i-1})$, then the Harder-Narasimhan $\mathbb{R}$-filtrations on $E(V_n, V_{n,K_i})$ are super-multiplicative, namely, for any $(n_1, n_2) \in \mathbb{N}^2$ and any $(t_1, t_2) \in \mathbb{R}^2$, one has

$$\mathcal{F}_{\mathrm{HN}}^{t_1}(E(V_{n_1}, V_{n_1,K_i})) \cdot \mathcal{F}_{\mathrm{HN}}^{t_2}(E(V_{n_2}, V_{n_2,K_i})) \subset \mathcal{F}_{\mathrm{HN}}^{t_1+t_2}(E(V_{n_1+n_2}, V_{n_1+n_2,K_i})).$$

For any $t \in \mathbb{R}$, let

$$V_{\bullet,K_i}^{(t)} := \bigoplus_{n\in\mathbb{N}} \mathcal{F}^{nt}(E(V_n, V_{n,K_i})).$$

The above super-multiplicativity shows that $V_{\bullet,K_i}^{(t)}$ is a graded linear series of K/K_i. Clearly this graded linear series is of sub-finite type. Let

$$\mu^* := \limsup_{n\to+\infty} \frac{\mu_{\max}(E(V_n, V_{n,K_i}))}{n}.$$

For $t > \mu^*$, the graded linear series $V_{\bullet,K_i}^{(t)}$ is trivial, namely $V_{n,K_i}^{(t)} = \{0\}$ for any $n \in \mathbb{N}_{\geqslant 1}$. For $t < \mu^*$, the graded linear series $V_{\bullet,K_i}^{(t)}$ is birational (see [7, Lemma 4.2]) and hence belongs to the family $\mathcal{A}(K/K_i)$. Thus the induction hypothesis applies and leads to a decreasing family $\big(\Delta^{(i)}(V_{\bullet,K_i}^{(t)})\big)_{t<\mu^*}$ of convex bodies in $\mathbb{R}^{d-i}$. Moreover, the induction hypothesis (notably conditions (a)–(c)) and the above super-additivity also imply that, for any $\varepsilon \in [0,1]$ and any $(t_1,t_2) \in \mathbb{R}^2_{<\mu^*}$ one has

$$\varepsilon\Delta^{(i)}(V_{\bullet,K_i}^{(t_1)}) + (1-\varepsilon)\Delta^{(i)}(V_{\bullet,K_i}^{(t_2)}) \subset \Delta^{(i)}(V_{\bullet,K_i}^{\varepsilon t_1+(1-\varepsilon t_2)}).$$

Therefor, the function $G_{V_\bullet} : \Delta^{(i)}(V_{\bullet,K_i}) \to [0,\mu^*]$ sending $x \in \Delta^{(i)}(V_\bullet, K_i)$ to

$$\sup\{t < \mu^* \,|\, x \in \Delta^{(i)}(V_{\bullet,K_i}^{(t)})\}$$

is concave. We call it the *concave transform* of $V_\bullet$. The convex body associated with $V_\bullet$ is then defined as the graph of the positive part of the concave transform, namely

$$\Delta^{(i-1)}(V_\bullet) := \{(x,t) \,|\, x \in \Delta^{(i)}(V_{\bullet,K_i}),\ 0 \leqslant t \leqslant G_{V_\bullet}(x)\}.$$

We have constructed above a map $\Delta^{(i-1)}$ from $\mathcal{A}(K/K_{i-1})$ to the set of convex bodies in $\mathbb{R}^{d-i+1}$. By definition is not hard to check that the map satisfies the conditions (a)–(c). We refer to [7, §4.4] for details. The condition (d) results from Riemann-Roch theorem, which implies that, for any non-zero vector bundle F on C_i, one has

$$\left| \dim_{K_{i-1}}(H^0(C_i, F)) - \int_0^{+\infty} \dim_{K_i}(\mathcal{F}_{\mathrm{HN}}^t(F))\,\mathrm{d}t \right| \leqslant \mathrm{rk}(F)\max(g(C_i)-1, 1),$$

where $g(C_i)$ denotes the genus of C_i relatively to K_{i-1}. We refer to [6, Theorem 2.4] for a proof of this inequality. Indeed, the measure of the convex body $\Delta^{(i-1)}(V_\bullet)$ can be written as

$$\int_{\Delta^{(i)}(V_{\bullet,K_i})} \int_0^{+\infty} \mathbb{1}_{\{t\leqslant G_{V_\bullet}(x)\}}\,\mathrm{d}t\,\mathrm{d}x = \int_0^{+\infty} \mathrm{vol}(\Delta^{(i)}(V_{\bullet,K_i}^{(t)}))\,\mathrm{d}t.$$

Note that the induction hypothesis shows that

$$\mathrm{vol}(\Delta^{(i)}(V_{\bullet,K_i}^{(t)})) = \lim_{n\to+\infty} \frac{\dim_{K_i}(\mathcal{F}_{\mathrm{HN}}^{nt}(E(V_n, V_{n,K_i})))}{n^{d-i}}.$$

Therefore we obtain

$$\begin{aligned}\mathrm{vol}(\Delta^{(i-1)}(V_\bullet)) &= \lim_{n\to+\infty} \frac{1}{n^{d-i}} \int_0^{+\infty} \dim_{K_i}(\mathcal{F}^{nt}_{\mathrm{HN}}(E(V_n, V_{n,K_i})))\,\mathrm{d}t \\ &= \lim_{n\to+\infty} \frac{1}{n^{d-i+1}} \int_0^{t} \dim_{K_i} \mathcal{F}^{t}_{\mathrm{HN}}(E(V_n, V_{n,K_i})))\,\mathrm{d}t.\end{aligned}$$

Combining with the above inequality resulting from Rimann-Roch theorem, we obtain

$$\mathrm{vol}(\Delta^{(i-1)}(V_\bullet)) = \lim_{n\to+\infty} \frac{\dim_{K_{i-1}}(H^0(C_i, E(V_n, V_{n,K_i})))}{n^{d-i+1}} = \lim_{n\to+\infty} \frac{\dim_{K_{i-1}}(V_n)}{n^{d-i+1}}.$$

The Fujita approximation property of $V_\bullet$ can be proved by using [1, Theorem 1.14].

By the above induction procedure, we construct a map Δ from $\mathcal{A}(K/k)$ to the set of convex bodies in $\mathbb{R}^d$ which satisfies the conditions (a)–(d). Observe that the construction in each induction step is the function field analogue of the arithmetic Newton-Okounkov body. Similar construction can be done in the positive characteristic case: it suffices to replace the Harder-Narasimhan $\mathbb{R}$-filtration by the $\mathbb{R}$-filtration of minima, and replace the geometry of vector bundles (notably Riemann-Roch theorem) by function field arithmetic (notably the analogue of Minkowski's second theorem by Roy and Thunder [17, Theorem 2.1]). We refer to [7] for more details.

Given a flag of intermediate extensions of K/k, it seems to be a difficult problem to compute explicitly the map Δ from $\mathcal{A}(K/k)$ to the set of convex bodies in $\mathbb{R}^d$. The computations made in [5] suggest that, even in the case where K/k is purely transcendental and $V_\bullet$ comes from a toric divisor on a toric variety model, the convex body $\Delta(V_\bullet)$ may often have a non-linear boundary. However, from the point of view of birational geometry, the additional data on which the map $\Delta(\cdot)$ depends seems to be more natural. It can be hoped that the convex body $\Delta(V_\bullet)$ contains more intrinsic information about the graded linear series $V_\bullet$ than the classic Newton-Okounkov body, and thus have potential applications in the study of birational algebraic geometry.

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