Symplectic singularities and nilpotent orbits

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Introduction

This is an exposition of the result proved in [Na 5].

Symplectic singularities have been playing important roles both in algebraic geometry and geometric representation theory ever since Beauville introduced their notion in [Be]. Most examples of symplectic singularities admit natural $\mathbf{C}^*$-actions with only positive weights. Kaledin [Ka] conjectured that any symplectic singularity admits such a $\mathbf{C}^*$-action.

If a symplectic singularity has a $\mathbf{C}^*$-action with positive weights, it can be globalized to an affine variety with a $\mathbf{C}^*$-action. Such an affine variety is called a *conical symplectic variety*. More precisely, an affine normal variety $X = \mathrm{Spec}\ R$ is a conical symplectic variety if

(i) R is positively graded: $R = \oplus_{i \geq 0} R_i$ with $R_0 = \mathbf{C}$;

(ii) the smooth part X_{reg} admits a homogeneous symplectic 2-form ω that extends to a regular 2-form on a resolution $\tilde{X}$ of X.

Denote the $\mathbf{C}^*$-action by $t : X \to X$ ($t \in \mathbf{C}^*$). By the assumption we have $t^*\omega = t^l\omega$ for some integer l. This integer l is called the weight of ω and is denoted by $wt(\omega)$. By the extension property (ii) we have $wt(\omega) > 0$ (cf. [Na 3], Lemma (2.2)).

Let $\{x_0, ..., x_n\}$ be a set of minimal homogeneous generators of the $\mathbf{C}$-algebra R and put $a_i := \deg\ x_i$. We put $N := \max\{a_0, ..., a_n\}$ and call N the *maximal weight* of X. It is uniquely determined by the conical symplectic variety X. By [Na 1], there are only finitely many conical symplectic varieties (X, ω) of a fixed dimension $2d$ and with a fixed maximal weight N, up to an isomorphism. In this sense it would be important to classify conical symplectic varieties with maximal weight 1. By the homogeneous generators $\{x_i\}$, we can embed X into an affine space $\mathbf{C}^{n+1}$. In [Na 2] we treat the case where $X \subset \mathbf{C}^{n+1}$ is a complete intersection of homogeneous polynomials. The main theorem of [Na 2] asserts that (X, ω) is isomorphic to the nilpotent cone (N, ω_{KK}) of a complex semisimple Lie algebra $\mathfrak{g}$ together with the Kirillov-Kostant 2-form provided that X is singular. However, there are a lot of examples of maximal weight 1 which are not of complete intersection. In fact, a nilpotent orbit O of a complex semisimple Lie algebra $\mathfrak{g}$ admits the Kirillov-Kostant form ω_{KK} and if its closure $\bar{O}$ is normal, then $(\bar{O}, \omega_{KK})$ is a conical symplectic variety with maximal weight 1 by Panyushev [Pa] and Hinich [Hi].

The main result of [Na 5] is the following, which claims that nilpotent orbit closures actually exhaust all conical symplectic varieties with maximal weight 1.

Theorem ([Na 5]). *Let (X, ω) be a conical symplectic variety with maximal weight 1. Then (X, ω) is isomorphic to one of the following:*

(i) $(\mathbf{C}^{2d}, \omega_{st})$ *with* $\omega_{st} = \Sigma_{1 \leq i \leq d} dz_i \wedge dz_{i+d}$,

(ii) $(\bar{O}, \omega_{KK})$ *where $\bar{O}$ is a normal nilpotent orbit closure of a complex semisimple Lie algebra $\mathfrak{g}$ and ω_{KK} is the Kirillov-Kostant form.*

There exist non-normal nilpotent orbit closures in complex semisimple Lie algebras (cf. [K-P], [Kra], [L-S], [Bro], [So]). The normalization $\tilde{O}$ of such an orbit closure $\bar{O}$ is also a conical symplectic variety.[1] But the maximal weight of $\tilde{O}$ is usually larger than 1.[2]

§1. **Preliminaries**

(1.1) *What is a nilpotent orbit ?*

Let us consider

$$sl(n, \mathbf{C}) := \{A \in \mathrm{End}(\mathbf{C}^n) \mid tr(A) = 0\}.$$

As is well known, this is a simple Lie algebra of type A_{n-1} where the Lie bracket is given by $[A, B] := AB - BA$, and $sl(n, \mathbf{C})$ is the Lie algebra of $SL(n, \mathbf{C})$. We define the nilpotent cone $\mathcal{N}$ by

$$\mathcal{N} := \{A \in sl(n, \mathbf{C}) \mid A \,\text{is nilpotent}\}.$$

A partition of n is a decreasing sequence $d_1 \geq d_2 \geq ... \geq d_r$ of positive integers such that $\sum d_i = n$. For a partition $[d_1, ..., d_r]$ of n, we put

$$O_{[d_1,...,d_r]} := \{A \in sl(n, \mathbf{C}) \mid \text{A is conjugate to the matrix} \begin{pmatrix} J_{d_1} & 0 & \cdots & 0 \\ 0 & J_{d_2} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & J_{d_s} \end{pmatrix}\}.$$

Here J_d is the Jordan matrix of size d:

$$J_d := \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Then we have

$$\mathcal{N} = \cup O_{[d_1,...,d_r]},$$

where $[d_1, ..., d_r]$ runs through all possible partitions of n. Recall that the adjoint action of $SL(n, \mathbf{C})$ on $sl(n, \mathbf{C})$ is given by

$$A \to PAP^{-1},\ A \in sl(n, \mathbf{C}),\ P \in SL(n, \mathbf{C}).$$

Then each $O_{[d_1,...,d_r]}$ is actually an orbit of the adjoint action. Since $O_{[d_1,...,d_r]}$ consists of nilpotent elements, it is called a *nilpotent orbit*. The closure $\bar{O}_{[d_1,...,d_r]}$ of $O_{[d_1,...,d_r]}$ is an affine variety with singularities.

[1] By [K-P], Proposition 7.4 $\bar{O}$ is always resolved by a vector bundle Y over G/P with a parabolic subgroup P of the adjoint group G of $\mathfrak{g}$. Denote this resolution by $\pi : Y \to \bar{O}$. The map π factorizes as $Y \to \tilde{O} \to \bar{O}$. The fiber $\pi^{-1}(0)$ coincides with the zero section of Y, which is isomorphic to G/P. As G/P is connected, the fibre $\mu^{-1}(0)$ of the normalization map $\mu : \tilde{O} \to \bar{O}$ consists of just one point, say $x \in \tilde{O}$. The $\mathbf{C}^*$-action on $\bar{O}$ extends to a $\mathbf{C}^*$-action on $\tilde{O}$ with a unique fixed point x. It is easily checked that this $\mathbf{C}^*$-action has only positive weights and $\tilde{O}$ becomes a conical symplectic variety.

[2] It may happen that $\tilde{O}$ coincides with a normal nilpotent orbit closure of a different complex semisimple Lie algebra (cf. [B-K], Example 3.5). In such a case the maximal weight is 1.

These notion can be generalized to an arbitrary complex semisimple Lie algebra $\mathfrak{g}$. In fact, the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ is a complex Lie group and its identity component $G := \mathrm{Aut}^0(\mathfrak{g})$ is a connected complex semisimple Lie group. Then $\mathfrak{g}$ is the Lie algebra of G. G naturally acts on $\mathfrak{g}$. On the other hand, an element g of G induces an automorphism

$$Ad_g : G \to G \ (h \to ghg^{-1})$$

and it induces an automorphism of the tangent space $\mathfrak{g} := T_1 G$ at the identity element. In this way G acts on $\mathfrak{g}$. This action is called the adjoint action of G on $\mathfrak{g}$. We know that the first G-action agree with the adjoint action. An orbit $O \subset \mathfrak{g}$ is called an adjoint orbit. An element x of $\mathfrak{g}$ is called nilpotent if

$$ad(x) : \mathfrak{g} \to \mathfrak{g} \ \ (z \to [x, z])$$

is a nilpotent endomorphism. An adjoint orbit O is called a nilpotent orbit if it consists of nilpotent elements. The set $\mathcal{N}$ of all nilpotent elements of $\mathfrak{g}$ is called the nilpotent cone of $\mathfrak{g}$. It is known that $\mathcal{N}$ is an affine normal variety and $\mathcal{N}$ is the union of a finite number of nilpotent orbits.

(1.2) *Coadjoint orbits and symplectic structures*

Here let us start with an arbitrary (not necessarily semisimple) complex Lie group G and its Lie algebra $\mathfrak{g}$. For $g \in G$, let $Ad_g : \mathfrak{g} \to \mathfrak{g}$ be the adjoint action of g. Then G has a dual action on $\mathfrak{g}^*$ defined by

$$Ad^*_{g^{-1}} : \mathfrak{g}^* \to \mathfrak{g}^* \ \ \alpha \to Ad^*_{g^{-1}}(\alpha) := \alpha(Ad_{g^{-1}}(\cdot)).$$

This action is called the coadjoint action. An orbit $O' \subset \mathfrak{g}^*$ is called a coadjoint orbit. We shall explain that every coadjoint orbit has a canonical symplectic structure. Pick an element $\alpha \in O'$ and let us consider the surjective map

$$G \to O \ \ (g \to Ad^*_{g^{-1}}(\alpha)).$$

Put

$$G_\alpha := \{g \in G \mid Ad^*_{g^{-1}}(\alpha) = \alpha\}.$$

By the surjection we can identify G/G_α with O'. In particular, we have an identification

$$\mathfrak{g}/\mathfrak{g}_\alpha \cong T_\alpha O'.$$

For $x \in \mathfrak{g}$ we denote by $\bar{x} \in \mathfrak{g}/\mathfrak{g}_\alpha$ the class determined by x. We define a skew symmetric form

$$\omega_\alpha : T_\alpha O' \times T_\alpha O' \to \mathbf{C}$$

by

$$\omega_\alpha(\bar{x}, \bar{y}) := \alpha([x, y]).$$

Then ω_α is well-defined and is non-degenerate. Moreover, one can check that $\omega := \{\omega_\alpha\}_{\alpha \in O'}$ determine a d-closed form on O'. The ω constructed above is called the *Kirillov-Kostant form.*

Now let us assume that $\mathfrak{g}$ is semisimple. Then the Killing form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbf{C}$ is non-degenerate, and it identifies $\mathfrak{g}$ with $\mathfrak{g}^*$. Since this identification is G-equivariant, each adjoint orbit $O \subset \mathfrak{g}$ is identified with a coadjoint orbit $O' \subset \mathfrak{g}^*$. Therefore every adjoint orbit O of a complex semisimple Lie algebra $\mathfrak{g}$ admits a symplectic structure.

(1.3) *Symplectic varieties and Poisson schemes*

Let X be a normal complex algebraic variety of dimension $2d$ and let ω be an algebraic regular 2-form on the smooth part X_{reg}. Then (X, ω) is called a *symplectic variety* if

1) ω is a symplectic 2-form, that is, $d\omega = 0$ and $\wedge^d \omega$ is a nowhere vanishing $2d$-form on X_{reg}, and

2) ω can be extended to a regular 2-form on a resolution Y of X; more precisely, there is a proper birational morphism $\pi : Y \to X$ from a smooth variety Y such that the regular 2-form $\pi^*\omega$ on $\pi^{-1}(X_{reg})$ can be extended to a regular 2-form on Y.

Moreover, (X, ω) is called *conical* if X is an affine variety Spec(R) such that

3) R is positively graded, that is, $R = \oplus_{i \geq 0} R_i$ and $R_0 = \mathbf{C}$, and

4) ω is homogeneous with respect to the $\mathbf{C}^*$-action on X induced by the grading of R defined in 3). In other words, there is an integer l such that $t^*\omega = t^l \omega$ for $t \in \mathbf{C}^*$. Such an integer l is called the weight of ω and we often denote it by $wt(\omega)$.

Example. Let O be a nilpotent orbit of a complex semisimple Lie algebra $\mathfrak{g}$. By (1.2) O admits the Kirillov-Kostant form ω. Let $\tilde{O}$ be the normalization of the nilpotent orbit closure $\bar{O}$. Then $\tilde{O}_{reg}$ contains the original orbit O, and ω extends to a symplectic 2-form ω' on $\tilde{O}_{reg}$ because $\mathrm{Codim}_{\tilde{O}_{reg}} \tilde{O}_{reg} - O \geq 2$. By [Hi] and [Pa], $(\tilde{O}, \omega')$ satisfies the property 2) of a symplectic variety; hence it is a symplectic variety. Moreover, the scalar $\mathbf{C}^*$-action on $\mathfrak{g}$ preserves $\bar{O}$. The $\mathbf{C}^*$-action on $\bar{O}$ induces a $\mathbf{C}^*$-action on $\tilde{O}$. Since weight $wt(\omega)$ of the Kirillov-Kostant form is 1, we have $wt(\omega') = 1$. Therefore $(\tilde{O}, \omega')$ is a conical symplectic variety.

We next define a Poisson scheme. A scheme X over $\mathbf{C}$ is called a *Poisson scheme* if it admits a $\mathbf{C}$-bilinear skew-symmetric pairing

$$\{\ ,\ \} : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$$

such that

a) $\{\ ,\ \}$ is a biderivation, that is, $\{fg, h\} = f\{g, h\} + g\{f, h\}$ and $\{f, gh\} = g\{f, h\} + h\{f, g\}$ for f, g, $h \in \mathcal{O}_X$.

b) $\{\ ,\ \}$ satisfies the Jacobi identity, i.e.

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$$

for f, g, $h \in \mathcal{O}_X$.

Let Y be a closed subscheme of a Poisson scheme X defined by the ideal sheaf I_Y. Then Y is called a Poisson subscheme if $\{I_Y, \mathcal{O}_X\} \subset I_Y$. It is easily checked that the Poisson structure on X naturally induces a Poisson structure on Y; hence a Poisson subscheme Y is a Poisson scheme.

Let (X, ω) be a symplectic variety. We shall explain that X becomes a Poisson scheme.by the property 1). First, X_{reg} become a Poisson scheme by using ω. In fact, ω

identifies the sheaf $\Omega^1_{X_{reg}}$ of 1-form with the sheaf $\Theta_{X_{reg}}$ of vector fields. It also induces an identification

$$\Omega^2_{X_{reg}} \cong \wedge^2 \Theta_{X_{reg}}.$$

Notice that ω itself is a section of the left hand side. Hence it determines a section θ of $\wedge^2\Theta_{X_{reg}}$. By using θ we define the bracket

$$\{\ ,\ \} : \mathcal{O}_{X_{reg}} \times \mathcal{O}_{X_{reg}} \to \mathcal{O}_{X_{reg}}$$

by $\{f, g\} := \theta(df \wedge dg)$. It is well known that d-closedness of ω is equivalent to the Jacobi identity for the bracket $\{\ ,\ \}$. Hence X_{reg} is a Poisson scheme. By the normality of X, this bracket uniquely extends to the bracket

$$\{\ ,\ \} : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X.$$

This bracket makes X a Poisson scheme.

Example. Let G be a complex Lie group and let $\mathfrak{g}$ be the Lie algebra of G. Then $\mathfrak{g}^*$ has a natural Poisson structure. In fact, $\mathfrak{g}^* = \mathrm{Spec} \oplus_{i\geq 0} Sym^i(\mathfrak{g})$ as an affine variety. By using the Lie bracket $[\ ,\ ]$ on $\mathfrak{g}$, one can determine a Poisson bracket on $\oplus_{i\geq 0} Sym^i(\mathfrak{g})$. For example, let x, y and z be elements of $\mathfrak{g}$. Then $\{x \cdot y, z\}$ is defined to be $x[y, z] + y[x, z]$. The closure $\bar{O}'$ of a coadjoint orbit $O' \subset \mathfrak{g}^*$ is a Poisson closed subscheme.

Kaledin [Ka] showed that the property 2) of a symplectic variety X strongly constrains the Poisson structure on X. We here introduce one of his results, which will be used later. Let $\mathrm{Sing}^{(1)}(X)$ be the singular locus of X with reduced structure. Let $\mathrm{Sing}^{(2)}(X)$ be the singular locus of $\mathrm{Sing}^{(1)}(X)$ with reduced structure. We define inductively $\mathrm{Sing}^{(i)}(X)$ for $i \geq 3$.

Theorem([Ka]):

(i) An irreducible component of $\mathrm{Sing}^{(i)}(X)$ *is a Poisson integral subscheme of* X*. Conversely, every Poisson integral closed subscheme of* X *is of this form. In particular, there are only a finite number of Poisson integral closed subschemes of* X*.*

(ii) Denote by $U^{(0)}$*,* $U^{(i)}$ $(i \geq 1)$ *the smooth locus of* X *and* $\mathrm{Sing}^{(i)}(X)$*. Then* $X = \cup_{i\geq 0} U^{(i)}$*, where all connected components of* $U^{(i)}$ *are symplectic manifolds.*

The connected components of $U^{(i)}$ are called *symplectic leaves*. Thus, (ii) can be rephrased as "X is stratified into a finite number of symplectic leaves".

Example. Let $\mathcal{N}$ be the nilpotent cone in (1.1). Then $\mathcal{N} = \bar{O}_{[n]}$. Let ω be the Kirillov-Kostant form on $O_{[n]}$. Then $(\mathcal{N}, \omega)$ is a symplectic variety. All Poisson integral subschemes of $\mathcal{N}$ are of the form $\bar{O}_{[d_1,\ldots,d_r]}$. The decomposition $\mathcal{N} = \cup O_{[d_1,\ldots,d_r]}$ is nothing but the decomposition of $\mathcal{N}$ into symplectic leaves.

§2. **Proof of Theorem**

In the remainder, X is a conical symplectic variety with the maximal weight $N = 1$. First of all, we prove in Proposition 1 that $wt(\omega) = 2$ or $wt(\omega) = 1$. In the first case (X, ω) is isomorphic to an affine space $\mathbf{C}^{2d}$ together with the standard symplectic form ω_{st}. In the second case the Poisson bracket has degree -1 and R_1 has a natural Lie algebra structure. Then it is fairly easy to show that X is a coadjoint orbit closure of a complex Lie algebra $\mathfrak{g}$ (Proposition 3). If X has a crepant resolution, we can prove that $\mathfrak{g}$ is semisimple in the same way as in [Na 2]. But X generally does not have such a resolution and we need a new method to prove the semisimplicity. This is done in Proposition 4.

Proposition 1 *Assume that X is a conical symplectic variety with maximal weight $N = 1$. Then $wt(\omega) = 1$ or $wt(\omega) = 2$. If $wt(\omega) = 2$, then (X, ω) is isomorphic to an affine space $(\mathbf{C}^{2d}, \omega_{st})$ with the standard symplectic form..*

Remark. As is remarked in the beginning of [Na 2], §2, if X is a smooth conical symplectic variety with maximal weight 1, then $(X, \omega) \cong (\mathbf{C}^{2d}, \omega_{st})$. Hence X is singular exactly when $wt(\omega) = 1$.

Proof. Since $N = 1$, the coordinate ring R is generated by R_1. We put $l := wt(\omega)$. We already know that $l > 0$. If $l > 2$, then we have $\{R_1, R_1\} = 0$ and hence $\{R, R\} = 0$, which is absurd. We now assume that $l = 2$ and prove that X is an affine space with the standard symplectic form. Then the Poisson bracket induces a skew-symmetric form $R_1 \times R_1 \to R_0 = \mathbf{C}$. If this is a degenerate skew-symmetric form, we can choose a non-zero element $x_1 \in R_1$ such that $\{x_1, \cdot\} = 0$. Notice that $x_1 = 0$ determines a non-zero effective divisor D on X_{reg}. If we choose a general point $a \in D$, then the reduced divisor D_{red} is smooth around a. Consider an analytic open neighborhood $U \subset X_{reg}$ of a. Then there is a system of local coordinates $\{z_1, ..., z_{2d}\}$ of U such that x_1 can be written as $x_1 = z_1^m$ for a suitable $m > 0$. The Poisson structure on X induces a non-degenerate Poisson structure $\{\cdot, \cdot\}_U$ on U. But, by the choice of x_1, we have $\{z_1^m, \cdot\}_U = m z_1^{m-1}\{z_1, \cdot\}_U = 0$, which implies that $\{z_1, \cdot\}_U = 0$. This contradicts that the Poisson bracket $\{\cdot, \cdot\}_U$ is non-degenerate.

Therefore the skew-symmetric form is non-degenerate. In this case X is a closed Poisson subscheme of an affine space with a non-degenerate Poisson structure induced by the standard symplectic form. But such an affine space has no Poisson closed subscheme except the affine space itself because the Hamiltonian vector fields span the tangent space at each point. Therefore $X = \mathbf{C}^{2d}$. Q.E.D.

The regular part X_{reg} of a conical symplectic variety X is a smooth Poisson variety. Let $\Theta_{X_{reg}}$ denote the sheaf of vector fields on X_{reg}. By using the Poisson bracket we define the Lichnerowicz-Poisson complex

$$0 \to \Theta_{X_{reg}} \stackrel{\delta_1}{\to} \wedge^2 \Theta_{X_{reg}} \stackrel{\delta_2}{\to} ...$$

by

$$\delta_p f(da_1 \wedge ... \wedge da_{p+1}) := \sum_{i=1}^{p+1} (-1)^{i+1} \{a_i, f(da_1 \wedge ... \hat{da_i} \wedge ... \wedge da_{p+1})\}$$

$$+ \sum_{j<k} (-1)^{j+k} f(d\{a_j, a_k\} \wedge da_1 \wedge ... \wedge \hat{da_j} \wedge ... \wedge \hat{da_k} \wedge ... \wedge da_{p+1}).$$

In the Lichnerowicz-Poisson complex, $\wedge^p \Theta_{X_{reg}}$ is placed in degree p. The Lichnerowicz-Poisson complex of X_{reg} is closely related to the Poisson deformation of $(X, \{\ ,\ \})$. For details, see [Na 4].

In the remaining part we assume that $wt(\omega) = 1$. The Poisson bracket then defines a pairing map $R_1 \times R_1 \to R_1$ and R_1 becomes a Lie algebra. We denote this Lie algebra by $\mathfrak{g}$. As all generators have weight 1, we have a surjection $\oplus \mathrm{Sym}^i(R_1) \to R$. It induces a $\mathbf{C}^*$-equivariant closed embedding $X \to \mathfrak{g}^*$.

Recall that the adjoint group G of $\mathfrak{g}$ (cf. [Pro], p.86) is defined as a subgroup of $GL(\mathfrak{g})$ generated by all elements of the form $\exp(ad\ v)$ with $v \in \mathfrak{g}$. The adjoint group G is a complex Lie subgroup of $GL(\mathfrak{g})$, but it is not necessarily a closed algebraic subgroup of $GL(\mathfrak{g})$. Moreover, the Lie algebra $Lie(G)$ does not necessarily coincide with $\mathfrak{g}$. We have $Lie(G) = \mathfrak{g}$ if and only if the adjoint representation is a faithful $\mathfrak{g}$-representation, or equivalently, $\mathfrak{g}$ has trivial center.

Proposition 2. *Let* $\mathrm{Aut}^{\mathbf{C}^*}(X, \omega)$ *denote the group of* $\mathbf{C}^*$*-equivariant automorphisms preserving* ω. *Then the identity component of* $\mathrm{Aut}^{\mathbf{C}^*}(X, \omega)$ *can be identified with the adjoint group* G *of* $\mathfrak{g}$. *Moreover* $\mathfrak{g}$ *has trivial center. In particular,* $\mathfrak{g}$ *is the Lie algebra of the linear algebraic group* $\mathrm{Aut}^{\mathbf{C}^*}(X, \omega)$.

Proof . Let $(\wedge^{\geq 1}\Theta_{X_{reg}}, \delta)$ be the Lichnerowicz-Poisson complex for the smooth Poisson variety X_{reg}. The algebraic torus $\mathbf{C}^*$ acts on $\Gamma(X_{reg}, \wedge^p\Theta_{X_{reg}})$ and there is an associated grading

$$\Gamma(X_{reg}, \wedge^p\Theta_{X_{reg}}) = \oplus_{n \in \mathbf{Z}}\, \Gamma(X_{reg}, \wedge^p\Theta_{X_{reg}})(n).$$

Since the Poisson bracket of X has degree -1, the coboundary map δ has degree -1; thus we have a complex

$$\Gamma(X_{reg}, \Theta_{X_{reg}})(0) \xrightarrow{\delta_1} \Gamma(X_{reg}, \wedge^2\Theta_{X_{reg}})(-1) \xrightarrow{\delta_2} ...$$

The kernel $\mathrm{Ker}(\delta_1)$ of this complex is isomorphic to the tangent space of $\mathrm{Aut}^{\mathbf{C}^*}(X, \omega)$ at $[id]$. In fact, an element of $\mathrm{Ker}(\delta_1)$ corresponds to a derivation of $O_{X_{reg}}$ (or an infinitesimal automorphism of X_{reg}) preserving the Poisson structure, but it uniquely extends to a derivation of O_X preserving the Poisson structure (cf. [Na 4, Proposition 8]).

The Lichnerowicz-Poisson complex $(\wedge^{\geq 1}\Theta_{X_{reg}}, \delta)$ is identified with the truncated De Rham complex $(\Omega^{\geq 1}_{X_{reg}}, d)$ by the symplectic form ω (cf. [Na 4], Proposition 9, [Na 3], Section 3). The algebraic torus $\mathbf{C}^*$ acts on $\Gamma(X_{reg}, \Omega^p_{X_{reg}})$ and there is an associated grading

$$\Gamma(X_{reg}, \Omega^p_{X_{reg}}) = \oplus_{n \in \mathbf{Z}}\, \Gamma(X_{reg}, \Omega^p_{X_{reg}})(n).$$

The coboundary map d has degree 0; thus we have a complex

$$\Gamma(X_{reg}, \Omega^1_{X_{reg}})(1) \xrightarrow{d_1} \Gamma(X_{reg}, \Omega^2_{X_{reg}})(1) \xrightarrow{d_2} ...$$

Since ω has weight 1, this complex is identified with the the Lichnerowicz-Poisson complex above.

There is an injective map $d : \Gamma(X_{reg}, \mathcal{O}_{X_{reg}})(1) \to \Gamma(X_{reg}, \Omega^1_{X_{reg}})(1)$. We shall prove that $\mathrm{Ker}(d_1) = \mathrm{Im}(d)(\cong \Gamma(X_{reg}, \mathcal{O}_{X_{reg}})(1))$. The $\mathbf{C}^*$-action on X defines a vector field ζ on X_{reg}. For $v \in \Gamma(X_{reg}, \Omega^1_{X_{reg}})(1)$, the Lie derivative $L_\zeta v$ of v along ζ equals v. If moreover v is d-closed, then one has $v = d(i_\zeta v)$ by the Cartan relation

$$L_\zeta v = d(i_\zeta v) + i_\zeta(dv).$$

This means that $v \in \Gamma(X_{reg}, \mathcal{O}_{X_{reg}})(1)$. On the other hand, we have $\Gamma(X_{reg}, \mathcal{O}_{X_{reg}})(1) = \Gamma(X, \mathcal{O}_X)(1) = R_1 = \mathfrak{g}$.

It follows from the identification of $\mathrm{Ker}(\delta_1)$ and $\mathrm{Ker}(d_1)$ that every element of $\mathrm{Ker}(\delta_1)$ is a Hamiltonian vector field $H_f := \{f, \cdot\}$ for some $f \in R_1$. In particular, for $g \in R_1$, we have $H_f(g) = [f, g]$. Since $H_f \neq 0$ for a non-zero f, the map $ad : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is an injection. Notice that an element of $\mathrm{Aut}^{\mathbf{C}^*}(X, \omega)$ determines an automorphism of the graded $\mathbf{C}$-algebra R. In particular, it induces a $\mathbf{C}$-linear automorphism of $R_1 = \mathfrak{g}$. Since R is generated by R_1, this linear automorphism completely determines the automorphism of R. Hence, both G and $\mathrm{Aut}^{\mathbf{C}^*}(X, \omega)$ are subgroups of $GL(\mathfrak{g})$. The tangent spaces of both subgroups at $[id]$ coincide with $\mathfrak{g} \cong ad(\mathfrak{g}) \subset \mathrm{End}(\mathfrak{g})$. Therefore G is the identity component of $\mathrm{Aut}^{\mathbf{C}^*}(X, \omega)$ and $Lie(G) = \mathfrak{g}$. Q.E.D.

Proposition 3 *The symplectic variety (X, ω) coincides with the closure of a coadjoint orbit $(\bar{O}, \omega_{KK})$ of $\mathfrak{g}^*$ together with the Kirillov-Kostant form.*

Proof. Since G is the identity component of $\mathrm{Aut}^{\mathbf{C}^*}(X, \omega)$, X is stable under the coadjoint action of G on $\mathfrak{g}^*$. Hence X is a union of G-orbits. The G-orbits in X are symplectic leaves of the Poisson variety X. In our case, since X has only symplectic singularities, X has only finitely many symplectic leaves by [Ka]. Therefore X consists of finite number of G-orbits; hence there is an open dense G-orbit and X is the closure of such an orbit. Q.E.D.

Remark. We do not need the finiteness of symplectic leaves to prove that X contains an open G-orbit if we notice that the embedding $\mu : X \to \mathfrak{g}^*$ is the moment map for the G-action on X. In fact, the G-action on X determines a map $\mathfrak{g} \to \Gamma(X, \Theta_X)$; since μ is the moment map, it factorizes as $\mathfrak{g} \stackrel{\mu^*}{\to} \Gamma(X, O_X) \stackrel{H}{\to} \Gamma(X, \Theta_X)$. Here we regard $\mathfrak{g}$ as the space of linear functions on $\mathfrak{g}^*$ and μ^* is the natural map induced by μ. The map H associates to each $f \in \Gamma(X, O_X)$ the Hamiltonian vector field H_f. Let $x \in X$ be a smooth point. Choose $f_1, ..., f_r \in \mathfrak{g}$ so that df_1, ..., df_r spans $\Omega^1_X \otimes k(x)$. As ω is nondegenerate at x, the Hamiltonian vectors H_{f_1}, ..., H_{f_r} span $\Theta_X \otimes k(x)$. This impllies that the composite $\mathfrak{g} \to \Gamma(X, \Theta_X) \to \Theta_X \otimes k(x)$ is a surjection. Hence the G-orbit containing x is an open orbit.

For the unipotent radical U of G, let us denote by $\mathfrak{n}$ its Lie algebra. The ideal $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$ since we already know that $\mathfrak{g}$ has no center. Assume that $\mathfrak{n} \neq 0$. Then the center $z(\mathfrak{n})$ of $\mathfrak{n}$ is also non-trivial because $\mathfrak{n}$ is a nilpotent Lie algebra. Moreover $z(\mathfrak{n})$ is an ideal of $\mathfrak{g}$. In fact, it is enough to prove that, if $y \in \mathfrak{g}$ and $z \in z(\mathfrak{n})$, then $[x, [y, z]] = 0$ for any $x \in \mathfrak{n}$. Consider the Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

First, since $z \in z(\mathfrak{n})$, one has $[z, x] = 0$. Next, since $\mathfrak{n}$ is an ideal of $\mathfrak{g}$, we have $[x, y] \in \mathfrak{n}$; hence $[z, [x, y]] = 0$. It then follows from the Jacobi identity that $[x, [y, z]] = 0$.

Proposition 4. *Let $\mathfrak{g}$ be a complex Lie algebra with trivial center and whose adjoint group G is a linear algebraic group. Assume that $\mathfrak{n} \neq 0$. Let O be a coadjoint orbit of $\mathfrak{g}^*$ with the following properties*

(i) *O is preserved by the scalar $\mathbf{C}^*$-action on $\mathfrak{g}^*$;*

(ii) $T_0\bar{O} = \mathfrak{g}^*$, *where* $T_0\bar{O}$ *denotes the tangent space of the closure* $\bar{O}$ *of* O *at the origin.*

Then $\bar{O} - O$ *contains infinitely many coadjoint orbits; in particular* $\bar{O}$ *has infinitely many Poisson integral closed subschemes.*

Remark. This proposition shows that $(\bar{O}, \omega_{KK})$ cannot have symplectic singularities. In fact, if $(\bar{O}, \omega_{KK})$ is a symplectic variety, it has only finitely many Poisson integral closed subschemes [Ka].

Proof of Proposition 4. By a result of Mostow [Mos] (cf. [Ho], VIII, Theorem 3.5, Theorem 4.3), G is a semi-direct product of a reductive subgroup L and the unipotent radical U. Therefore we have a decomposition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{n}$. Take an element $\phi \in O$. Then ϕ is a linear function on $\mathfrak{g}$, which restricts to a *non-zero* function on $z(\mathfrak{n})$. In fact, if ϕ is zero on $z(\mathfrak{n})$, then $O \subset (\mathfrak{g}/z(\mathfrak{n}))^*$ and hence $\bar{O} \subset (\mathfrak{g}/z(\mathfrak{n}))^*$, which contradicts the assumption (ii). We put $\bar{\phi} := \phi|_{z(\mathfrak{n})} \neq 0$.

Notice that the adjoint group G is the subgroup of $\mathrm{GL}(\mathfrak{g})$ generated by all elements of the form $\exp(ad\ v)$ with $v \in \mathfrak{g}$. If $v \in z(\mathfrak{n})$, then $\exp(ad\ v) = id + ad\ v$ because $(ad\ v)^2 = 0$ for $v \in z(\mathfrak{n})$. Let $Z(U)$ be the identity component of the center of the unipotent radical U. Then one can write $Z(U) = 1 + ad\ z(\mathfrak{n})$. By the assumption, the map $ad : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is an injection. Now we identify $z(\mathfrak{n})$ with $ad\ z(\mathfrak{n})$; then one can write $Z(U) = 1 + z(\mathfrak{n})$ and its group law is defined by $(1+v)(1+v') := 1 + (v+v')$ for $v, v' \in z(\mathfrak{n})$. Fix an element $v \in z(\mathfrak{n})$ and consider $Ad^*_{1+v}(\phi) \in \mathfrak{g}^*$. Since $(1+v)^{-1} = 1-v$, the adjoint action

$$Ad_{(1+v)^{-1}} : \mathfrak{l} \oplus \mathfrak{n} \to \mathfrak{l} \oplus \mathfrak{n}$$

is defined by

$$x \oplus y \to x \oplus (-[v,x] + y)$$

because $(ad\ v)(y) = 0$. By definition

$$Ad^*_{1+v}(\phi)(x \oplus y) = \phi(Ad_{(1+v)^{-1}}(x \oplus y)) = \phi(x \oplus y) - \bar{\phi}([v,x]).$$

Here notice that $[v,x] \in z(\mathfrak{n})$ and hence $\phi([v,x]) = \bar{\phi}([v,x])$.

Since $z(\mathfrak{n})$ is an $\mathfrak{l}$-module by the Lie bracket, we decompose it into irreducible factors $z(\mathfrak{n}) = \bigoplus V_i$. Notice that it is the same as the irreducible decomposition of $z(\mathfrak{n})$ as a $[\mathfrak{l},\mathfrak{l}]$-module if $[\mathfrak{l},\mathfrak{l}] \neq 0$. In fact, the reductive Lie algebra $\mathfrak{l}$ is written as a direct sum of the semi-simple part and the center: $\mathfrak{l} = [\mathfrak{l},\mathfrak{l}] \oplus z(\mathfrak{l})$. Since $z(\mathfrak{l})$ is an Abelian Lie algebra, $z(\mathfrak{n})$ can be written as a direct sum $\oplus V_\alpha$ of the weight spaces for $z(\mathfrak{l})$. The semisimple part $[\mathfrak{l},\mathfrak{l}]$ acts on each weight space V_α; hence V_α is a direct sum of irreducible $[\mathfrak{l},\mathfrak{l}]$ modules. These irreducible $[\mathfrak{l},\mathfrak{l}]$ modules are stable under the $z(\mathfrak{l})$-action and, hence are irreducible $\mathfrak{l}$-modules.

When $\dim V_i = 1$ for some i, this V_i is an ideal of $\mathfrak{g}$. By the assumption (i), one can write $\bar{O} = \mathrm{Spec} R$ with a graded $\mathbf{C}$-algebra $R = \oplus_{j \geq 0} R_j$. By (ii) we see that $R_1 = \mathfrak{g}$. Take a generator x of a 1-dimensional space V_i. Then x generates a Poisson ideal I of R and $Y := \mathrm{Spec}(R/I)$ is a closed Poisson subscheme of $\bar{O}$ of codimension 1. Moreover, Y is stable under the G-action. Since $\dim Y$ is odd, Y contains infinitely many coadjoint orbits.

In the remainder we assume that $\dim V_i > 1$ for all i. In this case $[\mathfrak{l},\mathfrak{l}] \neq 0$. Since $\bar{\phi} \neq 0$, we can choose an i such that $\phi|_{V_i} \neq 0$. We fix a Cartan subalgebra $\mathfrak{h}$ of the semisimple Lie algebra $[\mathfrak{l},\mathfrak{l}]$ and choose a set Δ of simple roots from the root system Φ. We define $\mathfrak{n}^+ := \bigoplus_{\alpha\in\Phi^+}[\mathfrak{l},\mathfrak{l}]_\alpha$. Let $v_0 \in V_i$ be a highest weight vector of the irreducible $[\mathfrak{l},\mathfrak{l}]$-module V_i. Then one has $[v_0,\mathfrak{n}^+]=0$ and, in particular, $\phi([v_0,\mathfrak{n}^+])=0$. Moreover, we may assume that $\bar{\phi}(v_0)\neq 0$ by replacing ϕ by a suitable $Ad_g^*(\phi)$ with $g\in L$. This can be seen as follows. In fact, if $Ad_g^*(\bar{\phi})(v_0)=0$ for all g, then $\bar{\phi}$ is zero on the vector subspace of V_i spanned by all $Ad_g(v_0)$. But, since V_i is an irreducible L-representation, such a subspace coincides with V_i. This contradicts the fact that $\bar{\phi}|_{V_i}\neq 0$. Since v_0 is a highest weight vector of a non-trivial $[\mathfrak{l},\mathfrak{l}]$-irreducible module V_i, $[v_0,h]$ is a multiple of v_0 by a non-zero constant for an $h\in\mathfrak{h}$. Since $\bar{\phi}(v_0)\neq 0$, we also have $\bar{\phi}([v_0,h])\neq 0$ for this $h\in\mathfrak{h}$.

Let us consider $\bar{\phi}_{v_0} := \bar{\phi}([v_0,\cdot])|_{[\mathfrak{l},\mathfrak{l}]}$. By definition $\bar{\phi}_{v_0}$ is an element of $[\mathfrak{l},\mathfrak{l}]^*$. By the Killing form it is identified with an element of $[\mathfrak{l},\mathfrak{l}]$. The two facts $\bar{\phi}_{v_0}(\mathfrak{n}^+)=0$ and $\bar{\phi}_{v_0}(h)\neq 0$ mean that $\bar{\phi}_{v_0}$ is *not* a nilpotent element of $[\mathfrak{l},\mathfrak{l}]$.

For such v_0 and ϕ, we consider $Ad^*_{1+t^{-1}v_0}(t\phi)$, with $t\in\mathbf{C}^*$. One can write

$$Ad^*_{1+t^{-1}v_0}(t\phi)(x\oplus y) = t\phi(x\oplus y) - \bar{\phi}([v_0,x]).$$

Thus one has

$$\lim_{t\to 0} Ad^*_{1+t^{-1}v_0}(t\phi)(x\oplus y) = -\bar{\phi}([v_0,x]).$$

By definition $Ad^*_{1+t^{-1}v_0}(t\phi)\in O$. Thus $\lim_{t\to 0} Ad^*_{1+t^{-1}v_0}(t\phi)\in\bar{O}$. Moreover, by the equality above, we see that $\lim_{t\to 0} Ad^*_{1+t^{-1}v_0}(t\phi)|_{\mathfrak{n}}=0$; thus it can be regarded as an element of $(\mathfrak{g}/\mathfrak{n})^* = \mathfrak{l}^*$.

Furthermore, we have

$$\lim_{t\to 0} Ad^*_{1+t^{-1}v_0}(t\phi)|_{[\mathfrak{l},\mathfrak{l}]} = -\bar{\phi}_{v_0},$$

which can be regarded as an element of $[\mathfrak{l},\mathfrak{l}]$ by the identification $[\mathfrak{l},\mathfrak{l}]^*\cong[\mathfrak{l},\mathfrak{l}]$. As remarked above, this is not a nilpotent element.

Let us write $\mathfrak{l}$ as a direct sum of the semi-simple part and the center: $\mathfrak{l}=[\mathfrak{l},\mathfrak{l}]\oplus z(\mathfrak{l})$. There is an L-equivariant isomorphism $\mathfrak{l}^*\cong[\mathfrak{l},\mathfrak{l}]^*\oplus z(\mathfrak{l})^*$. Here L acts trivially on the second factor $z(\mathfrak{l})^*$. Therefore, every coadjoint orbit of $\mathfrak{l}^*$ is a pair of a coadjoint orbit of $[\mathfrak{l},\mathfrak{l}]^*$ and an element of $z(\mathfrak{l})^*$.

In our situation, we can write

$$\bar{\phi}([v_0,\cdot]) = \bar{\phi}_{v_0}\oplus\bar{\phi}([v_0,\cdot])|_{z(\mathfrak{l})}.$$

We can apply the same argument for $\lambda\phi$ with an arbitrary $\lambda\in\mathbf{C}^*$ to conclude that $\lambda\bar{\phi}([v_0,\cdot])\in\bar{O}$. One can write

$$\lambda\cdot\bar{\phi}([v_0,\cdot) = \lambda\bar{\phi}_{v_0}\oplus\lambda\cdot\bar{\phi}([v_0,\cdot])|_{z(\mathfrak{l})}.$$

Since $\bar{\phi}_{v_0}$ is not a nilpotent element, we see that $\lambda\bar{\phi}_{v_0}$ ($\lambda\in\mathbf{C}^*$) are contained in mutually different coadjoint orbits of $[\mathfrak{l},\mathfrak{l}]^*$.

Therefore, $\lambda\cdot\bar{\phi}([v_0,\cdot])$ ($\lambda\in\mathbf{C}^*$) are also contained in mutually different coadjoint orbits of $\mathfrak{l}^*$. Q.E.D.

Examples

(1) Let

$$G := \{g = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in SL(2, \mathbf{C})\}$$

be a Borel subgroup of $SL(2, \mathbf{C})$. Then

$$\mathfrak{g} := \{\begin{pmatrix} x & y \\ 0 & -x \end{pmatrix} \mid x,\ y \in \mathbf{C}\}.$$

Then

$$\mathfrak{n} = \{\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}\} \neq 0.$$

For the basis

$$h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \ f := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

of $\mathfrak{g}$, we take its dual basis $\{h^*,\ f^*\}$ of $\mathfrak{g}^*$. We denote by $O_{(\alpha,\beta)}$ the coadjoint orbit of $\mathfrak{g}^*$ passing through $\alpha h^* + \beta f^*$, $\alpha, \beta \in \mathbf{C}^*$. Then $O_{(\alpha,0)}$ consists of one point αh^* for any α. On the other hand, the set

$$O := \{\alpha h^* + \beta f^* \mid \alpha \in \mathbf{C},\ \beta \in \mathbf{C} - \{0\}\}$$

is a 2-dimensional orbit. This orbit O satisfies the conditions of Proposition 4. In this case $\bar{O} = \mathfrak{g}^*$ and $\bar{O} - O = \cup O_{(\alpha,0)}$.

(2) Put $\mathfrak{g} := sl(n, \mathbf{C})$ and let us consider the coadjoint orbits of $\mathfrak{g}^*$. By the Killing form every coadjoint orbit of $\mathfrak{g}^*$ is identified with an adjoint orbit $\mathfrak{g}$. Let O be the coadjoint orbit of $\mathfrak{g}^*$ corresponding to the nilpotent orbit $O_{[n]}$ of $\mathfrak{g}$. Then O satisfies the conditions (i) and (ii) of Proposition 4 except that $\mathfrak{g}$ is semisimple. In this case $\bar{O} - O$ consists of a finite number of coadjoint orbits corresponding to the nilpotent orbits $O_{[d_1,\dots,d_r]}$.

Proof of Theorem. We already know that $wt(\omega) = 1$ or $wt(\omega) = 2$. In the latter case (X, ω) is isomorphic to $(\mathbf{C}^{2d}, \omega_{st})$. So we assume that $wt(\omega) = 1$. By Propositions 2, 3 and 4, (X, ω) is isomorphic to a coadjoint orbit closure $(\bar{O}, \omega_{KK})$ of a complex semisimple Lie algebra $\mathfrak{g}$ together with the Kirillov-Kostant form. For a semisimple Lie algebra, a coadjoint orbit is identified with an adjoint orbit by the Killing form. A coadjoint orbit preserved by the scalar $\mathbf{C}^*$-action corresponds to a nilpotent orbit by this identification.

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