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Tilting Cohen-Macaulay representations

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Aim : Study Cohen-Macaulay modules by using tilting theory and cluster tilting theory

Throughout k is an algebraically closed field of characteristic zero for simplicity

1/1

Part 1 : Tilting theory (Derived category)
⇒ graded Cohen-Macaulay modules

Theorem [Gabriel '72]

Q : an acyclic connected quiver, kQ : the path algebra of Q

• kQ is **representation-finite** (i.e. there are only finitely many isoclasses of indecomposable kQ -modules)

⇔ Q is a **Dynkin quiver**

A_n ($n \geq 1$) • - • - • - • - ... - • - • - •

D_n ($n \geq 4$) •
 |
• - • - • - • - ... - • - •

E_n ($n = 6, 7, 8$) •
 |
• - • - • - • - ... - • - •

• In this case, \exists a bijection between isoclasses of indecomposable kQ -modules and positive roots

2/1

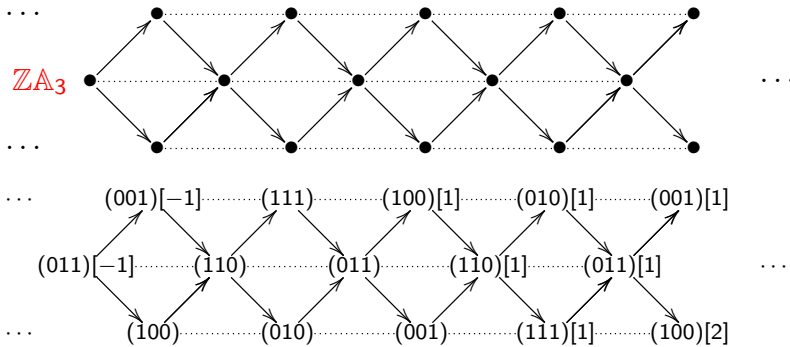
- $\text{mod } kQ$: the category of finitely generated kQ -modules
- $D^b(\text{mod } kQ)$: the bounded derived category of $\text{mod } kQ$

Theorem [Happel '87]

Q : a Dynkin quiver

The Auslander-Reiten quiver of $D^b(\text{mod } kQ)$ is $\mathbb{Z}Q$

- $Q = \mathbb{A}_3 = [\bullet \longrightarrow \bullet \longrightarrow \bullet]$



Cohen-Macaulay representations

- R : a **Gorenstein ring** \iff a Noetherian ring satisfying $\text{inj.dim } R_R < \infty$ and $\text{inj.dim } {}_R R < \infty$
- $X \in \text{mod } R$ is **Cohen-Macaulay** $\iff \text{Ext}_R^i(X, R) = 0$ for all $i > 0$
- $\text{CM } R$: the category of Cohen-Macaulay R -modules

Example

R : a commutative local Gorenstein ring

- $X \in \text{mod } R$ is Cohen-Macaulay $\iff \text{depth } X = \dim R$ or $X = 0$

Definition [Auslander-Bridger]

$\underline{\text{CM}} R$: the **stable category**

- The objects are the same as $\text{CM } R$
- $\text{Hom}_{\underline{\text{CM}} R}(X, Y) := \text{Hom}_R(X, Y) / P(X, Y)$, where $P(X, Y)$ consists of morphisms factoring through projective R -modules

Properties [Happel, Buchweitz, Eisenbud]

- $\underline{\text{CM}} R$ has a canonical structure of a triangulated category
- \exists a triangle equivalence $\underline{\text{CM}} R \simeq D^b(\text{mod } R)/K^b(\text{proj } R)$
- $\text{proj } R$: the category of finitely generated projective R -modules
- $K^b(\text{proj } R)$: the bounded homotopy category of $\text{proj } R$
- R : a hypersurface
- $\implies \underline{\text{CM}} R$ is described by matrix factorizations

More generally

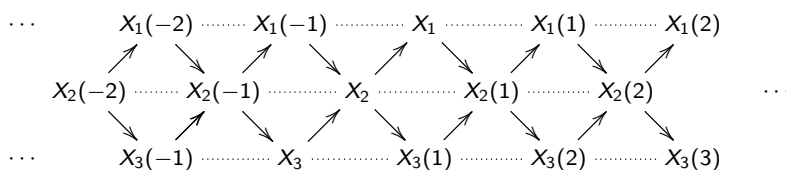
- G : an abelian group
- R : a G -graded Gorenstein ring
- $\text{CM}^G R$: the category of G -graded Cohen-Macaulay R -modules
- $\underline{\text{CM}}^G R$: the **stable category** is defined similarly

• $\underline{\text{CM}}^G R$ has a canonical structure of a triangulated category

- Other properties also hold after suitable modifications

Example

- $R := k[x]/(x^n)$ with $\deg x = 1$
- R is **selfinjective** ($\iff \text{inj.dim } R = 0$) and hence Gorenstein
- $\text{CM}^{\mathbb{Z}} R = \text{mod}^{\mathbb{Z}} R$
- Indecomposable objects in $\text{mod } R$ are $X_i = k[x]/(x^i)$ ($1 \leq i \leq n$)
- Indecomposable objects in $\text{mod}^{\mathbb{Z}} R$ are $X_i(j)$ ($1 \leq i \leq n, j \in \mathbb{Z}$)
- For $n = 4$, the Auslander-Reiten quiver of $\text{mod}^{\mathbb{Z}} R$ is



This is $\mathbb{Z}\mathbb{A}_3$, the Auslander-Reiten quiver of $D^b(\text{mod } k\mathbb{A}_3)$

\exists a triangle equivalence $\text{mod}^{\mathbb{Z}}(k[x]/(x^n)) \simeq D^b(\text{mod } k\mathbb{A}_{n-1})$

Theorem [Buchweitz-Greuel-Schreyer '87]

R : a commutative complete local Gorenstein k -algebra

- R is **CM-finite** (i.e. there are only finitely many isoclasses of indecomposable objects in $\text{CM } R$) $\iff R$ is a **simple singularity**

$$R = k[[x, y, z_2, \dots, z_d]]/(f), f = \begin{cases} x^{n+1} + y^2 + z_2^2 + \dots + z_d^2 & A_n \\ x^{n-1} + xy^2 + z_2^2 + \dots + z_d^2 & D_n \\ x^4 + y^3 + z_2^2 + \dots + z_d^2 & E_6 \\ x^3y + y^3 + z_2^2 + \dots + z_d^2 & E_7 \\ x^5 + y^3 + z_2^2 + \dots + z_d^2 & E_8. \end{cases}$$

Theorem [Geigle-Lenzing '91, Kajiura-Saito-Takahashi '07]

$R = k[x, y, z_2]/(f)$: a simple singularity with $\dim R = 2$ and \mathbb{Z} -grading given by

R and Q	$A_n : x^{n+1} - yz_2$	D_n	E_6	E_7	E_8
deg x	1	2	3	4	6
deg y	p	$n - 2$	4	6	10
deg z_2	$n + 1 - p$	$n - 1$	6	9	15

$\implies \exists$ a triangle equivalence $\underline{\text{CM}}^{\mathbb{Z}} R \simeq D^b(\text{mod } kQ)$

Gabriel's Theorem \longleftrightarrow Buchweitz-Greuel-Schreyer's Theorem

Tilting theory

\mathcal{T} : a triangulated category

Definition [Rickard '89]

$U \in \mathcal{T}$ is a **tilting object** \iff

- $\forall i \neq 0, \text{Hom}_{\mathcal{T}}(U, U[i]) = 0$
- U generates \mathcal{T} as a thick subcategory

- $\text{K}^b(\text{proj } \Lambda)$ has a tilting object $\Lambda = (\dots \rightarrow 0 \rightarrow 0 \rightarrow \Lambda \rightarrow 0 \rightarrow \dots)$

Proposition [Keller '94]

\mathcal{T} : an algebraic triangulated category (e.g. $\underline{\text{CM}}^G R$)

$U \in \mathcal{T}$: a tilting object

$\implies \exists$ a triangle equivalence $\mathcal{T} \simeq \text{K}^b(\text{proj } \Lambda)$ for $\Lambda := \text{End}_{\mathcal{T}}(U)$

- If moreover $\text{gl.dim } \Lambda < \infty$, then $\text{K}^b(\text{proj } \Lambda) = D^b(\text{mod } \Lambda)$ and we have a triangle equivalence $\mathcal{T} \simeq D^b(\text{mod } \Lambda)$

Dimension zero

Theorem [Yamaura '13]

- R : a \mathbb{Z} -graded finite dimensional selfinjective k -algebra
 - Assume $R_i = 0$ for $i < 0$ and $\text{gl.dim } R_0 < \infty$
- $\implies \underline{\text{mod}}^{\mathbb{Z}} R$ has a tilting object $\bigoplus_{i \geq 0} R(i)_{\geq 0}$
 where $X_{\geq 0} := \bigoplus_{i \geq 0} X_i$ for $X \in \underline{\text{mod}}^{\mathbb{Z}} R$

- $R = k[x]/(x^n)$ with $\deg x = 1$
- $\implies \exists$ a triangle equivalence $\underline{\text{mod}}^{\mathbb{Z}} (k[x]/(x^n)) \simeq \text{D}^b(\text{mod } k\mathbb{A}_{n-1})$

Corollary [Happel '87]

- Λ : a finite dimensional k -algebra with $\text{gl.dim } \Lambda < \infty$
 - $T(\Lambda) = \Lambda \oplus D\Lambda$: the **trivial extension algebra** (D : the k -dual)
- $(x_1, f_1) \cdot (x_2, f_2) = (x_1 x_2, x_1 f_2 + f_1 x_2)$ for $x_1, x_2 \in \Lambda, f_1, f_2 \in D\Lambda$
 $T(\Lambda)_0 = \Lambda, T(\Lambda)_1 = D\Lambda, T(\Lambda)_i = 0$ for $i \neq 0, 1$
 $\implies \exists$ a triangle equivalence $\underline{\text{mod}}^{\mathbb{Z}} T(\Lambda) \simeq \text{D}^b(\text{mod } \Lambda)$



9/1

Dimension one

Theorem [Buchweitz-I-Yamaura (arXiv:1803.05269)]

- R : a commutative Gorenstein ring with $\dim R = 1$
 - Assume $R_i = 0$ for $i < 0, R_0 = k$ and R is reduced
- $\implies \underline{\text{CM}}^{\mathbb{Z}} R$ has a tilting object $\bigoplus_{i=1}^n R(i)_{\geq 0}$ ($n \gg 0$)

- There is a result for non-reduced case

Example (cf. [Dieterich-Wiedemann, Araya])

$R = k[x, y]/(f)$: a simple singularity with $\dim R = 1$ and \mathbb{Z} -grading given by

R	A_{2n-1}	A_{2n}	D_{2n}	D_{2n+1}	E_6	E_7	E_8
$\deg x$	1	2	1	2	3	2	3
$\deg y$	n	$2n+1$	$n-1$	$2n-3$	4	3	5
Q	D_{n+1}	A_{2n}	D_{2n}	A_{4n-1}	E_6	E_7	E_8

$\implies \exists$ a triangle equivalence $\underline{\text{CM}}^{\mathbb{Z}} R \simeq \text{D}^b(\text{mod } kQ)$



10/1

Geigle-Lenzing complete intersection (arXiv:1409.0668)

[joint work with Herschend, Minamoto, Oppermann]

- $C := k[T_0, \dots, T_d]$: a polynomial algebra over a field k ($d \geq 0$)
- $\ell_1, \dots, \ell_n \in C$: linear forms ($n \geq 0$)
- $p_1, \dots, p_n \geq 2$: integers

Definition

- $R := C[X_1, \dots, X_n]/(X_i^{p_i} - \ell_i \mid 1 \leq i \leq n)$: a ring
- $\mathbb{L} := \langle \vec{c}, \vec{x}_1, \dots, \vec{x}_n \rangle / \langle p_i \vec{x}_i - \vec{c} \mid 1 \leq i \leq n \rangle$: an abelian group

Properties

- R is \mathbb{L} -graded by $\deg T_j := \vec{c}$ and $\deg X_i := \vec{x}_i$
- \mathbb{L} is an abelian group of rank one
- R is a complete intersection ring with $\dim R = d + 1$
- R has the a -invariant $\vec{\omega} := (n - d - 1)\vec{c} - \sum_{i=1}^n \vec{x}_i \in \mathbb{L}$
i.e. $\text{Ext}_R^{d+1}(k, R(\vec{\omega})) \simeq k$ in $\text{mod}^{\mathbb{L}} R$



11 / 1

Definition

(R, \mathbb{L}) : a **Geigle-Lenzing complete intersection** \iff

\forall at most $d + 1$ elements from ℓ_1, \dots, ℓ_n are linearly independent

- $n \leq d + 1 \iff R$ is a polynomial algebra
- $n = d + 2 \iff R$ is a hypersurface

$$R \simeq k[X_1, \dots, X_n]/(\alpha_1 X_1^{p_1} + \dots + \alpha_n X_n^{p_n})$$

Classical case $d = 1$ [Geigle-Lenzing '87, '91]

- (R, \mathbb{L}) was used to study the **weighted projective line**

$$R \simeq k[X_1, \dots, X_n]/(X_i^{p_i} - \alpha_{i1} X_1^{p_1} - \alpha_{i2} X_2^{p_2})_{3 \leq i \leq n}$$

- (R, \mathbb{L}) is **domestic** $\iff n - 2 \leq \frac{1}{p_1} + \dots + \frac{1}{p_n}$

In this case, the Veronese subring $R^{(\vec{\omega})}$ is a simple singularity with $\dim R^{(\vec{\omega})} = 2$ and $\text{CM}^{\mathbb{L}} R \simeq \text{CM}^{\mathbb{Z}} R^{(\vec{\omega})}$

(p_1, \dots, p_n)	$n \leq 2$	$(2, 2, p)$	$(2, 3, 3)$	$(2, 3, 4)$	$(2, 3, 5)$
$R^{(\vec{\omega})}$	$A_{p_1 + \dots + p_n - 1}$	D_{p+2}	E_6	E_7	E_8



12 / 1

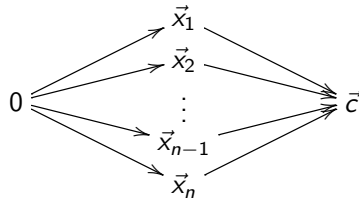
Theorem [HIMO '14, Futaki-Ueda '11 for $n = d + 2$, Kussin-Lenzing-Meltzer '13 for $d = 1, n = 3$]

(R, \mathbb{L}) : a Geigle-Lenzing complete intersection
 $\implies \exists$ a triangle equivalence $\underline{\text{CM}}^{\mathbb{L}} R \simeq D^b(\text{mod } A^{\text{CM}})$

- \mathbb{L} is a poset : $\vec{x} \geq \vec{y} \iff \vec{x} - \vec{y} \in \langle \vec{c}, \vec{x}_1, \dots, \vec{x}_n \rangle_{\text{monoid}}$
- $A^{\text{CM}} := (R_{\vec{x}-\vec{y}})_{0 \leq \vec{x}, \vec{y} \leq d\vec{c} + 2\vec{\omega}}$

This is a k -algebra by product in R and matrix multiplication

- $n = d + 2 \implies A^{\text{CM}} = \bigotimes_{i=1}^n k\mathbb{A}_{p_i-1}$
- $n = d + 3$ and $p_1 = \dots = p_n = 2$
 $\implies A^{\text{CM}}$ is given by the following quiver with 2 relations



Part 2 : Cluster category (Cluster tilting theory)

\implies ungraded Cohen-Macaulay modules

- Cluster categories were introduced in categorification of Fomin-Zelevinsky cluster algebras
- [Buan-Marsh-Reineke-Reiten-Todorov '06] for $\Lambda = kQ$
- [Amiot '09, Guo '11 based on Keller '05] for general Λ

Definition

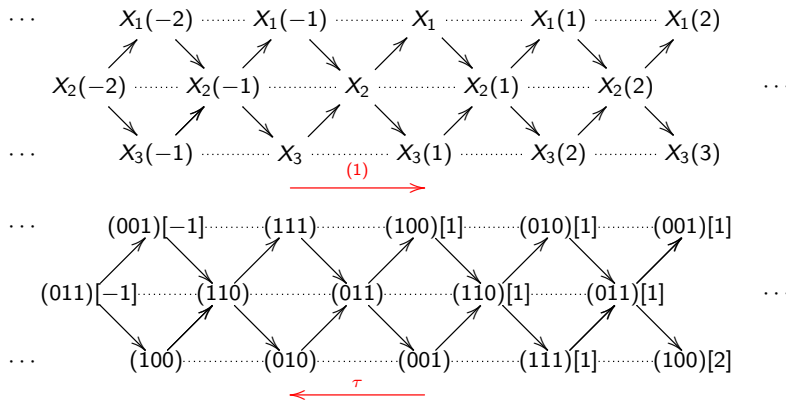
\mathcal{T} : an additive category

$F : \mathcal{T} \simeq \mathcal{T}$: an automorphism

- The orbit category \mathcal{T}/F has the same objects as \mathcal{T}
- $\text{Hom}_{\mathcal{T}/F}(X, Y) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{T}}(X, F^i(Y))$
 where the composition is defined naturally

Recall

- \exists a triangle equivalence $\text{mod}^{\mathbb{Z}}(k[x]/(x^n)) \simeq D^b(\text{mod } k\mathbb{A}_{n-1})$



- $(1) \simeq \tau^{-1}$ holds
- Their orbit categories are also equivalent

$$\begin{array}{ccc} \text{mod}^{\mathbb{Z}}(k[x]/(x^n))/(1) & \simeq & D^b(\text{mod } k\mathbb{A}_{n-1})/\tau \\ \parallel & & \parallel \\ \text{mod}(k[x]/(x^n)) & & C_1(k\mathbb{A}_{n-1}) \end{array}$$

Serre functor

- \mathcal{T} : a k -linear Hom-finite triangulated category

Definition

- An automorphism $\mathbb{S} : \mathcal{T} \rightarrow \mathcal{T}$ is a **Serre functor** \iff
 \exists a bifunctorial isomorphism $\text{Hom}_{\mathcal{T}}(X, Y) \simeq D\text{Hom}_{\mathcal{T}}(Y, \mathbb{S}X)$

- [Happel] Serre functor \implies Auslander-Reiten theory in \mathcal{T}

Example

- X : a smooth projective variety over k with $\dim X = d$
 Then $D^b(\text{coh } X)$ has a Serre functor $- \otimes \omega_X[d]$
- [Happel] Λ : a finite dimensional k -algebra with $\text{gl.dim } \Lambda < \infty$
 Then $D^b(\text{mod } \Lambda)$ has a Serre functor $\nu := - \otimes_{\Lambda}^L D\Lambda$
- [Auslander] R : a Gorenstein isolated singularity, $\dim R = d + 1$
 Then $\underline{\text{CM}} R$ is **d -Calabi-Yau** (i.e. $[d]$ is a Serre functor)

Cluster category

- $d \geq 1$
 - Λ : a finite dimensional k -algebra with $\text{gl.dim } \Lambda \leq d$
 - $\nu_d := \nu \circ [-d] : D^b(\text{mod } \Lambda) \simeq D^b(\text{mod } \Lambda)$ (e.g. $\tau = \nu_1$)
 - $C_d^\circ(\Lambda) := D^b(\text{mod } \Lambda) / \nu_d$
- This does not have a structure of a triangulated category in general

Theorem [AGK]

- \exists a triangulated category $C_d(\Lambda)$ (the d -cluster category) containing $C_d^\circ(\Lambda)$ as a full subcategory such that the composition $D^b(\text{mod } \Lambda) \rightarrow C_d^\circ(\Lambda) \subset C_d(\Lambda)$ is a triangle functor
- If $C_d(\Lambda)$ is Hom-finite, then it is d -Calabi-Yau



17 / 1

Cluster tilting

- Appeared in higher dimensional Auslander-Reiten theory

Definition [I '07]

$d \geq 1$

\mathcal{T} : a triangulated category or an exact category

$M \in \mathcal{T}$ is a d -cluster tilting object

\iff The following conditions for $X \in \mathcal{T}$ are equivalent

- X is a direct summand of $M^{\oplus n}$ for some $n \geq 0$
- $1 \leq \forall i \leq d - 1, \text{Hom}_{\mathcal{T}}(M, X[i]) = 0$
- $1 \leq \forall i \leq d - 1, \text{Hom}_{\mathcal{T}}(X, M[i]) = 0$

Example

- [AG] If $C_d(\Lambda)$ is Hom-finite, then $\Lambda \in C_d(\Lambda)$ is d -cluster tilting
- R is CM-finite \iff CM R has a 1-cluster tilting object
- [I] R : a Gorenstein isolated singularity with $\dim R = d + 1$
 $M \in \text{CM } R$ is a d -cluster tilting object
 $\iff \text{End}_R(M)$ is a **non-commutative crepant resolution** of R

18 / 1

Preprojective algebra

- Λ : a finite dimensional k -algebra with $\text{gl.dim } \Lambda \leq d$
- $E := \text{Ext}_{\Lambda}^d(D\Lambda, \Lambda)$ has a Λ -bimodule structure

Definition [I-Oppermann, Keller]

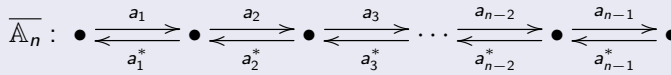
- $\Pi = \Pi_{d+1}(\Lambda) := T_{\Lambda}(E)$: the $(d+1)$ -preprojective algebra
- This is \mathbb{Z} -graded by $\Pi_i = E^{\otimes \Lambda^i}$ for $i \geq 0$

- $\Pi_0 = \Lambda$ and $\Pi_i = H^0(\nu_d^{-i}(\Lambda))$ for $i \geq 0$
- $\Pi = \text{End}_{C_d(\Lambda)}(\Lambda)$

Classical case $d = 1$

- Λ : a finite dimensional **hereditary** k -algebra (i.e. $\text{gl.dim } \Lambda \leq 1$)
 $\iff \Lambda = kQ$ for an acyclic quiver Q

- \overline{Q} : the **double** of Q



- $\Pi_2(kQ)$ is isomorphic to $k\overline{Q}/\langle \sum_{a \in Q_1} (aa^* - a^*a) \rangle$

19 / 1

- Λ : a finite dimensional k -algebra with $\text{gl.dim } \Lambda \leq d$
- $\Pi_{d+1}(\Lambda)$ enjoys nice homological properties under the condition :

Definition

Λ : **d -hereditary** $\iff \forall i \in \mathbb{Z}, \nu_d^i(\Lambda)$ is concentrated in degrees $d\mathbb{Z}$

Dichotomy Theorem [Herschend-I-Oppermann '14]

Assume that Λ is ring-indecomposable

Λ is d -hereditary \iff One of the following holds

- \exists a d -cluster tilting object $M \in \text{mod } \Lambda$ (**d -representation-finite**)
- $\forall i \geq 0, \nu_d^{-i}(\Lambda) \in \text{mod } \Lambda$ (**d -representation-infinite**)

Classical case $d = 1$

A finite dimensional k -algebra Λ is

- 1-representation-finite $\iff \Lambda = kQ$ for a Dynkin quiver Q
- 1-representation-infinite $\iff \Lambda = kQ$ for a non-Dynkin quiver Q

20 / 1

d-representation-finite case

- Λ : a d -representation-finite k -algebra
- $\Pi = \Pi_{d+1}(\Lambda)$

Theorem [IO '13, Amiot '09 for $d = 1$]

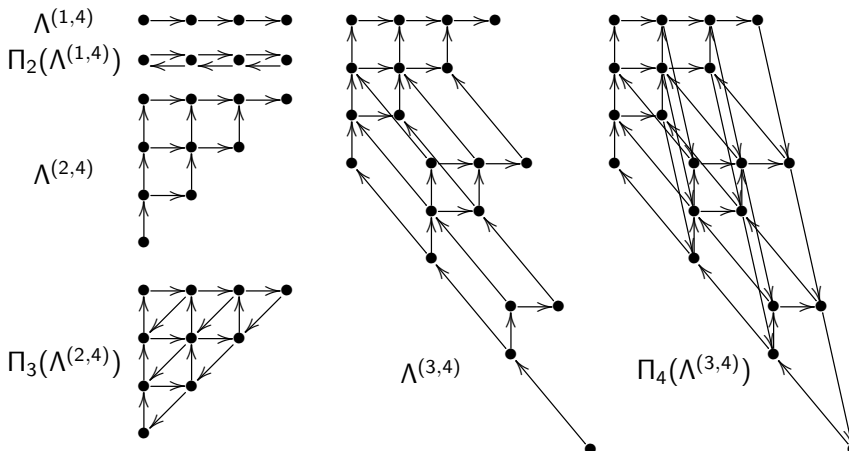
- Π is a finite dimensional selfinjective k -algebra
- \exists a triangle equivalence $\underline{\text{mod}}^{\mathbb{Z}} \Pi \simeq D^b(\underline{\text{mod}} \text{End}_{\underline{\text{mod}} \Lambda}(\Pi))$
- \exists a triangle equivalence $\underline{\text{mod}} \Pi \simeq C_{d+1}(\underline{\text{mod}} \text{End}_{\underline{\text{mod}} \Lambda}(\Pi))$

In particular,

- [Crawley-Boevey '00 for $d = 1$]
- $\underline{\text{mod}} \Pi$ is a $(d + 1)$ -Calabi-Yau triangulated category
- [Geiss-Leclerc-Schröer '06 for $d = 1$]
- $\underline{\text{mod}} \Pi$ has a $(d + 1)$ -cluster tilting object

Example of type A

- $\forall d, n \geq 1, \exists \Lambda^{(d,n)}$: a d -representation-finite k -algebra s.t.
- $\Lambda^{(1,n)} = k\mathbb{A}_n$ and $\Lambda^{(d+1,n)} = \text{End}_{\Lambda^{(d,n)}}(\Pi_{d+1}(\Lambda^{(d,n)}))$



- \exists a triangle equivalence $\underline{\text{mod}} \Pi_{d+1}(\Lambda^{(d,n)}) \simeq C_{d+1}(\Lambda^{(d+1,n-1)})$

d -representation-infinite case

Setting

- Λ : a d -representation-infinite k -algebra
- Assume $\Pi := \Pi_{d+1}(\Lambda)$ is noetherian
- $e \in \Lambda$: an idempotent s.t. $\dim_k(\Pi/(e)) < \infty$ and $e\Lambda(1-e) = 0$

Theorem [Amiot-I-Reiten '15]

(cf. [de Thanhoffer de Völcsey-Van den Bergh '16])

Under the above setting, let $R := e\Pi e$ and $\Lambda' := \Lambda/(e)$

- R is a Gorenstein ring
- Λ' is a finite dimensional k -algebra with $\text{gl.dim } \Lambda' \leq d$
- $\underline{\text{CM}}^{\mathbb{Z}} R$ has a tilting object Πe
- \exists a triangle equivalence $\underline{\text{CM}}^{\mathbb{Z}} R \simeq D^b(\text{mod } \Lambda')$
- $\underline{\text{CM}} R$ has a d -cluster tilting object Πe
- \exists a triangle equivalence $\underline{\text{CM}} R \simeq C_d(\Lambda')$



23 / 1

Example ($d = 1$)

Q : an extended Dynkin quiver, e : the extended vertex

$\Pi := \Pi_2(kQ)$, $R := e\Pi e$

- [Geigle-Lenzing] R is a simple singularity with $\dim R = 2$
- [Auslander] $\text{CM } R$ has a 1-cluster tilting object Πe
- For the Dynkin quiver $Q' = Q \setminus e$, \exists triangle equivalences $\underline{\text{CM}}^{\mathbb{Z}} R \simeq D^b(\text{mod } kQ')$ and $\underline{\text{CM}} R \simeq C_1(kQ')$

Theorem [Minamoto-Mori '11, Keller '11, Amiot-I-Reiten '15]

\exists a bijection between

(1) d -representation-infinite k -algebras Λ , and

(2) $(d+1)$ -Calabi-Yau k -algebras Γ of a -invariant -1 , i.e.

- Γ is a \mathbb{Z} -graded k -algebra with $\Gamma_i = 0$ for $i < 0$ and $\dim_k \Gamma_i < \infty$
- $\text{proj.dim } (\Gamma)_{\Gamma^e} < \infty$ for $\Gamma^e := \Gamma^{\text{op}} \otimes_k \Gamma$
- $\text{Ext}_{\Gamma^e}^i(\Gamma, \Gamma^e) \simeq \begin{cases} 0 & i \neq d+1 \\ \Gamma(-1) & i = d+1 \end{cases}$ in $\text{Mod } {}^{\mathbb{Z}}\Gamma^e$

The bijection is given by $\Lambda \mapsto \Pi_{d+1}(\Lambda)$, and the converse is $\Gamma \mapsto \Gamma_0$

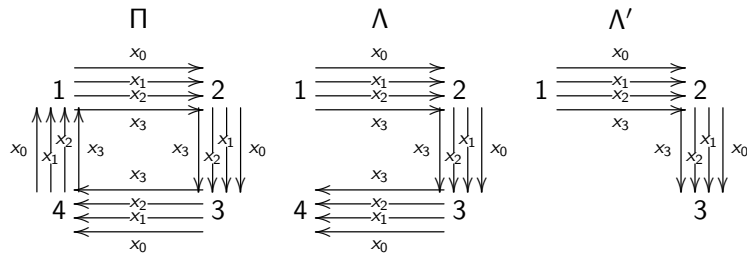


24 / 1

Example (Quotient singularity) (cf. [Ueda '08])

ζ : a primitive n -th root of unity
 $G := \langle \text{diag}(\zeta^{a_0}, \dots, \zeta^{a_d}) \rangle \subset \text{SL}_{d+1}(k)$ ($0 \leq a_i < n$)
 Assume $(n, a_i) = 1$ and $a_0 + \dots + a_d = n$
 $\implies \exists$ a triangle equivalence $\underline{\text{CM}}(k[x_0, \dots, x_d]^G) \simeq \text{C}_d(\Lambda')$
 for some explicit finite dimensional k -algebra Λ'

$d = 3, n = 4, a_0 = a_1 = a_2 = a_3 = 1$



- Dimer model [Broomhead, Ishii-Ueda, Davison, Bocklandt,...]

Example ($d = 2$, Dimer model) [Nakajima '18]

R : a Gorenstein toric singularity with $\dim R = 3$
 $\implies \exists$ a triangle equivalence $\underline{\text{CM}} R \simeq \text{C}_2(\Lambda')$
 for some explicit finite dimensional k -algebra Λ'

