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Hilbert squares of ADE surface singularities

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A normal algebraic variety X over $\mathbb{C}$ is called a *symplectic variety* if the regular part X_{reg} of X admits an algebraic symplectic form ω such that ω extends to a regular 2-form on Y for any resolution $Y \to X$. Symplectic varieties play important roles in various branches of mathematics and have been intensively studied since they were first introduced by Beauville [Bea2]. The aim of this article is to introduce a certain class of symplectic varieties and to study their properties. These varieties are obtained as the Hilbert schemes of two points on singular surfaces. We also consider their special desingularizations and give a characterization of singularities of the varieties using fibers of such desingularizations (cf. Theorem 4). Finally we will give an application of this result. For details, please see the paper [Y1].

Before we consider the Hilbert schemes of points, let us see examples of symplectic varieties, some of which will appear again later.

Example 1. (ADE surface singularities)
ADE surface singularities are the only examples of 2-dimensional symplectic singularities. Here, by a symplectic singularity, we mean the germ of a singular point of a symplectic variety. □

Example 2. (Nilpotent orbit closures)
Let G be a complex connected semisimple Lie group and $\mathfrak{g}$ its Lie algebra. Then G acts on $\mathfrak{g}$ by adjoint action. An orbit O_x of a nilpotent element $x \in \mathfrak{g}$ under this action is called a *nilpotent orbit*. O_x admits a symplectic form ω_{KK} called the Kostant-Killirov form, and the normalization of the closure $\overline{O_x}$ in $\mathfrak{g}$ is a symplectic variety. □

Example 3. (Symplectic quotient singularities)
For an integer $n \geq 1$, the vector space $\mathbb{C}^{2n}$ has a standard symplectic form

$$\omega_{st} = \sum_{i=1}^{n} dx_i \wedge dx_{i+n}$$

where x_i's are the standard coordinates of $\mathbb{C}^{2n}$. Let $Sp(2n) \subset GL(2n)$ be the subgroup consisting of elements which preserve ω_{st}. Then, for any finite subgroup G of $Sp(2n)$, the quotient $\mathbb{C}^{2n}/G$ is a symplectic variety. □

Symplectic varieties are most studied when they admit symplectic resolutions in particular from the viewpoint of geometric representation theory.

Definition 1. *A resolution $\pi : Y \to X$ of a symplectic variety X is called a symplectic resolution if the pullback $\pi^*\omega$ on $\pi^{-1}(X_{reg})$ extends to a symplectic form on the whole of Y.* □

Remark 1. A symplectic resolution is the same thing as a crepant resolution of a symplectic variety. □

Remark 2. Symplectic resolutions do not exist in general and are usually rare. For example, only several types examples of symplectic quotient singularities are known to have symplectic resolutions (cf. [BS]). □

Example 4. (Springer resolution)
Let G and $\mathfrak{g}$ be as in Example 2. Then the set $\mathcal{N}$ of all nilpotent elements of $\mathfrak{g}$ is called the *nilpotent cone* of $\mathfrak{g}$. There is a unique nilpotent orbit O_{reg}, called the *regular orbit*, such that $\mathcal{N} = \overline{O_{\text{reg}}}$, and $\mathcal{N}$ is a symplectic variety. $\mathcal{N}$ has a symplecic resolution

$$f : T^*(G/B) \to \mathcal{N}$$

called the *Springer resolution* where $B \subset G$ is a Borel subgroup of G. Note that the cotangent bundle of any manifold has a natural symplectic structure. □

One important problem in the study of symplectic singularities is to give a classification. It is shown that isomorphism classes of conical symplectic singularities appear only in a discrete way [Na2], and there are also some classification results [Bea2],[Na1],[Na3], and so on. However, complete classification seems still out of reach. In order to classify symplectic singularities, it seems useful to find a good invariant of them. In this article, I propose a fiber of a symplectic resolution as such an invariant. Fibers of symplectic resolutions have been studied in particular for the case of Springer resolutions where the fibers are called *Springer fibers*. In that case it is known that there is a relation between the geometry of fibers and the representation theory of the Weyl group of the Lie algebra $\mathfrak{g}$ [Sp]. Thus, the study of such fibers is important in representation theory, and the geometric structure of fibers is also interesting from the viewpoint of combinatorics.

When we consider a fiber of a symplectic resolution, we should note that any symplectic variety admits a natural stratification by finitely many symplectic leaves, which are connected symplectic manifolds [K, Thm. 2.3]. This stratification is obtained by successively taking the singular loci. The decomposition of the nilpotent cone into smaller nilpotent orbits

$$\mathcal{N} = \bigsqcup \mathcal{O}_x$$

is a typical example of this stratification.

We particularly consider fibers of the smallest stratum.

Definition 2. *Let $\pi : Y \to X$ be a symplectic resolution of a symplectic variety X and let $\{p\}$ be a 0-dimensional symplectic leaf of X. Then we call a fiber $\pi^{-1}(p)$ a central fiber of π.* □

Note that the usage of the term "central fiber" above is not standard. To consider 0-dimensional leaves only is in some sense essential since a symplectic variety always decomposes into the product of $\mathbb{C}^{2l}$ and a smaller symplectic variety in a (formal) neighborhood of a point on an $2l$-dimensional leaf for $l > 0$ (cf. *loc. cit.*). In the case of the

Springer resolution, the origin is the unique 0-dimensional leaf, and the central fiber is the rational homogeneous space G/B.

In this article we consider special symplectic varieties which are obtained as the Hilbert schemes of two points on singular surfaces. For a (possibly singular) surface S, the Hilbert scheme $\mathrm{Hilb}^n(S)$ of n points on S is a moduli space of closed subschemes $Z \subset S$ such that $H^0(Z, \mathcal{O}_Z)$ is a n-dimensional $\mathbb{C}$-vector space. If S is smooth and symplectic, it is well-known that $\mathrm{Hilb}^n(S)$ also smooth and symplectic [F],[Bea1]. It seems natural to consider a singular analogue and to allow S to have symplectic singularities.

Since two-dimensional symplectic singularities are ADE-singularities, let S_T be a surface with one ADE-singular point of type $T = A_k, D_k$ or E_k. The main object of this article is the Hilbert scheme $\mathrm{Hilb}^2(S_T)$ of 2 points on S_T. We can of course consider the Hilbert scheme of n points on S_T for $n \geq 3$, but the situation will be more complicated in that case and we concentrate on the two-points case from now on. Later we will mention the general case in Remark 3.

We have the following fact.

Fact 1. *The Hilbert-Chow morphism* $\mathrm{Hilb}^2(S_T) \to \mathrm{Sym}^2(S_T)$ *is identified with the blow-up of* $\mathrm{Sym}^2(S_T)$ *along the diagonal* $\Delta \subset \mathrm{Sym}^2(S_T)$ *where* $\mathrm{Sym}^2(S_T)$ *is the symmetric product* $(S_T \times S_T)/\mathfrak{S}_2$. □

Using this fact we can explicitly calculate $\mathrm{Hilb}^2(S_T)$. In particular we can show the following.

Proposition 3. $\mathrm{Hilb}^2(S_T)$ *is a symplectic variety.* □

The main result of this article is a characterization of singularities of $\mathrm{Hilb}^2(S_T)$ using central fibers of symplectic resolutions. Before we state the main result, we study the symplectic leaves and symplectic resolutions of $\mathrm{Hilb}^2(S_T)$.

Fact 1 enables us to describe the symplectic leaves of $\mathrm{Hilb}^2(S_T)$ explicitly. In this case the singular locus $\mathrm{Sing}(\mathrm{Hilb}^2(\mathrm{S_T}))$ is isomorphic to the blow-up $\mathrm{Bl}_o(S_T)$ of S_T at the singular point $o \in S_T$. In particular it is irreducible and hence $\mathrm{Hilb}^2(S_T)$ has a unique 2-dimensional leaf. We have the following list of the correspondence between T and the type of the singularity of $\mathrm{Bl}_o(S_T)$.

T	A_1, A_2	$A_n\,(n \geq 3)$	D_4	$D_n\,(n \geq 5)$	E_6	E_7	E_8
$\mathrm{Bl}_o(S_T)$	smooth	A_{n-2}	$3A_1$	$A_1 + D_{n-2}$	A_5	D_6	E_7

Table 1: Singularity type of $\mathrm{Bl_o(S_T)}$

Next we show how to get a symplectic resolution of $\mathrm{Hilb}^2(S_T)$. Let $\widetilde{S_T} \to S_T$ be the minimal resolution. Then, by the functoriality of Sym^2, we obtain a birational morphism $\mathrm{Sym}^2(\widetilde{S_T}) \to \mathrm{Sym}^2(S_T)$. Also, we can consider the Hilbert-Chow morphism $\mathrm{Hilb}^2(\widetilde{S_T}) \to \mathrm{Sym}^2(\widetilde{S_T})$. Note that $\mathrm{Hilb}^2(\widetilde{S_T})$ is a smooth symplectic variety since $\widetilde{S_T}$ is a smooth symplectic surface. However, note that Hilb^2 is not a functor, and we do not have regular morphism from $\mathrm{Hilb}^2(\widetilde{S_T})$ to $\mathrm{Hilb}^2(S_T)$. But, by applying Mukai flops to $\mathrm{Hilb}^2(\widetilde{S_T})$ certain times, we obtain a new symplectic manifold $\mathcal{H}$ such that $\mathcal{H}$ admits a regular morphism π to $\mathrm{Hilb}^2(S_T)$. This π is a symplectic resolution of $\mathrm{Hilb}^2(S_T)$.

$$
\begin{array}{ccccc}
\mathcal{H} & & \xrightarrow{\quad\quad\pi\quad\quad} & & \mathrm{Hilb}^2(S_T) \\
\text{Mukai flops}\ \updownarrow & & & & \downarrow \\
\mathrm{Hilb}^2(\widetilde{S}_T) & \xrightarrow{\pi_2} & \mathrm{Sym}^2(\widetilde{S}_T) & \xrightarrow{\pi_1} & \mathrm{Sym}^2(S_T)
\end{array}
$$

We can explicitly describe the Mukai flops. To explain this, we introduce the followings.

- $E_1, \ldots, E_k \subset \mathrm{Hilb}^2(\widetilde{S}_T)$: the irreducible exceptional divisors of π_1
- $E_0 \subset \mathrm{Hilb}^2(\widetilde{S}_T)$: the unique irreducible exceptional divisor of π_2
- $e_i \subset H_2(\mathrm{Hilb}^2(\widetilde{S}_T), \mathbb{R})$: the class of a general fiber ($\cong \mathbb{P}^1$) of $\pi_1 \circ \pi_2|_{E_i}$.
- $\mathcal{M}\mathrm{ov} \subset H^2(\mathrm{Hilb}^2(\widetilde{S}_T), \mathbb{R})$: the movable cone i.e., the cone generated by divisors of $\mathrm{Hilb}^2(\widetilde{S}_T)$ which have no fixed components in their linear systems.

$\mathcal{M}\mathrm{ov}$ has a natural wall-and-chamber structure such that the set of chambers bijectively corresponds to the set of symplectic resolutions of $\mathrm{Sym}^2(\widetilde{S}_T)$. The explicit wall-and-chamber structure was given by Bellamy [Bel] and is explained as follows: Since $E_1, \ldots, E_k$ come from the exceptional curves of the minimal resolution $\widetilde{S_T} \to S_T$, they bijectively correspond to the simple roots $\alpha_1, \ldots, \alpha_k$ of the root system of type T. For each positive root α of the root system, it is written as

$$\alpha = \sum_{i=1}^{k} c_i \alpha_i \quad (c_i \in \mathbb{Z}_{>0}).$$

Bellamy showed that every wall in $\mathcal{M}\mathrm{ov}$ is associated to a positive root α and that this wall is defined by the hyperplane which is orthogonal to $e_0 - \sum_{i=1}^{k} c_i e_i$ with respect to the intersection paring of curves and divisors.

Using this description, we deduce the fact that there is unique $\mathcal{H}$ with the desired property. Also, we can compute the central fibers of π. In particular we can show the following.

Fact 2. *Any irreducible component of the fiber $\pi^{-1}(p)$ of a 0-dimensional leaf $p \in \mathrm{Hilb}^2(S_T)$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or the second Hirzebruch surface $\Sigma_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$.* □

For the precise structure of the central fiber, see [Y1]. Now we state the main result of this article.

Theorem 4. *Let $\pi' : \widetilde{X} \to X$ be a symplectic resolution of a 4-dimensional symplectic variety X and take a point $x \in X$. Assume that $\pi'^{-1}(x)$ is isomorphic to a central fiber $\pi^{-1}(p)$ for some $p \in \mathrm{Hilb}^2(S_T)$. Then the germ (X, x) is isomorphic to $(\mathrm{Hilb}^2(S_T), p)$.* □

We expect that, for any symplectic variety admitting a symplectic resolution, central fibers determine their (local) singularities. This is clearly true for a 2-dimensional case since in this case the singularity (i.e. the ADE-type) is recovered from the exceptional curve, which is a tree of projective lines of the corresponding Dynkin type. Moreover, when a central fiber is a rational homogeneous space, the singularity is also uniquely determined using Lemma 5 below. The main result above suggests that considering central fibers is useful even if there are irreducible components which are not rational homogeneous spaces.

Outline of the proof of Theorem 4

In order to prove the theorem, it is sufficient to show that the isomorphism class of the formal neighborhood of $\pi'^{-1}(x)$ in $\widetilde{X}$ is uniquely determined just by the isomorphism class of $\pi'^{-1}(x)$. For this, we take the two steps:

Step 1

Determine the formal neighborhood of each irreducible component V of $\pi'^{-1}(x)$ in $\widetilde{X}$.

We have the following useful lemma.

Lemma 5. *Let Y be a nonsingular symplectic variety and $V \subset Y$ a nonsingular Lagrangian subvariety. Assume that*

$$H^1(V, T_V \otimes \operatorname{Sym}^k T_V) = 0,\ k \geq 1$$

holds where T_V denotes the tangent sheaf of V. Then we have an isomorphism $(Y, V) \cong (T^*V, V)$ of the formal neighborhoods which preserves the symplectic structures. □

By using this lemma, we see that the irreducible components of $\pi'^{-1}(x)$ which are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ have the same neighborhoods as the cotangent bundle. However, for Σ_2, the cohomology groups do not vanish and in fact the neighborhoods are not isomorphic to the cotangent bundles. But we still have the following lemma.

Lemma 6. *Take Y and V as in the previous lemma and assume that V is isomorphic to Σ_2. Let $s \subset V$ be the unique (-2)-curve of V. Then the isomorphism class of the formal neighborhood of (Y, V) is uniquely determined by the isomorphism class of (Y, s).* □

By looking at the structure of the central fiber, we can apply this lemma to every irreducible component which is isomorphic Σ_2 and can achieve *Step* 1. To complete the proof of the theorem, *Step* 1 is not sufficient but we also need the following.

Step 2

Show that the way of gluing the formal neighborhoods $(\widetilde{X}, V)$ is also unique up to isomorphism.

This is the most technical part of the proof of the theorem. To achieve *Step* 2, we need detailed analysis of the structure of the central fibers and also need some computation using an explicit model of the neighborhood of the central fiber. Finally, we can show the uniqueness of the neighborhood of the whole fiber and can prove the theorem.

Remark 3. We have considered the Hilbert scheme of two points so far, but it is also natural to consider the Hibert scheme of n points for higher n. Also in this case, we can apply basically the same arguments as the two-points case, and the analogous statement to Theorem 4 is expected to hold. In higher dimensions, however, the structure of the central fiber gets more complicated and it seems much more difficult to prove it using the same method. Thus, more conceptual method is desirable in order to prove the claim for general n. □

Application of Theorem 4

As an application of the characterization of the singularity of $\mathrm{Hilb}^2(S_T)$ using the central fibers, we show that a symplectic variety called a Slodowy slice has the same singularity as $\mathrm{Hilb}^2(S_T)$.

Let $\mathfrak{g}$ be a complex simple Lie algebra of type $T = A_k, D_k$, or E_k. For a nilpotent element $x \in \mathcal{N} \subset \mathfrak{g}$, we define a Slodowy slice $\mathcal{S}_x$ as follows:
By Jacobson-Morozov Theorem, there are $y, h \in \mathfrak{g}$ such that

$$[h, x] = 2x,\ [h, y] = -2y,\ [x, y] = h.$$

(Such a triple (x, y, h) is called an $\mathfrak{sl}_2$-triple.) Let $\mathrm{ad}(y) : \mathfrak{g} \to \mathfrak{g}$ be the $\mathbb{C}$-linear map defined by $v \to [y, v]$. Then the Slodowy slice is defined as

$$\mathcal{S}_x = x + \mathrm{Ker}\ \mathrm{ad}(y),$$

which is an affine subspace of $\mathfrak{g}$.

Properties of $\mathcal{S}_x$

1. Let O_x be the nilpotent orbit containing x. Then $\mathcal{S}_x \cap O_x = \{x\}$.
2. $\mathcal{S}_x$ and O_x meet transversally i.e., $T_xO_x \oplus T_x\mathcal{S}_x = \mathfrak{g}$.
3. $\mathcal{S}_x \cap \mathcal{N}$ is a symplectic variety with respect to the restriction of the Kostant-Killirov form on $\mathcal{N}$.
4. The restriction f' of the Springer resolution $f : T^*(G/B) \to \mathcal{N}$ gives a (unique) symplectic resolution of $\mathcal{S}_x \cap \mathcal{N}$:

$$\begin{array}{ccc} f : T^*(G/B) & \longrightarrow & \mathcal{N} \\ \cup & & \cup \\ f' : f^{-1}(\mathcal{S}_x \cap \mathcal{N}) & \longrightarrow & \mathcal{S}_x \cap \mathcal{N} \end{array}$$

Recall that the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$ is the closure of the nilpotent orbit O_{reg}. It is known that $\mathcal{N} \setminus O_{\mathrm{reg}}$ is also the closure of a nilpotent orbit O_{subreg}, which is called the *subregular orbit*. It is well-known that $\mathcal{S}_x \cap \mathcal{N}$ has an ADE singularity of type T when x is taken from the subregular orbit (see [Sl]).

In this article we consider a Slodowy slice for a next smaller nilpotent orbit. However, the third biggest orbits, which we call the *subsubregular orbits*, are not unique in general. More precisely, we have just one subsubregular orbit for $T = A_k$ and E_k, but there are three (resp. two) subsubregular orbits for $T = D_4$ (resp. D_n, $n \geq 5$). So, the situation is similar to the singular locus of $\mathrm{Hilb}^2(S_T)$ (see Table 1).

From now on, we choose x from a subsubregular orbit. In this case the stratification of $\mathcal{S}_x \cap \mathcal{N}$ is given as follows.

$$\mathcal{S}_x \cap \mathcal{N} = (\mathcal{S}_x \cap O_{\mathrm{reg}}) \sqcup (\mathcal{S}_x \cap O_{\mathrm{subreg}}) \sqcup \{x\}$$

This is the decomposition into symplectic leaves of dimension 4,2, and 0 respectively. Since a symplectic resolution of $\mathcal{S}_x \cap \mathcal{N}$ is obtained just as the restriction of the Springer resolution, the central fiber is nothing but a Springer fiber of the subsubregular orbit. This was studied by Lorist [L]. Using his result, we can show the following.

Theorem 7. *The Springer fiber $f^{-1}(x)$ is isomorphic to a central fiber for* $\mathrm{Hilb}^2(S_T)$. □

Applying the main result to $\mathcal{S}_x \cap \mathcal{N}$, we obtain the following.

Corollary 8. *$(\mathcal{S}_x \cap \mathcal{N}, x)$ is isomorphic to $(\mathrm{Hilb}^2(S_T), p)$ for a 0-dimensional leaf $p \in \mathrm{Hilb}^2(S_T)$.* □

Remark 4. The similar result was obtained by Manolescu for type A_k [M] and by Jackson for type D_k [J]. They gave open embeddings of transversal slices for certain nilpotent orbits into $\mathrm{Hilb}^n(S_T)$ for every n. □

Remark 5. As another application of the main result, we can show that certain compact singular symplectic varieties have the same singularities as $\mathrm{Hilb}^2(S_T)$. See [Y2] for details. □

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