Examples of Calabi–Yau 3-folds from projective joins of del Pezzo manifolds

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1. Introduction

The derived equivalence between Grassmannian and Pfaffian Calabi–Yau 3-folds is an interesting phenomenon discovered in the study of mirror symmetry of Calabi–Yau 3-folds. These Calabi–Yau 3-folds share the same mirror family due to Rødland and the derived equivalence is indicated in the two different boundary points of the family. We construct similar examples of Calabi–Yau 3-folds but with Picard number greater than one as an application of homological projective dualities by Kuznetsov–Perry [KP].

2. Linear dualities

Linear dualities of projective bundles are special cases of the homological projective dualities, but they often result in birational Calabi–Yau 3-folds, which are known to be Fourier–Mukai partners to each other. The following example is a special case of [Kuz, Section 8].

Example

Let $\mathcal{E}$ be a vector bundle on $G(2,4)$ such that $\mathcal{E}^*$ is globally generated and $c_1(\mathcal{E}) = -4H$. Let $\mathcal{E}^\perp$ be an orthogonal vector bundle of $\mathcal{E}$ defined by $0 \to \mathcal{E}^\perp \to H^0(G(2,4), \mathcal{E}^*) \otimes \mathcal{O}_{G(2,4)} \to \mathcal{E}^* \to 0$. We take a general linear subspace $L \subset H^0(G(2,4), \mathcal{E}^*)$ of codimension $r = \operatorname{rank} \mathcal{E}$. Let $L^\perp \subset H^0(G(2,4), (\mathcal{E}^\perp)^*)$ be the orthogonal linear subspace of L. Then the linear sections of projective bundles

$$X = \mathbb{P}_{G(2,4)}(\mathcal{E}) \cap \mathbb{P}(L^\perp), \quad Y = \mathbb{P}_{G(2,4)}(\mathcal{E}^\perp) \cap \mathbb{P}(L)$$

are Calabi–Yau 3-folds.
These X and Y are derived equivalent by the linear duality due to Kuznetsov. Also, it turns out X and Y are birational, hence they are derived equivalent due to Bridgeland's theorem.

$$\begin{tikzcd}
\mathbb{P}_{G(2,4)}(\mathcal{E}^\perp) \arrow[dr] & & \mathbb{P}_{G(2,4)}(\mathcal{E}) \arrow[dl] \\
Y \arrow[u, phantom, "\cup"] \arrow[dr] & G(2,4) & X \arrow[u, phantom, "\cup"] \arrow[dl] \\
& \bar{X} \arrow[u, phantom, "\cup"] &
\end{tikzcd}$$

Then, in these cases, the derived equivalences are also followed from the Bridgeland's theorem. Here $\bar{X}$ is an anti-canonical hypersurface of $G(2,4)$.

3. Categorical joins

In a recent paper [KP], Kuznetsov and Perry have formulated *categorical join* and found many new examples of homological projective dualities. By using their results, we can find new pairs of Calabi–Yau 3-folds whose derived categories are equivalent. We recall a definition of projective joins of projective varieties.

Def

For projective varieties $M_i \subset \mathbb{P}(V_i)$ $(i = 1, 2)$, a projective join of M_1 and M_2 is defined by

$$\mathrm{Join}(M_1, M_2) = \overline{\bigcup_{x_1 \in M_1, x_2 \in M_2} \langle x_1, x_2 \rangle} \subset \mathbb{P}(V_1 \oplus V_2)$$

where $\langle x_1, x_2 \rangle$ is the linear subspace spanned by $[x_1, 0]$ and $[0, x_2]$ in $\mathbb{P}(V_1 \oplus V_2)$.

When we take M_1, M_2 to be del Pezzo manifolds, we can construct Calabi–Yau 3-folds from linear sections of $\mathrm{Join}(M_1, M_2)$ (c.f. [G]). Let us take $M_1 = G(2,5)$ and M_2 to be one of the followings:

$$\text{(i) } \mathbb{P}^2 \times \mathbb{P}^2 \qquad \text{(ii) } \mathrm{Bl}_{\mathrm{pt}}\mathbb{P}^3 \qquad \text{(iii) } \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

For each choices of M_2, we consider the following projective bundles $\mathbb{P}_{M_1, M_2}$:

$$\begin{aligned}
&\text{(i) } \mathbb{P}_{G(2,5)\times\mathbb{P}^2}(\pi_1^*\mathcal{O}_{G(2,5)}(-1) \oplus \pi_2^*\mathcal{K}_1^{\oplus 3}) \\
&\text{(ii) } \mathbb{P}_{G(2,5)\times\mathbb{P}^2}(\pi_1^*\mathcal{O}_{G(2,5)}(-1) \oplus \pi_2^*\mathcal{K}_1 \oplus \pi_2^*\mathcal{K}_2) \\
&\text{(iii) } \mathbb{P}_{G(2,5)\times\mathbb{P}^1\times\mathbb{P}^1}(\pi_1^*\mathcal{O}_{G(2,5)}(-1) \oplus \pi_2^*\mathcal{K}_{1,1})
\end{aligned}$$

where π_1 is the projection to $G(2,5)$ and π_2 is the projection to the remaining factors. Here $\mathcal{K}_i$ $(i = 1, 2), \mathcal{K}_{1,1}$ are defined as follows:

$$0 \to \mathcal{K}_i \to H^0(\mathbb{P}^2, \mathcal{O}(i)) \otimes \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}(i) \to 0 \quad (i = 1, 2),$$
$$0 \to \mathcal{K}_i \to H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1)) \otimes \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \to \mathcal{O}(1,1) \to 0.$$

Main Result

(1) Take a general linear subspace $L \subset H^0(\mathrm{Join}(M_1, M_2), \mathcal{O}(1))$ with an appropriate codimension. Consider the following linear sections

$$X = \mathrm{Join}(M_1, M_2) \cap \mathbb{P}(L^\perp), \; Y = \mathbb{P}_{M_1, M_2} \cap \mathbb{P}(L),$$

then X and Y are both Calabi–Yau 3-folds.
(2) These Calabi–Yau 3-folds X and Y are not birational, but derived equivalent. For the choices of M_1, M_2, the Hodge numbers are given as follows;
$(h^{i,j} = h_X^{i,j} = h_Y^{i,j})$

M_1	M_2	$h^{1,1}$	$h^{2,1}$
$G(2,5)$	$\mathbb{P}^2 \times \mathbb{P}^2$	2	47
$G(2,5)$	$\mathrm{Bl}_{\mathrm{pt}}\mathbb{P}^3$	2	47
$G(2,5)$	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	3	43

Example

By studying birational geometry of these Calabi–Yau 3-folds, we can see that these are not birational. For example, if $M_1 = G(2,5)$, $M_2 = \mathbb{P}^2 \times \mathbb{P}^2$, we have diagrams

$$\begin{tikzcd}
& X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathbb{P}^2 & & \mathbb{P}^2
\end{tikzcd}, \quad
\begin{tikzcd}
& Y \arrow[dl, "p_1"'] \arrow[dr, "\varphi_1"] & & Y' \arrow[dl, "\varphi_2"'] \arrow[dr, "p_2"] & \\
\mathbb{P}^2 & & \bar{Y} & & \mathbb{P}^2
\end{tikzcd}$$

where π_i (resp. p_i) $(i = 1, 2)$ are elliptic fibrations on each Calabi–Yau 3-folds. The morphisms φ_i $(i = 1, 2)$ are small contractions which contract 30 $\mathbb{P}^1$'s to points. The image $\bar{Y}$ is a complete intersection of type $(1,1,3)$ in $G(2,5)$ with 30 ordinary double points.

Remark

Main Result is based on the following well-known fact:
Fact: Let $E_i \subset \mathbb{P}(V_i)$ $(i = 1, 2)$ be projectively normal elliptic curves. Then the projective join $\mathrm{Join}(E_1, E_2) \subset \mathbb{P}(V_1 \oplus V_2)$ is a (singular) Calabi–Yau 3-fold.
Suppose E_1, E_2 are given by suitable linear sections of del Pezzo manifolds M_1 and M_2, respectively. Then the corresponding linear sections of $\mathrm{Join}(M_1, M_2)$ can be regarded as a smoothing of the singular Calabi–Yau 3-fold $\mathrm{Join}(E_1, E_2)$. As pointed out by [G], we can construct a lot of Calabi–Yau 3-folds in this way.

Example

There are some other possible choices of M_2 (with $M_1 = G(2,5)$). We can consider $\mathrm{Join}(M_1, M_2)$ with $M_1 = G(2,5)$ and $M_2 = \mathbb{P}^2$. The projective join is naturally resolved by the following projective bundle

$$\mathbb{P}_{G(2,5)\times\mathbb{P}^2}(\pi_1^*\mathcal{O}_{G(2,5)}(-1) \oplus \pi_2^*\mathcal{O}_{\mathbb{P}^2}(-3)).$$

Correspondingly to this, the dual projective bundle following to [KP] becomes

$$\mathbb{P}_{G(2,5)\times\mathbb{P}^2}(\pi_1^*\mathcal{O}_{G(2,5)}(-1) \oplus \pi_2^*\mathcal{K}_3)$$

where π_i and $\mathcal{K}_3$ are as before. We define X and Y by mutually orthogonal linear sections of these projective bundles. In this case, we can see that the Picard numbers of X and Y are greater than or equal to 6. I have not yet been able to determine whether X and Y are birational or not.

4. Mirror Calabi–Yau 3-folds: Fiber products of elliptic surfaces

S. Galkin pointed out some relations between projective joins and Hadamard products in [G]. Inspired by his result, we construct candidates of mirror families of Calabi–Yau 3-folds as fiber products of elliptic surfaces (c.f. Schoen's work).

Result

We construct elliptic surfaces S_1 and S_2:
(1) S_1 by a suitable smooth orbifold of Shioda modular surface of level 5.
(2) S_2 by closely related to Batyrev–Borisov toric mirror construction of $(1,1) \cap (1,1) \cap (1,1) \subset \mathbb{P}^2 \times \mathbb{P}^2$.
Then both S_1, S_2 are rational elliptic surfaces with sections. The fiber product $X^\vee = S_1 \times_{\mathbb{P}^1} S_2$ gives a family of Calabi–Yau 3-folds with Euler number $e(X^\vee) = 90$.

Conjecture

We conjecture that the above family of Calabi–Yau 3-folds is a mirror family of the linear section X of $\mathrm{Join}(G(2,5), \mathbb{P}^2 \times \mathbb{P}^2)$.

Indeed, this family naturally parametrized by $\mathbb{P}^2$ and have three maximally unipotent monodromy points. The following numbers are calculated from each maximally unipotent monodromy points by using mirror symmetry.

$d_1 \setminus d_2$	0	1	2	3
0	0	120	105	105
1	120	2085	15690	83400
2	105	15690	569475	9690270
3	105	83400	9690270	418812780
4	120	362850	107459880	10086474180
5	90	1365060	901887570	164859436335
6	120	4621020	6204484125	2041590595410
7	105	14399490	36701125005	20496053409240
8	105	41932200	192593575110	174405931797135
9	120	115485075	916315955820	1297448843314125
10	90	303166710	4015843886955	8630138044756890

$e_1 \setminus e_2$	0	1	2	3
0	0	30	0	0
1	105	330	105	0
2	120	2865	6585	2865
3	120	17400	151260	283755
4	105	87150	2141265	11044335
5	90	368670	22279830	256967580
6	105	1377840	186120810	4267143150
7	120	4644030	1311908070	55405726800
8	120	14441100	8065898475	594374999280
9	105	42003450	44272540830	5463083502630
10	90	115593255	220759120890	44140588111590

Table: BPS numbers of linear section Calabi–Yau 3-folds

We can identify these numbers with the counting invariants of X and those of its Fourier–Mukai partner Y in $\mathbb{P}_{G(2,5)\times\mathbb{P}^2}(\pi_1^*\mathcal{O}_{G(2,5)}(-1) \oplus \pi_2^*\mathcal{K}_1^{\oplus 3})$. Indeed, the number 30 in the right can be identified with the number of flopping curves.

References

[G] S. Galkin, *Joins and Hadamard products*, 2015, Talk presented at Categorical and analytic invariants in Algebraic geometry 1, Moscos, Steklov Mathematical Institute, September 17.

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