

ON THE COHOMOLOGY OF THE MODULI SPACE OF PARABOLIC CONNECTIONS

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Introduction

In this poster, we study the moduli space of logarithmic connections of rank 2 on $\mathbb{P}^1 \setminus \{t_1, \dots, t_n\}$ with fixed spectral data. We compute the cohomology of such moduli space, and this computation will be used to extend the results of geometric Langlands correspondence of [1] to the case where the parabolic connections have five simple poles on \mathbb{P}^1 .

Preliminaries

We introduce \mathfrak{sl}_2 -connections.

Fix complex numbers $\nu_1, \dots, \nu_n \in \mathbb{C}$. Suppose that $\nu_1 \cdots \nu_n \neq 0$ and

$$\sum_{i=1}^n \epsilon_i \nu_i \notin \mathbb{Z} \quad (1)$$

for any $(\epsilon_i), \epsilon_i \in \{1, -1\}$.

Definition 2.1 A ν - \mathfrak{sl}_2 -parabolic connection on \mathbb{P}^1 is a triplet (E, ∇, φ) such that

1. E is a rank 2 vector bundle on \mathbb{P}^1 ,
2. $\nabla: E \rightarrow E \otimes \Omega_{\mathbb{P}^1}^1(D)$ is a connection, where $D := t_1 + \cdots + t_n$,
3. $\varphi: \mathcal{N}^2 E \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$ is a horizontal isomorphism,
4. the residue $\text{res}_i(\nabla)$ of the connection ∇ at t_i has eigenvalues ν_i^{\pm} , $1 \leq i \leq n$.

Here, we put

$$\nu_i^{\pm} := \pm \nu_i \quad (i = 1, \dots, n-1), \quad \nu_n^+ := \nu_n, \quad \nu_n^- := 1 - \nu_n.$$

Denote by \mathcal{M} the moduli stack of ν - \mathfrak{sl}_2 -parabolic connections on \mathbb{P}^1 , and by M its coarse moduli space.

For such ν , the parabolic direction $l_i := \ker(\text{res}_i(\nabla) - \nu_i) \subset E|_{t_i}$ is uniquely determined. So, we can get the forgetful map

$$\text{Bun} : M \rightarrow \mathcal{P} := (E, \nabla, \varphi) \mapsto (E, \{l_i\})$$

where \mathcal{P} is the coarse moduli space of indecomposable quasi-parabolic bundles $(E, \{l_i\})$ on \mathbb{P}^1 .

Now, we consider the following stratification of M . By the irreducibility of $(E, \nabla, \varphi) \in M$, we have the following proposition.

Proposition 2.2 For $(E, \nabla, \varphi) \in M$, we have

$$E \simeq \mathcal{O}(k) \oplus \mathcal{O}(-k-1) \quad \text{where } 0 \leq k \leq \left\lfloor \frac{n-3}{2} \right\rfloor.$$

Denote by M^k the subvariety of M where $E \simeq \mathcal{O}(k) \oplus \mathcal{O}(-k-1)$. Then

$$M = M^0 \cup \cdots \cup M^{\lfloor (n-3)/2 \rfloor}.$$

Note that the stratum M^0 is a Zariski open dense of M .

Geometric description of M^0

Suppose $n = 5$. For computation of the cohomology of M , we introduce some blowing-up of the Hirzebruch surface \mathbb{F}_3 . Put $L := \mathcal{O}_{\mathbb{F}_3}(D)$. Let \mathbb{L} be the total space of the line bundle L . Note that $L = \mathbb{F}_3 \cup s_{\infty}$, where s_{∞} is the infinity section ($s_{\infty}^2 = -3$). Let $\pi: \mathbb{L} \rightarrow \mathbb{P}^1$ be the projection and let $\tilde{F}_i: \tilde{F}_i \rightarrow \mathbb{P}^1$ ($i = 1, 2$) be the residue map. Put $\nu_i^+ := \nu_i$, $\nu_i^- := -\nu_i$ for $i = 1, \dots, 4$, $\nu_5^+ := \nu_5$, $\nu_5^- := 1 - \nu_5$, and $\tilde{\nu}_i^{\pm} := \nu_i^{\pm}(\nu_i^{\pm})$. Set

$$\mathcal{K}_{5,0}^{\pm} := (\mathbb{B}_{\nu} \mathbb{L}) \setminus (\tilde{F}_1 \cup \cdots \cup \tilde{F}_5)$$

where $\mathbb{B}_{\nu} \mathbb{L}$ is the blowing-up of \mathbb{L} at \tilde{e}_i^{\pm} for $i = 1, \dots, n$, and \tilde{F}_i are the proper pre-images of the fiber F_i $i = 1, \dots, n$.

Lemma 3.1

$$H^i(\mathcal{K}_{5,0}^{\pm}, \mathcal{O}_{\mathcal{K}_{5,0}^{\pm}}) = \begin{cases} \mathbb{C}, & i = 0, \\ H_0^{\pm}(A) \neq 0, & i = 1, \\ 0, & i \geq 2, \end{cases}$$

where (A, \mathfrak{m}) is a local ring such that $\dim(A_{\mathfrak{m}}) = 2$.

We denote by \mathcal{K}_5 the image of $\mathcal{K}_{5,0}^{\pm}$ under the projection $\mathcal{K}_{5,0}^{\pm} \rightarrow \mathbb{F}_3 \setminus s_{\infty}$. We can define the following map

$$M^0 \rightarrow \text{Sym}^2(\mathcal{K}_5) \\ (E, \nabla, \varphi) \mapsto \{(q_1, p_1), (q_2, p_2)\}, \quad (2)$$

We consider the composite of the Hilbert-Chow morphism and the blowing-up

$$\text{Hilb}^2(\mathcal{K}_5) \rightarrow \text{Sym}^2(\mathcal{K}_5) \rightarrow \text{Sym}^2(\mathcal{K}_5).$$

We have the next proposition.

Proposition 3.2 ([3] Theorem 5.2) We can extend the map (2) to

$$M^0 \rightarrow \text{Hilb}^2(\mathcal{K}_5^{\pm})$$

and this map is injective.

Cohomology of $\mathcal{K}_{5,0}^{\pm}$

We denote by $Z \subset \text{Sym}^2(\mathcal{K}_5^{\pm})$ the proper pre-image of $\{(q_1, p_1), (q_1, -p_1)\} \subset \text{Sym}^2(\mathcal{K}_5^{\pm})$ under the blowing-up $\text{Sym}^2(\mathcal{K}_5^{\pm}) \rightarrow \text{Sym}^2(\mathcal{K}_5)$, and by $\tilde{Z} \subset \text{Hilb}^2(\mathcal{K}_5^{\pm})$ the proper pre-image of Z under the Hilbert-Chow morphism $\text{Hilb}^2(\mathcal{K}_5^{\pm}) \rightarrow \text{Sym}^2(\mathcal{K}_5^{\pm})$. Denote by

$$\tilde{\text{Hilb}}^2(\mathcal{K}_5^{\pm}) \rightarrow \text{Hilb}^2(\mathcal{K}_5^{\pm}) \quad (3)$$

the blowing-up along \tilde{Z} and by \tilde{Z} the strict transform of \tilde{Z} . We also denote by $(\mathcal{K}_5^{\pm} \times \mathcal{K}_5^{\pm})^{\sim}$ the blowing-up of $\mathcal{K}_5^{\pm} \times \mathcal{K}_5^{\pm}$ along the ideal $(q_1 - q_2, p_1 - p_2)$, and by $(\mathcal{K}_5^{\pm} \times \mathcal{K}_5^{\pm})^{\sim}$ the blowing-up of $(\mathcal{K}_5^{\pm} \times \mathcal{K}_5^{\pm})^{\sim}$ along the ideal $(q_1 - q_2, p_1 + p_2)$. Then $\text{Hilb}^2(\mathcal{K}_5^{\pm}) = (\mathcal{K}_5^{\pm} \times \mathcal{K}_5^{\pm})^{\sim} / \mathfrak{S}_2$, and $\tilde{\text{Hilb}}^2(\mathcal{K}_5^{\pm}) = (\mathcal{K}_5^{\pm} \times \mathcal{K}_5^{\pm})^{\sim} / \mathfrak{S}_2$. Now, using above description, we define another important blowing-up of Hirzebruch surface \mathbb{F}_3 . Fix $q_i \in \mathbb{P}^1$ (t_1, \dots, t_5) and define the fiber F_i over q_i . We denote by $(\mathbb{F}_3)^{\sim}$ the blowing-up of \mathbb{F}_3 at two points $\{(q_1, p_1), (q_1, -p_1)\}$ (when $p_1 = p_2 = 0$, blow up twice at $(q_1, 0)$). Set

$$\mathcal{K}_{5,q_1}^{\pm} := (\mathbb{F}_3)^{\sim} \setminus (s_{\infty} \cup \tilde{F}_1 \cup \cdots \cup \tilde{F}_5)$$

where \tilde{F}_i is the strict transform of F_i . We denote by \mathcal{K}_{5,q_1} the image of $\mathcal{K}_{5,q_1}^{\pm}$ under the projection $\mathcal{K}_{5,q_1}^{\pm} \rightarrow \mathbb{F}_3 \setminus s_{\infty}$.

Define

$$D_{q_1} := 2s_{\infty} + \tilde{F}_1 + \cdots + \tilde{F}_5 + \tilde{F}_6$$

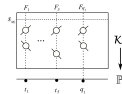
Then, we have

$$(s_{\infty}, \tilde{F}_6) = 1, (\tilde{F}_i, \tilde{F}_i) = 0, (s_{\infty}, s_{\infty}) = -3, (\tilde{F}_i, \tilde{F}_i) = -2$$

This implies

$$(D_{q_1}, D_{q_1}) = (D_{q_1}, s_{\infty}) = (D_{q_1}, \tilde{F}_i) = 0.$$

This is the same condition of Arinkin Lysenko [2], section 5.2.



Proposition 4.1

$$H^i(\mathcal{K}_{5,q_1}^{\pm}, \mathcal{O}_{\mathcal{K}_{5,q_1}^{\pm}}) = \begin{cases} \mathbb{C}, & i = 0, \\ 0, & i > 0. \end{cases}$$

Cohomology of M

Set $\tilde{M}_Z := M^0 \cup \tilde{Z}$. By Proposition 3.2, we have injective maps $\iota: M^0 \hookrightarrow \text{Hilb}^2(\mathcal{K}_5^{\pm})$ and $i: \tilde{M}_Z \hookrightarrow \tilde{\text{Hilb}}^2(\mathcal{K}_5^{\pm})$. We define the blowing-up parameter λ as $p_1 + p_2 = \lambda - (q_1 - q_2)$. Set $T := \tilde{\text{Hilb}}^2(\mathcal{K}_5^{\pm}) \setminus \tilde{M}_Z$. For a vector bundle \mathcal{F} on $\tilde{\text{Hilb}}^2(\mathcal{K}_5^{\pm})$,

$$H^i(\tilde{M}_Z, \mathcal{F}|_{\tilde{M}_Z}) = H^i(\tilde{\text{Hilb}}^2(\mathcal{K}_5^{\pm}), i_* \mathcal{F}) \\ = \varinjlim H^i(\tilde{\text{Hilb}}^2(\mathcal{K}_5^{\pm}), \mathcal{F}(kT)).$$

To compute $H^i(\tilde{\text{Hilb}}^2(\mathcal{K}_5^{\pm}), \mathcal{F}(kT))$, consider $H^i(\mathcal{K}_5^{\pm} \times \mathcal{K}_5^{\pm})^{\sim}, \mathcal{F}(kT^{\sim})$, where T^{\sim} is defined by $(\lambda = \infty)$. We can define a map

$$f: (\mathcal{K}_5^{\pm} \times \mathcal{K}_5^{\pm})^{\sim} \setminus T^{\sim} \rightarrow \mathcal{K}_{5,q_1}^{\pm} \\ (q_1, p_1, q_2, p_2) \mapsto (q_1, p_1),$$

and the fiber $f^{-1}(\{(q_1, p_1)\}) \simeq \mathcal{K}_{5,q_1}^{\pm}$. By Leray's spectral sequence, we have

$$H^i(\mathcal{K}_5^{\pm} \times \mathcal{K}_5^{\pm})^{\sim} \setminus T^{\sim}, \mathcal{F} \simeq \bigoplus_{p=0}^i H^p(\mathcal{K}_{5,q_1}^{\pm}, R^p f_* \mathcal{F}).$$

Using the Base change theorem, we have $(R^p f_* \mathcal{F})_{(q_1, p_1)} \simeq H^p(\mathcal{K}_{5,q_1}^{\pm}, \mathcal{F}_{(q_1, p_1)})$.

By using Lemma 3.1 and Proposition 4.1, we have

$$H^i(\mathcal{K}_5^{\pm} \times \mathcal{K}_5^{\pm})^{\sim} \setminus T^{\sim}, \mathcal{O} = \begin{cases} \mathbb{C}, & i = 0, \\ H_0^{\pm}(A) \neq 0, & i = 1, \\ 0, & i > 1. \end{cases}$$

Moreover, the action of \mathfrak{S}_2 on $H^i(\mathcal{K}_5^{\pm} \times \mathcal{K}_5^{\pm})^{\sim} \setminus T^{\sim}, \mathcal{O}$ is nontrivial. Therefore,

$$H^i(\tilde{M}_Z, \mathcal{O}_{\tilde{M}_Z}) = \begin{cases} \mathbb{C}, & i = 0, \\ 0, & i > 0. \end{cases}$$

Since $\text{codim}_{\text{Hilb}^2(\mathcal{K}_5^{\pm})}(\tilde{Z}) = 2$, and $M^1 = M \setminus M^0 \simeq A^2$, we have

Theorem 5.1

$$H^i(M, \mathcal{O}_M) = \begin{cases} \mathbb{C}, & i = 0, \\ 0, & i > 0. \end{cases}$$

References

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- [3] A. Kanouo, M.-H. Saito. Explicit description of jumping phenomena on moduli spaces of parabolic connections and Hilbert schemes of points on surfaces, accepted in Kyoto Journal of Mathematics. (arXiv:math/1611.00971)