A p-adic analytic approach to the absolute Grothendieck conjecture

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Introduction

Let K be a field of characteristic 0, $\overline{K}$ an algebraic closure of K, G_K the absolute Galois group of K, U a hyperbolic curve over K and $\pi_1(U)$ the étale fundamental group of U. We have the following natural exact sequence:

$$1 \to \pi_1(U_{\overline{K}}) \to \pi_1(U) \overset{\mathrm{pr}}{\to} G_K \to 1.$$

The relative and absolute Grothendieck conjectures ask:

$$\text{relative}: \quad \pi_1(U) \twoheadrightarrow G_K \underset{\text{recoverable}}{\overset{?}{\rightsquigarrow}} U,$$

$$\text{absolute}: \quad \pi_1(U) \underset{\text{recoverable}}{\overset{?}{\rightsquigarrow}} U.$$

When K is a p-adic field (i.e. a finite extension of $\mathbb{Q}_p$), the absolute version of Grothendieck conjecture (the absolute p-adic Grothendieck conjecture) is unsolved. On the other hand, the absolute p-adic Grothendieck conjecture is reduced to the problem of rational points of curves as follows:

$\boxed{\text{Group-theoretic determination of whether or not } U \text{ and its coverings admit rational points}}$

$\Downarrow$ [Tamagawa]

$\boxed{\text{Group-theoretic characterization of decomposition groups}}$

$\Downarrow$ [Mochizuki2]

$\boxed{\text{The absolute } p\text{-adic Grothendieck conjecture}}$

To consider this problem, we shall introduce the "i-invariant" of **compact** analytic manifold over a p-adic field K.

Theorem-Definition [Serre]

Let q be the cardinality of the residue field of K and Y a nonempty compact analytic manifold over K. Then Y is the disjoint union of a finite number of balls and the number of (closed) balls is well determined mod $(q-1)$. We call the number of balls $i_K(Y) \in \mathbb{Z}/(q-1)\mathbb{Z}$ the i-invariant of Y over K. Moreover, we set $i_K(\emptyset) \equiv 0 \mod (q-1)$.

Let X be a **proper** hyperbolic curve over K and $X(K)$ the set of K-rational points of X. In terms of the i-invariants, the absolute p-adic Grothendieck conjecture (for hyperbolic curves **not necessarily proper**) is reduced to the following two problems:

Question

(A) Data of i-invariants of sets of rational points of X and its étale coverings $\rightsquigarrow$ decomposition groups?

(B) Group-theoreticity of the i-invariant of $X(K)$?

Theorem A below gives a complete affirmative answer to (A) and Theorem B below gives a partial affirmative answer to (B).

Theorem A [M]

Let K be a p-adic field, X a proper hyperbolic curve over K and q the cardinality of the residue field of K. Assume that $q \neq 2$ and let $m > 1$ be a divisor of $q-1$. Then the following conditions are equivalent:

(i) $X(K) \neq \emptyset$.

(ii) $\exists X'$: a finite étale covering of X such that $X'(K) \neq \emptyset$.

(iii) $\exists X'$: a finite étale covering of X such that $i_K(X'(K)) \not\equiv 0 \mod (q-1)$.

(iv) $\exists X'$: a finite étale covering of X such that $i_K(X'(K)) \not\equiv 0 \mod m$.

(v) $\exists X'$: a finite étale covering of X such that $i_K(X'(K)) \equiv$ (a power of p) $\mathrm{mod}\ (q-1)$.

Theorem B [M]

For $i = 1, 2$, let p_i be an odd prime number, K_i a p_i-adic field and X_i a proper hyperbolic curve over K_i which has log smooth reduction (i.e. has stable reduction after tame base extension). Suppose that we are given an isomorphism of profinite groups $\alpha : \pi_1(X_1) \overset{\sim}{\to} \pi_1(X_2)$. Then we have

$$i_{K_1}(X_1(K_1)) \equiv i_{K_2}(X_2(K_2)) \mod 2.$$

In fact, by [Mochizuki1], if we are given an isomorphism $\alpha : \pi_1(X_1) \overset{\sim}{\to} \pi_1(X_2)$, we have $p_1 = p_2$ (more precisely, we have $q_1 = q_2$ where q_i is the cardinality of the residue field of K_i). Moreover, X_1 has log smooth reduction if and only if X_2 has log smooth reduction.

Remark

If we prove Theorem B without assuming that X_i has log smooth reduction, we can prove the absolute p-adic Grothendieck conjecture for p odd.

References

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