TITLE:
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CITATION:

ISSUE DATE:
2018

URL:
http://hdl.handle.net/2433/236416

RIGHT:
A \(p\)-adic analytic approach to the absolute Grothendieck conjecture

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Introduction

Let \(K\) be a field of characteristic 0, \(\overline{K}\) an algebraic closure of \(K\), \(G_K\) the absolute Galois group of \(K\), \(U\) a hyperbolic curve over \(K\) and \(\pi_1(U)\) the étale fundamental group of \(U\). We have the following natural exact sequence:

\[1 \rightarrow \pi_1(U_p) \rightarrow \pi_1(U) \rightarrow G_K \rightarrow 1.\]

The relative and absolute Grothendieck conjectures ask:

- relative: \(\pi_1(U) \rightarrow G_K \rightarrow \text{recoverable}\)
- absolute: \(\pi_1(U) \rightarrow U\) \(\text{recoverable}\)

When \(K\) is a \(p\)-adic field (i.e. a finite extension of \(\mathbb{Q}_p\)), the absolute version of Grothendieck conjecture (the absolute \(p\)-adic Grothendieck conjecture) is unsolved.

On the other hand, the absolute \(p\)-adic Grothendieck conjecture is reduced to the problem of rational points of curves as follows:

- Group-theoretic determination of whether or not \(U\) and its coverings admit rational points
- Group-theoretic characterization of decomposition groups
- The absolute \(p\)-adic Grothendieck conjecture

To consider this problem, we shall introduce the “\(i\)-invariant” of \(\text{compact} \) analytic manifold over a \(p\)-adic field \(K\).

Theorem-Definition [Serre]

Let \(q\) be the cardinality of the residue field of \(K\) and \(Y\) a nonempty compact analytic manifold over \(K\). Then \(Y\) is the disjoint union of a finite number of balls and the number of (closed) balls is well determined \(\text{mod } (q - 1)\). We call the number of balls \(i_K(Y) \in \mathbb{Z}/(q - 1)\mathbb{Z}\) the \(i\)-invariant of \(Y\) over \(K\). Moreover, we set \(i_K(\emptyset) \equiv 0 \mod (q - 1)\).

Let \(X\) be a \(\text{proper} \) hyperbolic curve over \(K\) and \(X(K)\) the set of \(K\)-rational points of \(X\). In terms of the \(i\)-invariants, the absolute \(p\)-adic Grothendieck conjecture (for hyperbolic curves \(\text{not necessarily proper}\)) is reduced to the following two problems:

\[\text{(A) Data of } i\text{-invariants of sets of rational points of } X \text{ and its étale coverings } \\ \text{reduce to decomposition groups?}\]

\[\text{(B) Group-theoreticity of the } i\text{-invariant of } X(K)?\]

Question

Theorem A below gives a complete affirmative answer to (A) and Theorem B below gives a partial affirmative answer to (B).

Theorem A [M]

Let \(K\) be a \(p\)-adic field, \(X\) a proper hyperbolic curve over \(K\) and \(q\) the cardinality of the residue field of \(K\). Assume that \(q \neq 2\) and let \(m > 1\) be a divisor of \(q - 1\). Then the following conditions are equivalent:

\[(i) \quad X(K) \neq 0.\]

\[(ii) \quad \exists X': \text{a finite étale covering of } X \text{ such that } X'(K) \neq 0.\]

\[(iii) \quad \exists X': \text{a finite étale covering of } X \text{ such that } i_K(X'(K)) \neq 0 \mod (q - 1).\]

\[(iv) \quad \exists X': \text{a finite étale covering of } X \text{ such that } i_K(X'(K)) \neq 0 \mod m.\]

\[(v) \quad \exists X': \text{a finite étale covering of } X \text{ such that } i_K(X'(K)) \equiv (a \text{ power of } p) \mod (q - 1).\]

Theorem B [M]

For \(i = 1, 2\), let \(p_i\) be an odd prime number, \(K_i\) a \(p_i\)-adic field and \(X_i\) a proper hyperbolic curve over \(K_i\) which has log smooth reduction (i.e. has stable reduction after tame base extension). Suppose that we are given an isomorphism of profinite groups \(\alpha : \pi_1(X_1) \xrightarrow{\sim} \pi_1(X_2)\). Then we have

\[i_{K_1}(X_1(K_1)) \equiv i_{K_2}(X_2(K_2)) \mod 2.\]

In fact, by [Mochizuki1], if we are given an isomorphism \(\alpha : \pi_1(X_1) \xrightarrow{\sim} \pi_1(X_2)\), we have \(p_1 = p_2\) (more precisely, we have \(q_1 = q_2\) where \(q_i\) is the cardinality of the residue field of \(K_i\)). Moreover, \(X_1\) has log smooth reduction if and only if \(X_2\) has log smooth reduction.

Remark

If we prove Theorem B without assuming that \(X_1\) has log smooth reduction, we can prove the absolute \(p\)-adic Grothendieck conjecture for \(p\) odd.

References


