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# SNC log symplectic structures on Fano products

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## Introduction to Poisson structures

### Definition (Poisson bracket).

The holomorphic **Poisson bracket** on  $X$ :

- (bilinear form)  $\{-, -\} : \mathcal{O}_X \times \mathcal{O}_X \rightarrow \mathcal{O}_X$
- (skew-symmetric)  $\{f, g\} = -\{g, f\}$ ,
- (Jacobi identity)  $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ ,
- (Leibniz rule)  $\{f, g \cdot h\} = \{f, g\}h + \{f, h\}g$ .

### Definition (Poisson structure).

The holomorphic **Poisson structure** on the smooth variety  $X$ :

$\Pi \in \Gamma(X, \wedge^2 T_X)$  s.t.  $[\Pi, \Pi] = 0 \in \wedge^3 T_X$ , where  $[-, -]$  is the Schouten bracket.

$$[\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j}, \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial w_l}] = \frac{\partial^2 \Pi}{\partial x_i \partial y_j} \wedge \frac{\partial^2 \Pi}{\partial z_k \partial w_l} - \frac{\partial^2 \Pi}{\partial y_j \partial z_k} \wedge \frac{\partial^2 \Pi}{\partial x_i \partial w_l} + \frac{\partial^2 \Pi}{\partial z_k \partial x_i} \wedge \frac{\partial^2 \Pi}{\partial y_j \partial w_l} + \frac{\partial^2 \Pi}{\partial w_l \partial x_i} \wedge \frac{\partial^2 \Pi}{\partial y_j \partial z_k} - \frac{\partial^2 \Pi}{\partial w_l \partial y_j} \wedge \frac{\partial^2 \Pi}{\partial x_i \partial z_k}$$

### Remark.

- {Poisson structures on  $X$ }  $\leftrightarrow$  {Poisson brackets on  $X$ }
- ( $\cdot$ ):  
 $\rightarrow : \Pi : \mathcal{O}_X \times \mathcal{O}_X \rightarrow \mathcal{O}_X; (f, g) \mapsto \Pi^\#(df, dg)$   
 $\leftarrow : \Pi = \sum_{i,j} \{x_i, x_j\} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$
- $[\Pi, \Pi] = 0 \iff$  Jacobi identity

### Definition (Degeneracy divisor).

- Poisson structure  $(X, \Pi)$  is **generically symplectic**  
 $\iff \dim X = 2n$  and  $\Pi^n \neq 0$ .

Suppose that  $(X, \Pi)$  is a generically symplectic.

- $D(\Pi) := \{x \in X \mid \Pi^n(x) = 0\}$  forms a divisor called the **degeneracy divisor**.
- $(X, \Pi)$  is a **log symplectic structure**  
 $\iff D(\Pi)$  is a reduced divisor,
- $(X, \Pi)$  is a **SNC log symplectic structure**  
 $\iff (X, \Pi)$  is a log symplectic structure and  $D(\Pi)$  is a simple normal crossing divisor.  
 $\Pi^n \in \Gamma(X, \wedge^{2n} T_X) \rightsquigarrow D(\Pi) \sim -\mathcal{K}_X$ .

## Motivations & Main Result

### Main Theorem

$X_i$ : Fano variety over  $\mathbb{C}$ ,  $\text{Pic}(X_i) = \mathbb{Z}$ ,  
 $\dim X_i = n_i \geq 3$ ,  
 $X = \prod_{i=1}^m X_i$ ,  $\dim X = 2n$ ,  
 $\Pi$ : SNC log symplectic structure.

- $X_i = \mathbb{P}^{n_i}$
- $\Pi$ : **diagonal Poisson structure**

## Background

### Question

#### How many $(X, \Pi)$ with conditions

- $X$ : smooth projective variety
- $D(\Pi)$ : reduced SNC

### Theorem (Lima, Pereira).

$X$ : Fano variety over  $\mathbb{C}$ ,  $\text{Pic}(X) = \mathbb{Z}$ ,  $\dim X = 2n \geq 4$ ,  
 $\Pi$ : SNC log symplectic structure on  $X$ .

- $\Rightarrow$  •  $X = \mathbb{P}^n$
- $\Pi$ : **diagonal Poisson structure on  $\mathbb{P}^n$**

#### How about if $\rho(X) \geq 2$

### Corollary (O).

$X$ : Fano variety over  $\mathbb{C}$ ,  $\text{Pic}(X) = \mathbb{Z}$ ,  
 $\dim X \geq 3$ ,  
 $X = \mathbb{P}^n \iff \exists \Pi$ : SNC log symplectic structure on  $X \times X$

## Diagonal Poisson structures and form as bivector fields

### Definition (Diagonal Poisson structure).

$X : \mathbb{A}^{2n}$  or (product of)  $\mathbb{P}^n$ ,  $\dim X = 2n$   
 $(X, \Pi)$  is a **diagonal Poisson structure**  
 $\iff D(\Pi)$  is composed of all coordinate hyperplanes.

### Theorem (Polishchuk).

There is a surjective map of bivector fields:

$$\varphi_n : \{(\mathbb{A}^{n+1}, \bar{\Pi}) \mid \bar{\Pi} : \text{quadratic}\} \rightarrow \{(\mathbb{P}^n, \Pi)\}$$

Furthermore,  $\bar{\Pi}$  is Poisson on  $\mathbb{A}^{n+1} \Rightarrow \varphi_n(\bar{\Pi})$  is also Poisson on  $\mathbb{P}^n$ .

( $\cdot$ ):

$$\left\{ \frac{x_i}{x_k}, \frac{x_j}{x_k} \right\} = \frac{1}{x_k^2} \{x_i, x_j\} - \frac{x_i}{x_k^3} \{x_k, x_j\} - \frac{x_j}{x_k^3} \{x_i, x_k\}$$

$\{x_i, x_j\}$  is quadratic,  $\rightsquigarrow \left\{ \frac{x_i}{x_k}, \frac{x_j}{x_k} \right\} \in k \left[ \frac{x_i}{x_k}, \dots, \frac{x_i}{x_k} \right]$   $\square$

### Fact (Pym).

$X := \prod_{i=1}^m \mathbb{P}^{n_i}$ ,  $\dim X = \sum_{i=1}^m n_i = 2n$ ,  $\Pi$ : diagonal Poisson str. on  $X$ ,  
 coordinate:  $[x_{10}, \dots, x_{1n_1}; x_{20}, \dots, x_{2n_2}; \dots; x_{m0}, \dots, x_{mn_m}]$ ,  
 $\Rightarrow \exists \sigma = \sum_{1 \leq i, k \leq m, 1 \leq j \leq n_i, 1 \leq l \leq n_k} \Delta_{ijkl}$ : diagonal Poisson str. on  $\mathbb{A}^{2n-m} \simeq \prod_{i=1}^m \mathbb{A}^{n_i-1}$ , where  $\Delta_{ij} = c_{ijk} x_{ij} x_{kl} \frac{\partial}{\partial x_{ij}} \wedge \frac{\partial}{\partial x_{kl}}$   
 s.t.  $\sigma$  induces  $\Pi$  on  $X$ .

( $\cdot$ ):

### Definition. (r-matrix construction)

$\Pi$  is constructed by **r-matrix construction** w.r.t a Lie group  $G$ :

$\Pi$  is a image of  $r$  along  $\mathfrak{g} \rightarrow \Gamma(X, T_X)$ ,

where  $\mathfrak{g}$ : Lie algebra of  $G$ ,  $r$ : r-matrix for  $G$ , i.e.  $[r, r] = 0$ .

The Fact comes from r-matrix construction for

$$G = (\mathbb{C}^*)^m \times (\mathbb{C}^*)^{n_2} \times \dots \times (\mathbb{C}^*)^{n_m}. \quad \square$$

### Key lemma (Pym).

$(X, \Pi)$ : SNC log symplectic structure

$D(\Pi) = \sum_{j=1}^k D_j$ : irreducible decomposition of the degeneracy divisor.

$$\Rightarrow \text{ch}(T_X) - \text{ch}(T_X^\vee) = 2 \sinh[D_j]$$

( $\cdot$ ): We have 2 exact sequences;

$$0 \rightarrow \Omega_X^1 \rightarrow \Omega_X^1(\log D) \rightarrow \bigoplus_{j=1}^m \mathcal{O}_{D_j} \rightarrow 0,$$

$$0 \rightarrow \mathcal{O}_X(-D_j) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{D_j} \rightarrow 0.$$