$\mathrm{Hilb}^G(\mathbb{C}^4)$ and crepant resolutions of certain abelian groups in SL(4,$\mathbb{C}$)

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Background

Question

Let G be a finite subgroup of $SL(n,\mathbb{C})$, then the quotient $\mathbb{C}^n/G$ has a Gorenstein canonical singularity. When does $\mathbb{C}^n/G$ have a crepant resolution?

- When $n=2$, the quotient $\mathbb{C}^2/G$ has a hypersurface singularity which is called a rational double point or ADE singularity. $\mathbb{C}^2/G$ has the minimal resolution(It is a crepant resolution).
- In the case $n=3$, it is known that $\mathbb{C}^3/G$ has crepant resolutions.
- However, in higher dimension, $\mathbb{C}^n/G$ does not always have crepant resolutions, and few examples of crepant resolutions are known.
  In this poster, we will show several examples of crepant resolutions in SL(4,$\mathbb{C}$) by $\mathrm{Hilb}^G(\mathbb{C}^4)$

Definition

A resolution $f : Y \to X$ is called a crepant resolution if the adjunction formula $K_Y = f^*K_X + \sum_{i=1}^n a_iD_i$ is satisfy $a_i = 0$ for all i

Definition

$$\mathrm{Hilb}^G(\mathbb{C}^n) = \{I \subset \mathbb{C}[x_1,\ldots,x_n] \mid I : G\text{−invariant ideal}, \mathbb{C}[x_1,\ldots,x_n]/I \cong \mathbb{C}[G]\}$$

- When $n=2$, $\mathrm{Hilb}^G(\mathbb{C}^2)$ is a crepant resolution for any finite subgroup in $SL(2,\mathbb{C})$
- In the case $n=3$, for any finite subgroup $SL(3,\mathbb{C})$, $\mathrm{Hilb}^G(\mathbb{C}^3)$ is one of crepant resolutions .
- If $n \geq 4$, the relationship between $\mathrm{Hilb}^G(\mathbb{C}^n)$ and crepant resolutions is not well known.

Crepant resolution as toric varieties

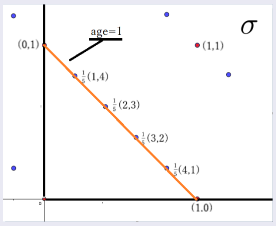
G denote a finite abelian subgroups of $SL(n,\mathbb{C})$. Any $g \in G$ is of the form $g = \begin{pmatrix} \varepsilon_r^{a_1} & & 0 \\ & \cdots & \\ 0 & & \varepsilon_r^{a_n} \end{pmatrix}$, where ε_r is a primitive rth root of unity.

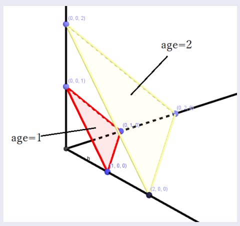
Then we can write $g = \frac{1}{r}(a_1,\ldots,a_n)$. Also, we define $\bar{g} = \frac{1}{r}(a_1,\ldots,a_n) \in \mathbb{R}$ Let $N := \mathbb{Z}^n + \mathbb{Z}\bar{g}$ be a free $\mathbb{Z}$-module of rank n, M be the dual $\mathbb{Z}$-module of N, and σ be the region of $\mathbb{R}^n$ whose all entries are non-negative.

Then the toric variety determined by σ is isomorphic to $\mathbb{C}^n/G$

Remark

When a cone σ to corresponding to $\mathbb{C}^n/G$ can be subdivided into Δ corresponding to smooth variety by lattice points of $age(g)=1$, then the toric variety determined by Δ is a crepant resolution of $\mathbb{C}^n/G$, where we define $age(g) = \frac{1}{r}\sum_{i=1}^n a_i$.





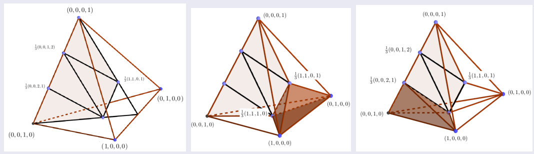
When $n=2$, lattice points of $age=1$ are on straight line. If $n=3$, they are on triangle.

So we consider tetrahedron in the case of $SL(4,\mathbb{C})$

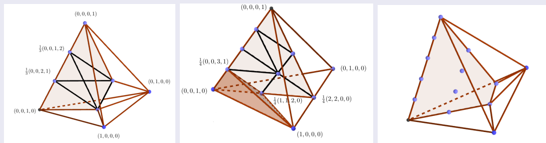
Main Result

Result 1

Let $r \geq 2$, $G = <\frac{1}{r}(1,1,0,r-2), \frac{1}{r}(0,0,1,r-1)>$.Then $\mathbb{C}^4/G$ has crepant resolutions. If r is even, then $\mathrm{Hilb}^G(\mathbb{C}^4)$ is one of crepant resolutions. When r is odd, $\mathrm{Hilb}^G(\mathbb{C}^4)$ is blow-up of certain crepant resolutions.



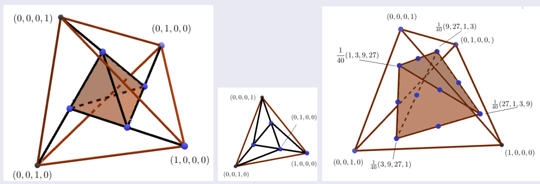
One of crepant resolutions for $G = <\frac{1}{3}(1,1,0,1), \frac{1}{3}(0,0,1,2)>$ and some examples of cone.



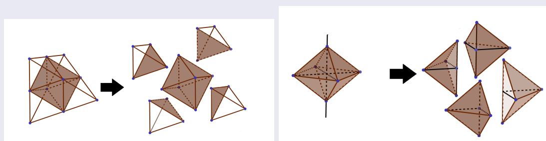
$\mathrm{Hilb}^G(\mathbb{C}^4)$ for $G = <\frac{1}{3}(1,1,0,1), \frac{1}{3}(0,0,1,2)>$ and lattice points of age(g)=1 of $G = <\frac{1}{4}(1,1,0,2), \frac{1}{4}(0,0,1,3)>$ and $G = <\frac{1}{5}(1,1,0,3), \frac{1}{5}(0,0,1,4)>$

Result 2

Let $r = 1+k+k^2+k^3$, $G = <\frac{1}{r}(1,k,k^2,k^3)>$.Then $\mathbb{C}^4/G$ has crepant resolutions. If $k=2$, then $\mathrm{Hilb}^G(\mathbb{C}^4)$ is one of crepant resolutions for $\mathbb{C}^4/G$. When $k \geq 3$, $\mathrm{Hilb}^G(\mathbb{C}^4)$ is blow-up of certain crepant resolutions.



Lattice point of $age=1$ for $G = <\frac{1}{15}(1,2,4,8)>$ and $G = <\frac{1}{40}(1,3,9,27)>$
$G = <\frac{1}{15}(1,2,4,8)>$ is a 4 dimensional version of $G = <\frac{1}{7}(1,2,4)> \subset SL(3,\mathbb{C})$

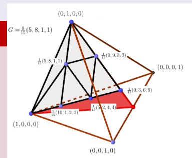


Subdivision of inside tetrahedron of $G = <\frac{1}{40}(1,3,9,27)>$ seems like 'orange-slice'

$\mathrm{Hilb}^G(\mathbb{C}^4)$ is a half of each pyramid of orange-slice and other cones are the same as crepant resolution.

Other examples

If $G = <\frac{1}{15}(5,8,1,1)>$ or $G = <\frac{1}{15}(1,8,3,3)>$, then $\mathbb{C}^4/G$ has crepant resolutions.
$\mathrm{Hilb}^G(\mathbb{C}^4)$ is one of crepant resolutions for $\mathbb{C}^4/G$.
The lattice points of G are on a blue triangle and are similar to lattice points of $\frac{1}{6}(1,2,3) \subset SL(3,\mathbb{C})$



Reference

[HIS] T.Hayashi, Y.Ito, Y.Sekiya, Existence of crepant resolutions, Advanced Study in Pure Mathematics, vol.74 (2017), 185-202.