<table>
<thead>
<tr>
<th>Title</th>
<th>Theta functions and the tropical Riemann-Roch inequality on tropical abelian surfaces</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Sumi, Ken</td>
</tr>
<tr>
<td>Citation</td>
<td>代数幾何学シンポジウム記録 (2018), 2018: 151-151</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2018</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/236419">http://hdl.handle.net/2433/236419</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Theta functions and the tropical Riemann-Roch inequality on tropical abelian surfaces

Ken Sumi (Department of Mathematics, Kyoto University) e-mail: ksumi@math.kyoto-u.ac.jp

1. Abstract

We show that the space of theta functions on tropical tori is identified with the space of sections of a certain line bundle. We also show a Riemann-Roch inequality for tropical abelian surfaces by calculating the self-intersection numbers of divisors.

2. Preliminaries

Tropical geometry: algebraic geometry over the tropical semifield \( T = \mathbb{R} \cup \{ -\infty \} \) equipped with the tropical sum \( x + y = \max \{ x, y \} \) and the tropical product \( "x \cdot y" = x + y \).

- Tropical polynomial: \( \sum_n a_n x^n = \max \{ a_n + n \cdot x \} \).
- Regular function: a convex piecewise linear function with integral slopes.
- Tropical line bundle: a \( \mathbb{T} \)-bundle with \( \mathbb{Z} \)-affine linear transition functions.

We show that the space of theta functions on tropical tori is identified with a space generated by just \( \mathbb{Z} \)-linear transitions.

3. Tropical line bundles on tropical tori

**Definition**

A theta function for \( L = L(Q, \alpha) \) is a regular function on \( \mathbb{R}^n \) satisfying the following quasi-periodicity:

\[ \Theta(x + \lambda) = \Theta(x) + \frac{1}{2} Q(\lambda, \lambda) + \alpha(\lambda), \quad \lambda \in \Lambda, \ x \in \mathbb{R}^n \]

The space of theta functions is identified with a subset \( \mathcal{H}(X, L) \) of \( \mathcal{H}(X) \) of sections of the line bundle \( L \). This is independent of the choice of \( \lambda \).

4. Example

For a tropical line bundle \( L \) on a tropical torus \( X \), we can take a symmetric form \( Q \) on \( \mathbb{R}^a \) such that \( Q(\lambda, \cdot) \in (\mathbb{Z}_+)^a \) and a linear function \( \alpha \) on \( \mathbb{R}^a \) such that \( L \) is isomorphic to the line bundle \( \mathbb{R}^a \times X \). We then denote \( L = L(\alpha, \lambda) \). \( Q \) is called the first Chern class of \( L \) and denoted by \( c_1(L) \).

5. Divisors and intersection numbers

**Theorem**

The tropical projectivization of \( \mathcal{H}(X, L(Q, 0)) \) is identified with

\[ \left\{ (r, s, t) + (a, a, a) \mid |r| \leq \frac{1}{6}, |s| \leq \frac{1}{6}, |t| \leq \frac{1}{6} \right\} \]

This is the convex polyhedron that has 14 vertices.

6. Main result 2

Combining Main result 1 and the above proposition, we have

**Theorem**

The classical Riemann-Roch formula for abelian surfaces is

\[ h^0(X, L) - h^0(X, \mathcal{O}(D)) + h^0(X, \mathcal{O}(0)) \geq 1 \frac{\text{deg}(D)^2}{2} \]

We can call this the real analogue of the classical Riemann-Roch inequality.