Theta functions and the tropical Riemann-Roch inequality on tropical abelian surfaces

Ken Sumi (Department of Mathematics, Kyoto University) e-mail: ksumi@math.kyoto-u.ac.jp

1. Abstract

We show that the space of theta functions on tropical tori is identified with a convex polyhedron. We also show a Riemann-Roch inequality for tropical abelian surfaces by calculating the self-intersection numbers of divisors.
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2. Preliminaries

Tropical geometry: algebraic geometry over the tropical semifield $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ equipped with the tropical sum "$x + y$" $= \max\{x, y\}$ and the tropical product "xy" $= x + y$.

- Tropical polynomial : "$\sum_n a_n x^n$" $= \max\{a_n + n \cdot x\}$.
- Regular function : a convex piecewise linear function with integral slopes.
- Tropical line bundle : a $\mathbb{T}$-bundle with $\mathbb{Z}$-affine linear transition functions.

Fix the standard lattice $\mathbb{Z}^n$.
Tropical torus : $\mathbb{R}^n/\Lambda$,where Λ is a lattice of rank n.
Facts:

- For a tropical line bundle L on a tropical torus X, we can take a symmetric form Q on $\mathbb{R}^n$ which satisfies $Q(\lambda, \cdot) \in (\mathbb{Z}^n)^*$ and a linear function α on $\mathbb{R}^n$ such that L is isomorphic to the line bundle $\mathbb{R}^n \times \mathbb{R}/(x, r) \sim (x + \lambda, r + Q(\lambda, x) + \frac{1}{2}Q(\lambda, \lambda) + \alpha(\lambda))$. Then we denote $L = L(Q, \alpha)$. Q is called the first Chern class of L and denoted by $c_1(L)$.
- For $\gamma \in (\mathbb{Z}^n)^*$, $L(Q, \alpha + \gamma) \cong L(Q, \alpha)$.
- Let $q\colon \mathbb{R}^n \to (\mathbb{R}^n)^*$ be a linear function given by $q(x) = Q(x, \cdot)$ and let $\iota_c\colon \mathbb{R}^n/\Lambda \to \mathbb{R}^n/\Lambda$ be a translation given by $x \mapsto x + c$ for $c \in \mathbb{R}^n$. Then $\iota_c^* L(Q, \alpha) \cong L(Q, \alpha + q(c))$.

3. Tropical line bundles on tropical tori

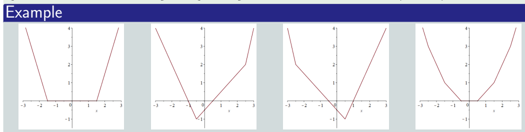
$X = \mathbb{R}^n/\Lambda$: a tropical tori.
$L = L(Q, \alpha)$: a tropical line bundle on X.

Definition
A theta function for $L = L(Q, \alpha)$ is a regular function on R^n which satisfies the following quasi-periodicity:

$$\Theta(x + \lambda) = \Theta(x) + Q(\lambda, x) + \frac{1}{2}Q(\lambda, \lambda) + \alpha(\lambda), \ \lambda \in \Lambda, \ x \in \mathbb{R}^n$$

Theta functions are identified with global sections of L which are not $-\infty$.
$\{$Theta functions for L$\} \cup \{-\infty\}$ is identified with $H^0(X, L)$.

Example



Graphs of some theta functions for a line bundle $L(1, 0)$ on $\mathbb{R}/3\mathbb{Z}$.

Remark
Theta functions are regular, and thus convex. If Q has a negative eigenvalue, the convexity is broken. Thus then $H^0(X, L) = \{-\infty\}$.

Mikhalkin-Zharkov(2008) states the following proposition without proof:

Proposition
Suppose that L is ample, that is, Q is positive definite. Then the $\mathbb{T}$-module $H^0(X, L)$ is generated by just $|(\mathbb{Z}^n)^*/q(\Lambda)|$ elements.

By analyzing theta functions, we get more detailed statement. We set $l = |$torsion part of $(\mathbb{Z}^n)^*/q(\Lambda)|$.

Theorem (Main result 1)
Suppose Q : positive semidefinite.

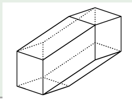
- If α does not lie in $(\mathbb{Z}^n)^* + \mathrm{Im}(q)$, then $H^0(X, L) = \{-\infty\}$.
- If α lies in $(\mathbb{Z}^n)^* + \mathrm{Im}(q)$, then $H^0(X, L)$ is generated by just l elements as a $\mathbb{T}$-module. Moreover, we can define a "natural" embedding $\varphi\colon H^0(X, L) \to \mathbb{T}^l$. Then $H^0(X, L)$ **is identified with an l-dimensional convex polyhedron** in $\mathbb{T}^l$ via φ and its projectivization is identified with a compact convex polyhedron in $\mathbb{R}^{l-1}$.

4. Example

$Q := \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $\Lambda := q^{-1}(2(\mathbb{Z}^2)^*)$, $X := \mathbb{R}^2/\Lambda$.
The tropical projectivization of $\varphi(H^0(X, L(Q, 0)))$ is identified with

$$\bigcup_{-\frac{1}{6} \le a \le \frac{1}{6}} \left\{ (r, s, t) + (a, a, a) \;\middle|\; |r| \le \frac{1}{6},\ |s| \le \frac{1}{6},\ |t| \le \frac{1}{6} \right\}.$$

This is the convex polyhedron that has 14 vertices.



5. Divisors and intersection numbers

$X = \mathbb{R}^2/\Lambda$:a 2-dim tropical torus.

Definition
A divisor on X is a tropical curve on X, that is, a graph Γ on X with weight function $w\colon$ (edges of Γ) $\to \mathbb{Z}$ satisfying "the balancing condition".

There exists a 1 to 1 correspondence between equivalent classes of divisors and equivalent classes of line bundles as in the classical case.
$\mathcal{O}(D)$: line bundle given by D.

Definition
D_1, D_2 : divisors on a 2-dimensional tropical tori X,
p : an intersection point of D_1, D_2,
E_i : an edge of D_i containing p.
The intersection number of D_1 and D_2 at p is
$(D_1, D_2)_p = w(E_1)w(E_2)|$cross product of slopes of E_1 and $E_2|$.
This is independent on the choice of E_i.
The intersection number of D_1 and D_2 is $(D_1, D_2) = \sum_{p \in D_1 \cap D_2} (D_1, D_2)_p$

Tropical abelian surface : a 2-dim tropical torus that has an ample line bundle.

Proposition
$X = \mathbb{R}^2/\Lambda$: a tropical abelian surface , D : a divisor on X.
$\Rightarrow D^2 = 2 \det Q \cdot \mathrm{vol}(\mathbb{R}^2/\Lambda)$.
Here $Q = c_1(\mathcal{O}(D))$ and $\det Q$ is the determinant associated to the standard basis of $\mathbb{R}^2$

Remark
$\mathrm{rk}(Q) = 2 \Rightarrow \det Q \cdot \mathrm{vol}(\mathbb{R}^2/\Lambda) = \pm|(\mathbb{Z}^n)^*/q(\Lambda)|$.

6. Main result 2

Combining Main result 1 and the above proposition, we have

Theorem
$X = \mathbb{R}^2/\Lambda$: a tropical abelian surface, D : a divisor on X.
If we define $h^0(X, \mathcal{O}(D)) := \dim H^0(X, \mathcal{O}(D))$, we have

$$h^0(X, \mathcal{O}(D)) + h^0(X, \mathcal{O}(-D)) \ge \frac{1}{2}D^2.$$

The classical Riemann-Roch formula for abelian surfaces is

$$h^0(X, L) - h^1(X, L) + h^2(X, L) = \frac{1}{2}L^2.$$

Thus we can call the above inequality Riemann-Roch inequality.

7. Validity

The Riemann-Roch theorem on tropical curves is the following.

Theorem (Gathmann-Kerber and Mikhalkin-Zharkov, 2008)
C: compact tropical curve with genus g, D: divisor on C,
K: canonical divisor of C. Then

$$r(D) - r(K - D) = \deg D - g + 1.$$

Here

$$r(D) = \begin{cases} \max\{k \in \mathbb{Z} \mid \forall E \in \mathrm{Div}_k^+(C), |D - E| \neq \emptyset\} & \text{if } |D| \neq \emptyset \\ = -1 & \text{if } |D| = \emptyset \end{cases}$$

In general, the number $r(D)$ is **different** from $\dim |D|$ as a polyhedral complex. Gathmann-Kerber(2008) gave an example where $r(D) + 1 \neq \dim H^0(C, \mathcal{O}(D))$. In their example, $H^0(C, \mathcal{O}(D))$ was not pure-dimensional; it seems that $r(D) + 1 = \dim H^0(C, \mathcal{O}(D))$ if $H^0(C, \mathcal{O}(D))$ is a pure-dimensional polyhedral complex. Thus we study the space of global sections of line bundles over tropical tori and show that it is always pure-dimensional.