Pathological quotient singularities which are not log canonical in positive characteristic

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Abstract

In characteristic zero, quotient singularities are log terminal. Moreover, we can check whether a quotient variety is canonical or not by using only the cyclic subgroups of the relevant finite group if the group does not have pseudo-reflections. In positive characteristic, a quotient variety is not log terminal, in general. In this paper, we give an example whose singularity cannot be determined by looking at proper subgroups.

1 Introduction

Quotient singularities form one of the most basic classes of singularities. They behave well in characteristic zero. In characteristic zero, any quotient variety has log terminal singularities. Moreover, for a finite group $G \subset \mathrm{GL}(d, \mathbb{C})$ without pseudo-reflection, we say that G is small, we can use Reid-Shepherd-Barron-Tai criterion [2, Theorem 3.21] to know the singularity of $\mathbb{C}^d/G$. Namely, the following two conditions are equivalent.

- $\mathbb{C}^d/G$ is canonical (resp. terminal).
- $\mathbb{C}^d/C$ is canonical (resp. terminal) for all cyclic subgroup C of G.

In positive characteristic, if the given finite group is tame, then the quotient variety is again log terminal and we can use the Reid-Shepherd-Barron-Tai criterion.

2 Main Theorem

If the group is wild, there exists a quotient variety which is not log terminal. We give an even more pathological example.

Theorem (Main Theorem)**.** Let C_3 be the cyclic group of order three and C_3^2 be the product of two copies of it. Suppose that the group C_3^2 is embedded in $\mathrm{SL}(3, K)$ and this embedding makes C_3^2 small, where K is algebraically closed field of characteristic three. Then the quotient variety $\mathbb{A}_K^3/C_3^2$ is not log canonical.

The pathological point of this example is that:

$$\mathbb{A}_K^3/C \text{ is canonical for all cyclic subgroup } C \subset C_3^2$$

because $\#C = 3$ for any cyclic subgroup C, see [3]. But

$$\mathbb{A}_K^3/C_3^2 \text{ is not log canonical.}$$

This is in contrast to the fact that, in characteristic zero, we can use the Ried-Shepherd-Barron-Tai criterion.

We give the proof of the main theorem by explicit computation. The all small action of C_3^2 are given in [1]. The actions are parametrized by $a \in K\backslash\mathbb{F}_3, b \in K$. Then, we can compute the explicit form of the quotient varieties X for each action of C_3^2. We will find that the quotient varieties are classified in two types separating by whether $b = 0$ or not about the parameter b of the action. Finally, we construct the proper birational morphism $Y \to X$ with exceptional divisors which discrepancy is smaller than -1, which shows the quotient varieties are not log canonical. This construction given by a few times blow up along the singular loci in each case.

3 Application

As an application of the main result, we give a criterion when a quotient variety associated to a small wild finite group is log terminal in dimension three and characteristic three. According to the criterion, we can judge the singularity of a quotient variety by seeing the order of the acting group.

Corollary. Let G be a wild small finite group of $\mathrm{GL}(3, K)$ where K is an algebraically closed field. We write $\#G = 3^r n$ where r, n are positive integer and n is not divided by three.

(i) If $r = 1$ then $\mathbb{A}_K^3/G$ is log terminal.

(ii) If $r \geq 2$ then $\mathbb{A}_K^3/G$ is not log canonical.

References

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