Singularities of non-$\mathbb{Q}$-Gorenstein varieties admitting a polarized endomorphism

Shou Yoshikawa

Graduate School of Mathematical Sciences, The University of Tokyo

Conjectures and Main Results

Let X be a normal complex projective variety admitting a non-invertible polarized endomorphism f. We are interested in the following two conjectures.

Conjectures

1. If X has the log canonical model $\mu : Y \to X$, then μ is an isomorphism in codimension one. [BH14]
2. X is of Calabi-Yau type. [BG17]

Broustet and Höring showed in [BH14] that if X is $\mathbb{Q}$-Gorenstein, then Conjecture 1 holds true i.e. X has log canonical singularities. And Broustet and Gongyo proved in [BG17] that if X is $\mathbb{Q}$-Gorenstein and f is étale in codimension one, then Conjecture 2 holds true. Our main results are removing the assumption that X is $\mathbb{Q}$-Gorenstein.

Main Results

- Conjecture 1 holds true.
- Conjecture 2 holds true if f is étale in codimension one and X has the log canonical model.

Notations and Properties

X : normal complex projective variety admitting a non-invertible polarized endomorphism f.
W : normal integral scheme essentially of finite type over a field of characteristic 0.
$\mathrm{Env}_W(D)$: nef envelope of Weil divisor D on W.

- $\mathrm{Env}_W(D)_Y$ is a divisor on birational model Y over W and if $\mu : Y' \to Y$ is birational morphism over W, then $\mu_* \mathrm{Env}_W(D)_{Y'} = \mathrm{Env}_W(D)_Y$
- If D is $\mathbb{Q}$-Cartier divisor, then $\mathrm{Env}_W(D)_Y = \pi^* D$ for any birational morphism $\pi : Y \to W$
- D is $\mathbb{Q}$-Cartier if and only if $\mathrm{Env}_W(D) + \mathrm{Env}_W(-D) = 0$ and $\oplus_m \mathcal{O}_W(mD)$ is finitely generated.

Definition and Key Theorems

We say that W has valuative log canonical singularities if

$$\mathrm{ord}_E(K_Y - \mathrm{Env}_X(K_W)_Y) + 1 \geq 0$$

for any birational model Y and prime divisor E on Y.
Thanks to the following theorem, we can reduce Conjecture 1 to prove that X has valuative log canonical singularities.

Key Theorem 1

The following are equivalent to each other.

1. W has valuative log canonical singularities.
2. For any birational model $\pi : Y \to W$ and positive number m, we have
$$\pi_* \mathcal{O}_Y(m(K_Y + E^\pi)) = \mathcal{O}_W(mK_W)$$
where, E^π is the exceptional prime divisors on Y.
Furthermore, if W has the log canonical model, the following condition is also equivalent.
3. The log canonical model of W is an isomorphism in codimension one.

The following theorems are local problems corresponding to main results.

Key Theorem 2

$(R, \mathfrak{m}, k)$: normal local ring of essentially of finite type over $\mathbb{C}$.
$\phi : R \to R$: finite injective local homomorphism. Suppose that

- Spec $R \backslash \{\mathfrak{m}\}$ has valuative log canonical singularities, and
- $\deg(\phi) > [\phi_* k : k]$.

Then Spec R has valuative log canonical singularities.

We further assume the following conditions.

- $\oplus R(mK_R)$ is finitely generated.
- Spec $R \backslash \{\mathfrak{m}\}$ is $\mathbb{Q}$-Gorenstein.
- ϕ is étale in codimension one.

Then R is $\mathbb{Q}$-Gorenstein.

Sketch of the proof of Main Result 1

We assume that non-valuative log canonical locus is not empty, and take an irreducible component Z. First, we prove the following claim.

Claim
Z is totally invariant up to replacing f by some iterate.

Next, by this claim, we may assume f induces an endomorphism of the local ring $\mathcal{O}_{X,\eta}$ of the generic point η of Z. Applying Key Theorem 2, we have

$$\deg(f) = [f_* \kappa(Z) : \kappa(Z)],$$

where $\kappa(Z)$ is the residue field of Z. Since $[f_* \kappa(Z) : \kappa(Z)]$ is nothing but $\deg(f|_Z)$, we see that

$$\deg(f) = \deg(f|_Z),$$

but it contradicts the fact that f is a non-invertible polarized endomorphism.

Sketch of the proof of Main Result 2

We assume that non-$\mathbb{Q}$-Gorenstein locus is not empty, and take an irreducible component Z. By Main Result 1, X has the amall log canonical model, so

$$\oplus \mathcal{O}_X(mK_X)$$

is finitely generated. By a similar argument, we may assume Z is totally invariant, and we can apply Key Theorem 2. Therefore we also see that

$$\deg(f) = \deg(f|_Z),$$

and it is a contradiction.

References

[BdFF12] S.Boucksom, T. de Fernex, C.Favre, The volume of an isolated singularity, Duke Math. J. **161** (2012),no. 8, 1455–1520.
[BG17] A. Broustet and Y. Gongyo, Remarks on Log Calabi- Yau structure of varieties admitting polarized endomorphisms, Taiwan J. Math. **21** (2017), no. 3, 569–582.
[BH14] A. Broustet and A. Höring, Singularities of varieties admitting an endomorphism. Math. Ann. **360** (2014), no. 1-2, 439–456.