Analysis of the planar exterior Navier-Stokes problem with effects related to rotation of the obstacle

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Chapter 1

Introduction

1.1 Background

Let $\mathcal{B}$ be a two-dimensional compact rigid body. Then the planar stationary motion of a viscous incompressible fluid filling the domain exterior to the body $\mathcal{B}$ is governed by a system of nonlinear partial differential equations, called the Navier-Stokes equations

$$\begin{cases} -\Delta w + \nabla r = -w \cdot \nabla w + h\,, \quad \operatorname{div} w = 0\,, \quad x \in \Omega\,, \\ w = 0\,, \quad x \in \partial\Omega\,, \end{cases} \tag{NS}$$

where we have set $\Omega = \mathbb{R}^2 \setminus \mathcal{B}$ and assumed that the boundary $\partial\Omega$ is smooth. Here $w = (w_1(x), w_2(x))^\top$ and $r = r(x)$ are respectively the velocity field and the pressure field of the fluid at the point $x = (x_1, x_2)^\top \in \Omega$, and $h = (h_1(x), h_2(x))^\top$ is a prescribed external force. At the boundary we impose the typical no-slip condition for viscous fluids implying that the velocity of the fluid at the boundary is equal to the velocity of the boundary.

The two-dimensional Navier-Stokes equations naturally appear when one considers the motion of a fluid in the three-dimensional domain where one direction is small enough compared to the other two directions, and therefore the flow behaves in an essentially two-dimensional manner. For example, the motion of the atmosphere or the ocean on large scales is sometimes treated as two-dimensional for many purposes. The study of two-dimensional flows has been motivated by such applications in addition to mathematical interests.

A specific difficulty related to the problem (NS) is the asymptotic behavior of solutions at spatial infinity. To explain this let us consider (NS) under the Dirichlet condition in the far field which describes that the motion of the fluid w is at rest at infinity:

$$w \to 0\,, \quad |x| \to \infty\,. \tag{Di}$$

We firstly give a brief review of the three-dimensional results. The mathematical analysis of (NS)–(Di) is started by Leray [43] in 1933, where the existence of solutions is established in the class of finite Dirichlet integral $\|\nabla w\|_{L^2(\Omega)} < \infty$ for an arbitrary external force h in $H^{-1}(\Omega) = W_0^{1,2}(\Omega)^*$. The condition (Di) is also verified in [43] but in a weak sense by the summability of the solutions in some weighted Lebesgue spaces. The uniform pointwise convergence (Di) is later proved by Finn [13], Fujita [16], and Ladyzhenskaya [42] for the case when an external force h has a compact support in Ω. In order to investigate the asymptotic behavior of solutions to (NS)–(Di) more in detail, Finn proved in [13] the unique existence of small solutions with the following spatial decay at infinity

$$w(x) \,=\, O(|x|^{-1})\,, \quad |x| \to \infty\,, \tag{1.1}$$

for a given external force $h(x)$ decaying sufficiently fast as $|x| \to \infty$. This class of solutions has a special importance for it is invariant under the Navier-Stokes scaling for $\lambda \in (0, \infty)$:

$$w_\lambda(x) \,=\, \lambda w(\lambda x)\,, \qquad r_\lambda(x) \,=\, \lambda^2 r(\lambda x)\,, \tag{1.2}$$

which is an invariant scaling for the nonlinear problem (NS) considered in the whole space. Thus, the solutions in the class (1.1) indicate the qualitative balance between the nonlinearity $w \cdot \nabla w$ and the dissipation Δw. For the derivative estimates of the solutions constructed in [13], which is important when one considers these stability properties, the optimal rate $\nabla w = O(|x|^{-2})$ is obtained by Novotny and Padula [49], and Borchers and Miyakawa [5] if the external force h is given by $h = \operatorname{div} H$ with some matrix $H = (H_{ij})_{1 \le i,j \le 3}$ and the quantity $\sup_{x \in \Omega}(1 + |x|)^2 |H(x)| + \sup_{x \in \Omega}(1 + |x|)^3 |\nabla H(x)|$ is sufficiently small. The asymptotic profiles of the solutions satisfying (1.1) are studied by Šverák [57], and Korolev and Šverák [40] where it is proved that they are described by the Landau solutions, stationary scale-invariant solutions to the Navier-Stokes equations in $\mathbb{R}^3 \setminus \{0\}$.

Contrary to the three-dimensional case, there have been only a few results for the two-dimensional problem. In fact, the existence of solutions to (NS)–(Di) still remains open in general. One reason for the difficulty is due to the lack of certain embeddings on unbounded domains in the two-dimensional case. Actually, the method of Leray leads to the existence of a vector field w satisfying (NS) with finite Dirichlet integral, however, we cannot verify the condition (Di) only by using the regularity $\|\nabla w\|_{L^2(\Omega)} < \infty$. Indeed, regardless of the domain shape, one can always find a function $W(x)$ of finite Dirichlet integral such that $|W(x)|$ diverges as $|x|$ goes to infinity. This fact provides a significant difference between the two- and three-dimensional cases. As for the study of the convergence of Leray's solutions at spatial infinity, we refer to Gilbarg and Weinberger [25, 26] and Amick [2, 3].

Another difficulty for the two-dimensional problem (NS)–(Di) is an over-determined aspect of the linearized equations. A glimpse of this inconvenience, which is known as the Stokes paradox, is firstly given by Stokes himself in 1851 for the case when the domain Ω is the exterior to a disk; if we consider the linearized problem of (NS)–(Di) under a nonhomogeneous Dirichlet condition $w_* \in \mathbb{R}^2 \setminus \{0\}$ on the boundary $\partial\Omega$

$$\begin{cases} -\Delta w + \nabla r \,=\, 0\,, \quad \operatorname{div} w \,=\, 0\,, \quad x \in \Omega\,, \\ w \,=\, w_*\,, \quad x \in \partial\Omega\,, \qquad w \to 0\,, \quad |x| \to \infty\,, \end{cases} \tag{1.3}$$

then, however, this problem does not admit a solution for any $w_* \in \mathbb{R}^2 \setminus \{0\}$. The reason for this phenomenon is clearly explained by Chang and Finn [7] later on in 1961 by means of the asymptotic expansion of the solutions to (1.3). They prove that the solution w converges to zero at spatial infinity only if the total net force to the boundary vanishes:

$$\int_{\partial\Omega} T(w, r)\nu \,\mathrm{d}\sigma \,=\, 0\,, \tag{1.4}$$

where $T(u, p) = \nabla u + (\nabla u)^\top - p\,\mathbb{I}\,, \mathbb{I} = (\delta_{ij})_{1 \le i,j \le 2}$, denotes the Cauchy stress tensor, and $\nu = (\nu_1, \nu_2)^\top$ is the outward unit normal vector to $\partial\Omega$. This condition and the energy equation for (1.3) yield that the flow w is a rigid motion, and hence that $w = 0$ which cannot satisfy (1.3) since $w_* \in \mathbb{R}^2 \setminus \{0\}$. The additional requirement (1.4), unfortunately, also appears when we consider the linearization of (NS)–(Di) with $h = \operatorname{div} H$ written as

$$\begin{cases} -\Delta w + \nabla r = \operatorname{div} H\,, \quad \operatorname{div} w = 0\,, \quad x \in \Omega\,, \\ w = 0\,, \quad x \in \partial\Omega\,, \qquad w \to 0\,, \quad |x| \to \infty\,, \end{cases} \tag{1.5}$$

even in the class $H \in C_0^\infty(\Omega)^{2\times 2}$ (we note that $w \cdot \nabla w = \operatorname{div}(w \otimes w)$ holds if $\operatorname{div} w = 0$). Thus, in proving the existence of small solutions to (NS)–(Di), while the analysis of the linearized problem is a rather standard procedure, it is already difficult to obtain qualitative estimates of the linear approximation (1.5) due to its over-determined feature. Partial results related to the solvability of (NS)–(Di) have been obtained by Galdi [18], Russo [53], Yamazaki [58], and Pileckas and Russo [52], where the solutions are constructed under some symmetry conditions on both domains and given data. In particular, the two-dimensional Navier-Stokes flows decaying in the scale-critical order $O(|x|^{-1})$ are obtained in [58]. The reader is also referred to a work by Hillairet and Wittwer [32] discussing the problem (NS)–(Di) in an exterior disk when the no-slip condition in (NS) is replaced as $w = \alpha x^\perp + b$, $x^\perp = (-x_2, x_1)^\top$, with a smooth function $b = b(x)$. We note that the flow $\alpha \frac{x^\perp}{|x|^2}$ exactly solves this problem if $h = b = 0$. When α is large enough and b is small, the solutions are constructed in [32] around the explicit solution $\tilde{\alpha} \frac{x^\perp}{|x|^2}$, where $\tilde{\alpha}$ is a number close to α.

In addition to stationary flows around a still rigid body, it is also interesting to study the dynamics of the fluid when the body moves in a prescribed manner. As we shall see below, the latter case is more manageable mathematically in view of the asymptotic behavior of the flows at large distances. In fact, in the two-dimensional case, the motion of the body attributes a radical change in the decay structure of the flows, which especially implies the resolution of the Stokes paradox. One of the most typical motions is a translation. The flows around a body $\mathcal{B}$ translating with a given constant velocity $\xi \in \mathbb{R}^2 \setminus \{0\}$ are described by

$$\begin{cases} \partial_t v - \Delta v + \nabla q = -v \cdot \nabla v\,, \quad \operatorname{div} v = 0\,, \quad t > 0\,,\ y \in \Omega(t)\,, \\ v = \xi\,, \quad t > 0\,,\ y \in \partial\Omega(t)\,, \qquad v \to 0\,, \quad t > 0\,,\ |y| \to \infty\,, \end{cases} \tag{1.6}$$

where $v = (v_1(y,t), v_2(y,t))^\top$ and $r = r(y,t)$ respectively denote the velocity field and the pressure field of the fluid at the time t and the point $y = (y_1, y_2)^\top \in \Omega(t)$, and the external force is assumed to be absent for simplicity. The time-dependent domain $\Omega(t)$ exterior to the translating body $\mathcal{B}$ (or, equivalently, occupied by the fluid) is expressed as

$$\Omega(t) = \{ y \in \mathbb{R}^2 \mid y = x + \xi t\,,\ x \in \Omega \}\,.$$

In order to consider the stationary flows in the coordinates attached to the translating body $\mathcal{B}$, it is appropriate to introduce the following reference frame:

$$y = x + \xi t\,, \quad u(x,t) = v(y,t)\,, \quad p(x,t) = q(y,t)\,. \tag{1.7}$$

Then the flows solving (1.6) which do not depend on time in this frame are subjecting to

$$\begin{cases} -\Delta u - \xi \cdot \nabla u + \nabla p = -u \cdot \nabla u\,, \quad \operatorname{div} u = 0\,, \quad x \in \Omega\,, \\ u = \xi\,, \quad x \in \partial\Omega\,, \qquad u \to 0\,, \quad |x| \to \infty\,. \end{cases} \tag{1.8}$$

The linearized equations of (1.8) are known as the Oseen equations. To perform a detailed analysis of the equations, Oseen [50] in 1910 introduces the associated fundamental solution. Then by using the representation formula, one can prove that a solution $(u_{\mathrm{os}}, p_{\mathrm{os}})$

to the Oseen equations has an anisotropic decay structure. More precisely, there exists a parabolic wake region $\mathcal{W} \subset \mathbb{R}^2$ in the direction $-\xi$ such that we have for large $|x|$,

$$u_{\rm os}(x) \,=\, O(|x|^{-\frac{1}{2}})\,, \quad x \in \mathcal{W}\,, \qquad u_{\rm os}(x) \,=\, O(|x|^{-1})\,, \quad x \notin \mathcal{W}\,. \tag{1.9}$$

We note that this decay structure is valid even when the condition (1.4) with (w, r) replaced by $(u_{\rm os}, p_{\rm os})$ fails. Hence the Stokes paradox does not appear in the Oseen equations case. Moreover, based on the decay estimate (1.9), Finn and Smith in [14, 15] prove the unique existence of small solutions to the nonlinear problem (1.8) for small but nonzero ξ.

The translation case above suggests that the motion of a rigid body leads to a localizing effect on its surrounding fluid, which is strong enough that one can construct corresponding stationary state solutions. Recently, an important progress is made by Hishida [35] when the prescribed motion on the body is a rotation. Let us assume that the body $\mathcal{B}$ rotates around the origin with a constant angular velocity $\alpha \in \mathbb{R} \setminus \{0\}$. The Navier-Stokes equations for the viscous incompressible fluid occupying the exterior to the rotating body $\mathcal{B}$ are given by

$$\begin{cases} \partial_t v - \Delta v + \nabla q \,=\, -v \cdot \nabla v + g\,, \quad {\rm div}\, v \,=\, 0\,, \quad t > 0\,,\, y \in \Omega(t)\,, \\ v \,=\, \alpha y^{\perp}\,, \quad t > 0\,,\, y \in \partial\Omega(t)\,, \qquad v \,\to\, 0\,, \quad t > 0\,,\, |y| \to \infty\,. \end{cases} \tag{1.10}$$

Here $g = (g_1(y,t), g_2(y,t))^\top$ denotes the given external force. The time-dependent fluid domain $\Omega(t)$ can be written by using a rotation matrix $O(\alpha t)$ as

$$\Omega(t) \,=\, \big\{ y \in \mathbb{R}^2 \mid y = O(\alpha t) x\,,\ x \in \Omega \big\}\,, \quad O(\alpha t) \,=\, \begin{pmatrix} \cos \alpha t & -\sin \alpha t \\ \sin \alpha t & \cos \alpha t \end{pmatrix}\,.$$

As is done in the translation case, we introduce the frame attached to the rotating body $\mathcal{B}$ by

$$\begin{gathered} y \,=\, O(\alpha t) x\,, \quad u(x,t) \,=\, O(\alpha t)^\top v(y,t)\,, \quad p(x,t) \,=\, q(y,t)\,, \\ f(x,t) \,=\, O(\alpha t)^\top g(y,t)\,. \end{gathered} \tag{1.11}$$

To describe stationary flows in this frame, we assume that $f = f(x)$ is independent of the time. Then the solutions of (1.10) time-independent in the frame (1.11) satisfy

$$\begin{cases} -\Delta u - \alpha(x^{\perp} \cdot \nabla u - u^{\perp}) + \nabla p \,=\, -u \cdot \nabla u + f\,, \quad {\rm div}\, u \,=\, 0\,, \quad x \in \Omega\,, \\ u \,=\, \alpha x^{\perp}\,, \quad x \in \partial\Omega\,, \qquad u \,\to\, 0\,, \quad |x| \to \infty\,. \end{cases} \tag{1.12}$$

Our first interest is focused on the asymptotic behavior of the solution $(u_{\rm lin}, p_{\rm lin})$ to the linearized problem of (1.12). In [35] the associated fundamental solution and its detailed estimates are obtained, and it is proved that if $\alpha \neq 0$ and the smooth external force f satisfies some suitable decay conditions, then $u_{\rm lin}$ obeys the asymptotic expansion

$$u_{\rm lin}(x) \,=\, \beta \frac{x^{\perp}}{|x|^2} + (1 + |\alpha|^{-1})\, o(|x|^{-1})\,, \quad |x| \to \infty\,, \tag{1.13}$$

where the constant $\beta = \beta(u, p, f)$ is given by

$$\beta \,=\, \frac{1}{4\pi} \bigg(\int_{\partial\Omega} y^{\perp} \cdot \big(T(u,p)\nu \big)\, {\rm d}\sigma_y + \int_{\Omega} y^{\perp} \cdot f \, {\rm d}y \bigg)\,. \tag{1.14}$$

Hence we again obtain the localizing effect arising from the rotation of the body which resolves the Stokes paradox as in the translation case. Particularly important here is the leading profile $\beta\frac{x^\perp}{|x|^2}$ of the asymptotics expansion in (1.13). It decays in the scale-critical order $O(|x|^{-1})$ and, moreover, is invariant under the Navier-Stokes scaling (1.2). This suggests that the similar asymptotics is valid also for the Navier-Stokes flow.

Motivated by the results in [35], it is natural to consider the following problems:

(I) Existence and uniqueness of solutions to the equations (1.12) for small $\alpha \in \mathbb{R} \setminus \{0\}$.
(II) Asymptotic behavior and leading profiles of these solutions at spatial infinity.
(III) Analysis of (1.12) when the rotation speed is sufficiently large $|\alpha| \gg 1$.

We shall prove the problems (I) and (II) in Chapter 2 by extending the results in [35]. The large rotation case as in the problem (III) is of importance in view of the localizing effect we have been discussing. It is natural to expect that a large rotation of the body would give a strong localizing effect on the motion of the fluid, however at the same time, it produces a strong shear near the boundary causing the appearance of boundary layers. Including this competitive mechanism, we will discuss the problem (III) in Chapter 3.

Let us consider the leading term $\beta\frac{x^\perp}{|x|^2}$ of (1.13) independently from the asymptotic expansion. The scale-critical rotating flow on the exterior unit disk $\Omega = \{x \in \mathbb{R}^2 \mid |x| > 1\}$

$$\big(\beta U,\, \beta^2 \nabla P\big) \quad \text{with} \quad U(x) \,=\, \frac{x^\perp}{|x|^2}\,, \qquad P(x) \,=\, -\frac{1}{2|x|^2}$$

is an exact solution to the Navier-Stokes equations (NS)–(Di) with $h = 0$ if we replace the no-slip boundary condition $w = 0$ on $\partial\Omega$ by the non-zero condition $w = \beta x^\perp$ on $\partial\Omega$. We recall that the general solvability of (NS)–(Di) in two dimensions is not established yet even if the condition $w = 0$ on $\partial\Omega$ is changed to $w = \beta x^\perp$ on $\partial\Omega$. Thus it is useful to study the property of such exact solutions in order to develop the theory of the two-dimensional Navier-Stokes equations. One of the most important topics concerning stationary solutions is stability. However, the stability analysis of the two-dimensional flows decaying in the order $O(|x|^{-1})$ is an open question in general. The difficulty is due to the fact that the Hardy inequality $\|\frac{f}{|x|}\|_{L^2(\Omega)} \le C\|\nabla f\|_{L^2(\Omega)}$, $f \in \dot{W}^{1,2}_0(\Omega)$, does not hold when Ω is an exterior domain in $\mathbb{R}^2$. The asymptotic stability of βU is treated in Maekawa [44] and the local L^2-stability is proved if $\beta \in \mathbb{R} \setminus \{0\}$ and initial perturbations are small enough. We generalize the result in [44] to the case when the domain loses symmetry in Chapter 4. Some related future works are also discussed at the end of the chapter.

1.2 Outline of results

In this section we quickly give an outline of the results gathered in this thesis.

1.2.1 On stationary Navier-Stokes flows around a rotating obstacle in two-dimensions (Chapter 2)

In Chapter 2 we consider the two-dimensional stationary Navier-Stokes equations describing viscous incompressible flows around a rotating rigid body (called the obstacle below):

$$\begin{cases} -\Delta u - \alpha(x^\perp \cdot \nabla u - u^\perp) + \nabla p \,=\, -u \cdot \nabla u + f\,, \quad \operatorname{div} u \,=\, 0\,, \quad x \in \Omega\,, \\ \qquad u \,=\, \alpha x^\perp\,, \quad x \in \partial\Omega\,, \\ \qquad u \,\to\, 0\,, \quad |x| \to \infty\,, \end{cases} \qquad (\mathrm{NS}_\alpha)$$

where Ω is an exterior domain with a smooth boundary, while the real number $\alpha \in \mathbb{R} \setminus \{0\}$ represents the rotation speed of the obstacle $\Omega^{\mathrm{c}} = \mathbb{R}^2 \setminus \Omega$. The equations ($\mathrm{NS}_\alpha$) is already derived as (1.12) in Section 1.1. As is also noticed in the same section, we shall prove the existence and uniqueness of solutions to (NS_α) by extending the result of Hishida [35]. To state the result let us introduce the function spaces. For a fixed number $s \in [0, \infty)$ we introduce the weighted L^∞ space $L^\infty_s(\Omega)$ and its subspace $L^\infty_{s,0}(\Omega)$ as

$$\begin{aligned} L^\infty_s(\Omega) &= \big\{ f \in L^\infty(\Omega) \mid (1+|x|)^s f \in L^\infty(\Omega) \big\}\,, \\ L^\infty_{s,0}(\Omega) &= \big\{ f \in L^\infty_s(\Omega) \mid \lim_{R\to\infty} \mathrm{ess.sup}_{|x|\geq R} |x|^s |f(x)| = 0 \big\}\,. \end{aligned} \tag{1.15}$$

These are Banach spaces equipped with the natural norm

$$\|f\|_{L^\infty_s(\Omega)} = \mathrm{ess.sup}_{x\in\Omega}(1+|x|)^s |f(x)|\,,$$

and the set of functions with compact support is dense in $L^\infty_{s,0}(\Omega)$. We denote by $L^2_{\mathrm{loc}}(\overline{\Omega})$ the set of functions which belong to $L^2(\Omega \cap K)$ for any compact set $K \subset \mathbb{R}^2$, and $W^{k,2}_{\mathrm{loc}}(\overline{\Omega})$, $k \in \mathbb{N}$, is defined in a similar manner. The main result in Chapter 2 is stated as follows.

Theorem 1.2.1 *There exists $\epsilon = \epsilon(\Omega) > 0$ such that the following statement holds. Assume that $f \in L^2(\Omega)^2$ is of the form $f = \mathrm{div}\, F = (\partial_1 F_{11} + \partial_2 F_{12}, \partial_1 F_{21} + \partial_2 F_{22})^\top$ with some $F = (F_{ij})_{1\leq i,j\leq 2} \in L^\infty_2(\Omega)^{2\times 2}$ and $F_{12} - F_{21} \in L^1(\Omega)$. If $\alpha \in \mathbb{R} \setminus \{0\}$ and*

$$|\alpha|^{\frac12} \big|\log|\alpha|\big| + |\alpha|^{-\frac12} \big|\log|\alpha|\big| \big(\|f\|_{L^2(\Omega)} + \|F\|_{L^\infty_2(\Omega)} + \|F_{12} - F_{21}\|_{L^1(\Omega)} \big) < \epsilon\,, \tag{1.16}$$

then there exists a solution $(u, \nabla p) \in \big(W^{2,2}_{\mathrm{loc}}(\overline{\Omega}) \cap L^\infty_1(\Omega) \big)^2 \times L^2_{\mathrm{loc}}(\overline{\Omega})^2$ to (NS_α)*, which is unique in a suitable class of functions (see Theorem 2.4.1 in Chapter 2 for the precise description). If $F \in L^\infty_{2,0}(\Omega)^{2\times 2}$ in addition, then the solution $u = u(x)$ behaves as*

$$u(x) = \tilde{\beta} \frac{x^\perp}{|x|^2} + o(|x|^{-1})\,, \quad |x| \to \infty\,, \tag{1.17}$$

where the constant $\tilde{\beta} = \tilde{\beta}(u, p, f)$ is given by

$$\tilde{\beta} = \frac{1}{4\pi} \bigg(\int_{\partial\Omega} y^\perp \cdot \big(T(u,p)\nu\big)\, \mathrm{d}\sigma_y + \lim_{\delta\to 0} \int_\Omega e^{-\delta|y|^2} y^\perp \cdot f\, \mathrm{d}y \bigg)\,. \tag{1.18}$$

Here $T(u,p) = \nabla u + (\nabla u)^\top - p\,\mathbb{I}$, $\mathbb{I} = (\delta_{ij})_{1\leq i,j\leq 2}$, denotes the Cauchy stress tensor, and ν is the outward unit normal vector to $\partial\Omega$.

Remark 1.2.2 (i) Both conditions $F \in L^\infty_2(\Omega)^{2\times 2}$ and $F_{12} - F_{21} \in L^1(\Omega)$ are critical in view of the Navier-Stokes equations scaling. Note that the L^1-summability of F is needed only for its antisymmetric part. These two conditions are not enough to ensure that u behaves like the circular flow $\tilde{\beta}\frac{x^\perp}{|x|^2}$ at spatial infinity, and the additional decay condition $F \in L^\infty_{2,0}(\Omega)^{2\times 2}$ as in Theorem 1.2.1 is required to achieve this asymptotic property.
(ii) The second term of the right-hand side of (1.18) is well-defined if $F \in L^\infty_{2,0}(\Omega)^{2\times 2}$ and $F_{12} - F_{21} \in L^1(\Omega)$. If $F \in L^\infty_{2+\gamma}(\Omega)^{2\times 2}$ with $\gamma \in (0,1)$ then the order $o(|x|^{-1})$ in (1.17) is replaced by $O(|x|^{-1-\gamma})$ at least when α and f are further small depending on γ.

(iii) The pressure p is determined uniquely up to a constant and belongs to $W^{1,2}_{\mathrm{loc}}(\overline{\Omega})$. Then the regularity $u \in W^{2,2}_{\mathrm{loc}}(\overline{\Omega})^2$ yields that the coefficient β in (1.18) is well-defined.
(iv) In Theorem 1.2.1 we assume that the external force f is of divergence form. In fact, this is not an essential assumption, and it is possible to deal with the external force f satisfying

$$x^{\perp} \cdot f \in L^1(\Omega)\,, \qquad f \in L^{\infty}_{3}(\Omega)^2\,, \tag{1.19}$$

with the smallness in these norms. Moreover, the asymptotic expansion (1.17) is verified if $f \in L^{\infty}_{3,0}(\Omega)^2$ in addition. This is obtained by using our recent result [30] in the whole space which solves the linearized problem for f satisfying (1.19).

As far as the author knows, Theorem 1.2.1 is the first general existence result of the Navier-Stokes flows around a rotating obstacle *in the two-dimensional case*. For the three-dimensional existence result, we refer to Borchers [4], Silvestre [55], Galdi [17], and Farwig and Hishida [9]. In particular, in [17] the stationary flows with the decay order $O(|x|^{-1})$ are obtained, while the work of [9] is based on the weak L^3-framework, which is another natural scale-critical space for the three-dimensional Navier-Stokes equations. Our Theorem 1.2.1 is considered as a two-dimensional counterpart of the result of [17].

Let us state the key idea for the proof of Theorem 1.2.1. Our approach is motivated by the linear analysis developed in [35], thus we recall its result more precisely than Section 1.1. The linearization of the Navier-Stokes equations (NS_{α}) is written as

$$\begin{cases} -\Delta u - \alpha(x^{\perp} \cdot \nabla u - u^{\perp}) + \nabla p \,=\, f\,, \quad \operatorname{div} u \,=\, 0\,, \quad x \in \Omega\,, \\ \qquad\qquad\qquad\qquad\qquad u \,=\, \alpha x^{\perp}\,, \quad x \in \partial\Omega\,, \\ \qquad\qquad\qquad\qquad\qquad u \,\to\, 0\,, \quad |x| \to \infty\,, \end{cases} \tag{S_{α}}$$

where the condition $\alpha \in \mathbb{R} \setminus \{0\}$ is imposed. In [35] it is proved that if the smooth external force $f = f(x)$ satisfies the decay conditions

$$\int_{\Omega} |x||f(x)| \,\mathrm{d}x < \infty\,, \qquad f(x) \,=\, o\big(|x|^{-3} (\log |x|)^{-1}\big)\,, \quad |x| \to \infty\,, \tag{1.20}$$

then the solution $u = u(x)$ to (S_{α}) obeys the asymptotic expansion

$$u(x) \,=\, \beta \frac{x^{\perp}}{|x|^2} + (1 + |\alpha|^{-1})\, o(|x|^{-1})\,, \quad |x| \to \infty\,, \tag{1.21}$$

where the constant $\beta = \beta(u, p, f)$ is given by

$$\beta \,=\, \frac{1}{4\pi} \bigg(\int_{\partial\Omega} y^{\perp} \big(T(u,p)\nu\big) \,\mathrm{d}\sigma_y + \int_{\Omega} y^{\perp} \cdot f \,\mathrm{d}y \bigg)\,. \tag{1.22}$$

Our strategy for the proof of Theorem 1.2.1 is summarized as follows; we derive at the same time the unique existence of solutions and their asymptotic behavior, under the smallness condition on the given data (α, f) in (NS_{α}). The solution in the form $u = \beta \frac{x^{\perp}}{|x|^2} + w$ is constructed by the fixed point theorem, where both the coefficient β and the remainder term w are sufficiently small corresponding to the size of (α, f). However, it is far from trivial to justify this idea directly from the results of [35], especially to ensure the smallness of (β, w) in the iteration scheme. Indeed, there are at least two difficulties for this procedure: (I) the condition (1.20) is slightly restrictive to handle the nonlinear term $u \cdot \nabla u$ in the scale-critical framework, and more seriously, (II) the singularity in (1.21) for small $|\alpha|$ may prevent us from closing the nonlinear estimates. In fact, a flow subject to the system (NS_α) is naturally pointwise bounded above by $|\alpha|$ near the boundary due to the boundary condition $u = \alpha x^\perp$.

In resolving the difficulty (I), the structure of the nonlinear term $u \cdot \nabla u = \nabla \cdot (u \otimes u)$ is essential. Indeed, the symmetry of the tensor $u \otimes u$ leads to a crucial cancellation in the coefficient "$\int_\Omega y^\perp \cdot (u \cdot \nabla u)\,\mathrm{d}y$", which removes a possible singularity caused by the scale-critical decay of the flow $u = O(|x|^{-1})$. To overcome the difficulty (II), we revisit the argument of [35] analyzing the fundamental solution to (S_α) in $\mathbb{R}^2$ and modify the singularity of α appearing in the estimates of the remainder term; see Theorem 2.3.1, Lemma 2.3.3, and Theorem 2.3.8 in Chapter 2. Applying these improved estimates, the problem (NS_α) is solved by the standard Banach fixed point theorem. However, the argument becomes quite complicated since we have to control two kinds of norms: the one bounds the local quantity, while the other one controls the spatial decay. This machinery is needed since the flow in a far field region ($|x| \gg 1$) exhibits a different dependence on $|\alpha|$ from the flow in a finite fluid region, and in principle, the problem becomes more singular at $|x| \gg 1$ as $|\alpha|$ is decreasing. In order to close the nonlinear estimates it is important to distinguish these two dependences on $|\alpha|$ and to estimate their interaction through the nonlinearity carefully.

The result in this chapter is based on a joint work with Yasunori Maekawa and Yuu Nakahara, which corresponds to the paper [31] published in Archive for Rational Mechanics and Analysis.

1.2.2 On stationary two-dimensional flows around a fast rotating disk (Chapter 3)

As we have seen in Section 1.1, the motion of a two-dimensional rigid body (obstacle) leads to a drastic change in the decay structure of its surrounding fluid. Moreover, it yields a significant localizing effect that enables one to construct corresponding stationary state solutions to the Navier-Stokes equations when the motion is slow enough. Although on the one hand a faster motion of the obstacle gives a stronger localizing and stabilizing effect, on the other hand it produces a rapid flow and creates a strong shear near the boundary that can be a source of instability. As a result, rigorous analysis becomes quite difficult for the nonlinear problem in general. Hence it is useful to study the problem under a simple geometrical setting and to understand a typical fluid structure that describes these two competitive mechanisms; localizing and stabilizing effects on the one hand, and the presence of a rapid flow and the boundary layer created by the fast motion of the obstacle on the other.

In Chapter 3 we study two-dimensional flows around a rotating obstacle assuming that the obstacle is a unit disk centered at the origin, especially in the case when the rotation speed is sufficiently fast. After following the same procedure as in Section 1.1, we consider the following stationary Navier-Stokes equations in the domain $\Omega = \{x \in \mathbb{R}^2 \mid |x| > 1\}$:

$$\begin{cases} -\Delta u - \alpha(x^\perp \cdot \nabla u - u^\perp) + \nabla p \,=\, -u \cdot \nabla u + f\,, \quad x \in \Omega\,, \\ \qquad\qquad\qquad\qquad\qquad\quad \operatorname{div} u \,=\, 0\,, \quad x \in \Omega\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\;\; u \,=\, \alpha x^\perp\,, \quad x \in \partial\Omega\,. \end{cases} \tag{NS_α}$$

Due to the symmetry of the fluid domain, there is an explicit solution to (NS_α) with $f = 0$:

$$\big(\alpha U,\, \alpha^2 \nabla P\big) \quad \text{with} \quad U(x) \,=\, \frac{x^\perp}{|x|^2}\,, \qquad P(x) \,=\, -\frac{1}{2|x|^2}\,.$$

Thus it is natural to consider an expansion around this explicit solution. By using the identity $u \cdot \nabla u = \frac{1}{2}\nabla|u|^2 + u^\perp \mathrm{rot}\, u$ with $\mathrm{rot}\, u = \partial_1 u_2 - \partial_2 u_1$ and the condition $\mathrm{rot}\, U = 0$ for $x \neq 0$, the equations for $v = u - \alpha U$ can be written as

$$\begin{cases} -\Delta v - \alpha(x^\perp \cdot \nabla v - v^\perp) + \nabla q + \alpha U^\perp \mathrm{rot}\, v \;=\; -v^\perp \mathrm{rot}\, v + f\,, \quad x \in \Omega\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{div}\, v \;=\; 0\,, \quad x \in \Omega\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\;\; v \;=\; 0\,, \quad x \in \partial\Omega\,. \end{cases} \tag{$\widetilde{\mathrm{NS}}_\alpha$}$$

The goal of Chapter 3 is to show the existence and uniqueness of solutions to ($\widetilde{\mathrm{NS}}_\alpha$) for arbitrary $\alpha \in \mathbb{R}\backslash\{0\}$ under a suitable condition on the external force f in terms of regularity and summability. Moreover, we shall give a detailed qualitative analysis for the fast rotation case $|\alpha| \gg 1$ that exhibits a boundary layer structure and an axisymmetrization of the flows. The novelty of the results in Chapter 3 can be summarized as follows:

(1) Existence and uniqueness of solutions to ($\widetilde{\mathrm{NS}}_\alpha$) for arbitrary $\alpha \in \mathbb{R} \setminus \{0\}$.
(2) Relaxed summability condition on f and on the class of solutions, which allows slow spatial decay with respect to the Navier-Stokes scaling.
(3) Qualitative analysis of solutions in the fast rotation case $|\alpha| \gg 1$.

As for (1), the result is new compared with the one obtained in Chapter 2 where the stationary solutions are obtained only for nonzero but small $|\alpha|$, though there is no restriction on the shape of the obstacle. The reason why we can construct solutions for all nonzero α in the exterior unit disk is a remarkable coercive estimate for the term $-\alpha(x^\perp \cdot \nabla v - v^\perp) + \alpha U^\perp \mathrm{rot}\, v$ in polar coordinates; see (1.33) below. As for (2), we note that the given data f and the class of solutions in Chapter 2 are in a scale critical space. A typical condition for f assumed in Chapter 2 is that $f = \mathrm{div}\, F$ with $F(x) = O(|x|^{-2})$, and then the solution v satisfies $|v(x)| \leq C|x|^{-1}$ for $|x| \gg 1$. In Chapter 3 the summability condition on f is weaker than this scaling, see (1.25) below. Moreover, the radial part of the constructed solution only behaves like $o(1)$ as $|x| \to \infty$ in general, which is considerably slow, while the nonradial part of the solution belongs to $L^2(\Omega)$ which is just in the scale critical regime. The point (3) is important both physically and mathematically. Understanding the fluid structure around the fast rotating obstacle is one of the main subjects of Chapter 3, and we show the appearance of a boundary layer as well as an axisymmetrization mechanism due to the fast rotation.

Let us state our functional setting. Thanks to the symmetry of the domain it is natural to introduce the relevant function spaces in terms of polar coordinates. As usual, we set

$$\begin{aligned} x_1 \;&=\; r\cos\theta\,, \quad x_2 \;=\; r\sin\theta\,, \qquad r \;=\; |x| \geq 1\,, \quad \theta \in [0, 2\pi)\,, \\ \mathbf{e}_r \;&=\; \frac{x}{|x|}\,, \quad \mathbf{e}_\theta \;=\; \frac{x^\perp}{|x|} \;=\; \partial_\theta \mathbf{e}_r\,, \end{aligned}$$

and

$$v \;=\; v_r\, \mathbf{e}_r \;+\; v_\theta\, \mathbf{e}_\theta\,, \qquad v_r \;=\; v \cdot \mathbf{e}_r\,, \quad v_\theta \;=\; v \cdot \mathbf{e}_\theta\,.$$

For each $n \in \mathbb{Z}$, we denote by $\mathcal{P}_n$ the projection on the Fourier mode n:

$$\mathcal{P}_n v \;=\; v_{r,n} e^{in\theta} \mathbf{e}_r + v_{\theta,n} e^{in\theta} \mathbf{e}_\theta\,, \tag{1.23}$$

where

$$v_{r,n}(r) \;=\; \frac{1}{2\pi}\int_0^{2\pi} v_r(r\cos\theta, r\sin\theta)e^{-in\theta}\,\mathrm{d}\theta\,,$$
$$v_{\theta,n}(r) \;=\; \frac{1}{2\pi}\int_0^{2\pi} v_\theta(r\cos\theta, r\sin\theta)e^{-in\theta}\,\mathrm{d}\theta\,.$$

We also set for $m \in \mathbb{N}\cup\{0\}$,

$$\mathcal{Q}_m v \;=\; \sum_{|n|=m+1}^{\infty} \mathcal{P}_n v\,. \tag{1.24}$$

For notational convenience we will often write v_n for $\mathcal{P}_n v$. Each $\mathcal{P}_n$ is an orthogonal projection in $L^2(\Omega)^2$, and the space $L^2_\sigma(\Omega) = \overline{\{f \in C_0^\infty(\Omega)^2 \mid \mathrm{div}\, f = 0\}}^{L^2(\Omega)^2}$ is invariant under the action of $\mathcal{P}_n$. Note that $v_0 = \mathcal{P}_0 v$ is the radial part of v, and thus, $\mathcal{Q}_0 v$ is the nonradial part of v. We will set $\mathcal{P}_n L^2(\Omega)^2 = \{f \in L^2(\Omega)^2 \mid f = \mathcal{P}_n f\}$, and similar notation will be used for $L^2_\sigma(\Omega)$ and $\mathcal{Q}_0$. A vector field f in Ω is formally identified with the pair $(\mathcal{P}_0 f, \mathcal{Q}_0 f)$. Then, for the class of external forces we introduce the product space

$$Y = \mathcal{P}_0 L^1(\Omega)^2 \times \mathcal{Q}_0 L^2(\Omega)^2\,. \tag{1.25}$$

For the class of solutions we set $W_0^{1,r}(\Omega) = \{f \in W^{1,r}(\Omega) \mid f = 0 \text{ on } \partial\Omega\}$ and

$$X = \mathcal{P}_0 W_0^{1,\infty}(\Omega)^2 \times \mathcal{Q}_0 W_0^{1,2}(\Omega)^2\,. \tag{1.26}$$

Then our first result in Chapter 3 is stated as follows.

Theorem 1.2.3 *There exists $\gamma \in (0,\infty)$ such that the following statements hold.*
(i) *Let $0 < |\alpha| < 1$. Then for any external force $f = (\mathcal{P}_0 f, \mathcal{Q}_0 f) \in Y$ satisfying*

$$\|(\mathcal{P}_0 f)_\theta\|_{L^1(\Omega)} \le \gamma|\alpha|\,, \qquad \|\mathcal{Q}_0 f\|_{L^2(\Omega)} \le \gamma|\alpha|^2\,, \tag{1.27}$$

there exists a unique solution $(v, \nabla q) \in X \cap W^{2,1}_{\mathrm{loc}}(\overline{\Omega})^2 \times L^1_{\mathrm{loc}}(\overline{\Omega})^2$ to $(\widetilde{\mathrm{NS}}_\alpha)$ satisfying

$$\|\mathcal{P}_0 v\|_{L^\infty(\Omega)} + \big\|\nabla \mathcal{P}_0 v\big\|_{L^\infty(\Omega)} \le C\|(\mathcal{P}_0 f)_\theta\|_{L^1(\Omega)} + C|\alpha|^{-\frac{3}{2}}\|\mathcal{Q}_0 f\|^2_{L^2(\Omega)}\,, \tag{1.28}$$
$$|\alpha|^{\frac{3}{4}} \sum_{|n|\ge 1} \|\mathcal{P}_n v\|_{L^\infty(\Omega)} + |\alpha|\|\mathcal{Q}_0 v\|_{L^2(\Omega)} + |\alpha|^{\frac{1}{2}}\|\nabla \mathcal{Q}_0 v\|_{L^2(\Omega)} \le C\|\mathcal{Q}_0 f\|_{L^2(\Omega)}\,. \tag{1.29}$$

(ii) *Let $|\alpha| \ge 1$. Then for any external force $f = (\mathcal{P}_0 f, \mathcal{Q}_0 f) \in Y$ satisfying*

$$\|(\mathcal{P}_0 f)_\theta\|_{L^1(\Omega)} \le \gamma\,, \qquad \|\mathcal{Q}_0 f\|_{L^2(\Omega)} \le \gamma\,, \tag{1.30}$$

there exists a unique solution $(v, \nabla q) \in X \cap W^{2,1}_{\mathrm{loc}}(\overline{\Omega})^2 \times L^1_{\mathrm{loc}}(\overline{\Omega})^2$ to $(\widetilde{\mathrm{NS}}_\alpha)$ satisfying

$$\|\mathcal{P}_0 v\|_{L^\infty(\Omega)} + \big\|\nabla \mathcal{P}_0 v\big\|_{L^\infty(\Omega)} \le C\|(\mathcal{P}_0 f)_\theta\|_{L^1(\Omega)} + C|\alpha|^{-\frac{1}{2}}\|\mathcal{Q}_0 f\|^2_{L^2(\Omega)}\,, \tag{1.31}$$
$$|\alpha|^{\frac{1}{4}} \sum_{|n|\ge 1} \|\mathcal{P}_n v\|_{L^\infty(\Omega)} + |\alpha|^{\frac{1}{2}}\|\mathcal{Q}_0 v\|_{L^2(\Omega)} + \|\nabla \mathcal{Q}_0 v\|_{L^2(\Omega)} \le C\|\mathcal{Q}_0 f\|_{L^2(\Omega)}\,. \tag{1.32}$$

Note that the summability of f assumed in Theorem 1.2.3 is much weaker than the scale-critical one. For the radial part $\mathcal{P}_0 v$ we can show $\lim_{|x|\to\infty} |\mathcal{P}_0 v(x)| = 0$ but there is no rate in general under the assumptions of Theorem 1.2.3. Theorem 1.2.3 already exhibits the axisymmetrization of the fast rotation in L^2 and L^∞, which will be further extended in Theorems 1.2.4 and 1.2.5 below. The proof of Theorem 1.2.3 consists in two ingredients: one is the analysis of the linearized equations of $(\widetilde{\mathrm{NS}}_\alpha)$

$$\begin{cases} -\Delta v - \alpha(x^\perp\cdot\nabla v - v^\perp) + \nabla q + \alpha U^\perp \mathrm{rot}\, v \,=\, f\,, \quad x\in\Omega\,,\\ \mathrm{div}\, v \,=\, 0\,, \quad x\in\Omega\,,\\ v\,=\,0\,, \quad x\in\partial\Omega\,. \end{cases} \tag{S_α}$$

The other is the estimate of the interaction between the radial part and the nonradial part in the nonlinear problem. The linear result used in Theorem 1.2.3 is proved based on an energy method. Although the proof of the linear result is not so difficult, there is a key observation for the term $-\alpha(x^\perp\cdot\nabla v - v^\perp) + \alpha U^\perp \mathrm{rot}\, v$. Indeed, for the linearized problem (S_α) the energy computation for $v_n = \mathcal{P}_n v$ with $n\neq 0$ gives the key identity

$$\begin{aligned} &\alpha n\Big(\|v_{r,n}\|^2_{L^2(\Omega)} - (1-\frac{2}{n^2})\|\frac{v_{r,n}}{r}\|^2_{L^2(\Omega)} + \|v_{\theta,n}\|^2_{L^2(\Omega)} - \|\frac{v_{\theta,n}}{r}\|^2_{L^2(\Omega)}\Big)\\ &\quad = -\mathrm{Im}\langle f_n\,, v_n\rangle_{L^2(\Omega)}\,. \end{aligned} \tag{1.33}$$

Here f_n denotes $\mathcal{P}_n f$ and the norm $\|g\|_{L^2(\Omega)}$ for the function $g:[1,\infty)\to\mathbb{C}$ is defined as $(2\pi)^{\frac12}\|g\|_{L^2((1,\infty);r\,\mathrm{d}r)}$. The key point here is that the bracket in (1.33) is nonnegative and provides a bound for $\|\frac{\sqrt{|x|^2-1}}{|x|}v_n\|^2_{L^2(\Omega)}$. Then by combining an interpolation inequality

$$\|g\|_{L^2(\Omega)} \le C\|\partial_r g\|^{\frac13}_{L^2(\Omega)}\|\frac{\sqrt{r^2-1}}{r}g\|^{\frac23}_{L^2(\Omega)} + C\|\frac{\sqrt{r^2-1}}{r}g\|_{L^2(\Omega)} \tag{1.34}$$

for any scalar function $g\in W^{1,2}((1,\infty);r\,\mathrm{d}r)$ with the dissipation from the Laplacian, we can close the energy computation for all $\alpha\in\mathbb{R}\setminus\{0\}$. In solving the nonlinear problem the key observation is that the product of the radial parts in the nonlinear term can always be written in a gradient form and thus regarded as a pressure term, which yields the identity

$$v^\perp \mathrm{rot}\, v = v_0^\perp \mathrm{rot}\,\mathcal{Q}_0 v + (\mathcal{Q}_0 v)^\perp \mathrm{rot}\, v_0 + (\mathcal{Q}_0 v)^\perp \mathrm{rot}\,\mathcal{Q}_0 v + \nabla\tilde{q} \tag{1.35}$$

for a suitable $\tilde{q}$. Since $\mathcal{P}_0\big(v_0^\perp \mathrm{rot}\,\mathcal{Q}_0 v + (\mathcal{Q}_0 v)^\perp \mathrm{rot}\, v_0\big) = 0$ as long as $\mathcal{Q}_0 v\in W_0^{1,2}(\Omega)^2$ the radial part of the velocity in the right-hand side of (1.35) (neglecting $\nabla\tilde{q}$) belongs to $L^1(\Omega)^2$, which is the same summability as the space Y. This is a brief explanation for the reason why we can close the nonlinear estimate and solve $(\widetilde{\mathrm{NS}}_\alpha)$ in X for a source $f\in Y$.

Our second result is focused on the fast rotation case $|\alpha|\gg 1$. In this regime there are three fundamental mechanisms in our system:

(I) Axisymmetrization due to the fast rotation of the obstacle,

(II) Presence of a boundary layer for the nonradial part of the flow due to the no-slip boundary condition,

(III) Diffusion in high angular frequencies due to the viscosity.

(I) and (II) are potentially in a competitive relation, for the no-slip boundary condition and the boundary layer can suppress the effect of the fast rotation to some extent.

(II) and (III) are also competitive. Indeed, it is natural that if the viscosity is strong enough then the boundary layer is diffused and is no longer observable. The important task here is to determine the regime of angular frequencies in which the boundary layer appears, and to estimate its thickness. We show that the boundary layer appears in the regime $1 \leq |n| \ll O(|\alpha|^{\frac{1}{2}})$, and the thickness is $(2|\alpha n|)^{-\frac{1}{3}}$ for each n in this regime. By performing the boundary layer analysis we can improve the result stated in (ii) of Theorem 1.2.3 in the regime $|\alpha| \gg 1$, which is briefly described as follows.

Theorem 1.2.4 *There exists $\gamma \in (0, \infty)$ such that the following statement holds. For all sufficiently large $|\alpha| \geq 1$ and for any external force $f = (\mathcal{P}_0 f, \mathcal{Q}_0 f) \in Y$ satisfying*

$$\|(\mathcal{P}_0 f)_\theta\|_{L^1(\Omega)} \leq \gamma |\alpha|^{\frac{1}{3}}, \qquad \|\mathcal{Q}_0 f\|_{L^2(\Omega)} \leq \gamma |\alpha|^{\frac{1}{3}}, \tag{1.36}$$

there exists a unique solution $(v, \nabla q) \in X \cap W^{2,1}_{\mathrm{loc}}(\overline{\Omega})^2 \times L^1_{\mathrm{loc}}(\overline{\Omega})^2$ to $(\widetilde{\mathrm{NS}}_\alpha)$ satisfying

$$\|\mathcal{P}_0 v\|_{L^\infty(\Omega)} + \|\nabla \mathcal{P}_0 v\|_{L^\infty(\Omega)} \leq C\|(\mathcal{P}_0 f)_\theta\|_{L^1(\Omega)} + C|\alpha|^{-1}\|\mathcal{Q}_0 f\|^2_{L^2(\Omega)}, \tag{1.37}$$

$$\begin{aligned}\big(\log|\alpha|\big)^{-\frac{1}{2}}|\alpha|^{\frac{1}{2}} \sum_{|n|\geq 1} \|\mathcal{P}_n v\|_{L^\infty(\Omega)} + |\alpha|^{\frac{2}{3}}\|\mathcal{Q}_0 v\|_{L^2(\Omega)} + |\alpha|^{\frac{1}{3}}\|\nabla \mathcal{Q}_0 v\|_{L^2(\Omega)} \\ \leq C\|\mathcal{Q}_0 f\|_{L^2(\Omega)}.\end{aligned} \tag{1.38}$$

By fixing the external force f we can state Theorem 1.2.4 in a different but more convenient way to understand the qualitative behavior of solutions in the fast rotation limit.

Theorem 1.2.5 *For any $f = (\mathcal{P}_0 f, \mathcal{Q}_0 f) \in Y$ there is sufficiently large $\alpha_0 = \alpha_0(\|f\|_Y) \geq 1$ such that the following statements hold. If $|\alpha| \geq \alpha_0$ then there exists a unique solution $(v^{(\alpha)}, \nabla q^{(\alpha)}) \in X \cap W^{2,1}_{\mathrm{loc}}(\overline{\Omega})^2 \times L^1_{\mathrm{loc}}(\overline{\Omega})^2$ to $(\widetilde{\mathrm{NS}}_\alpha)$ satisfying*

$$\|v^{(\alpha)} - v_0^{\mathrm{linear}}\|_{L^\infty(\Omega)} \leq C(\log|\alpha|)^{\frac{1}{2}}|\alpha|^{-\frac{1}{2}}. \tag{1.39}$$

Here v_0^{linear} is the solution to the linearized problem (S_α) with f replaced by $\mathcal{P}_0 f$ which is in fact independent of α, and C depends only on $\|f\|_Y$.

By going back to (NS_α), Theorems 1.2.4 and 1.2.5 show that there exists a unique solution $u = u^{(\alpha)}$ of (NS_α) which satisfies

$$\|u^{(\alpha)} - \alpha U - v_0^{\mathrm{linear}}\|_{L^\infty(\Omega)} \leq C(\log|\alpha|)^{\frac{1}{2}}|\alpha|^{-\frac{1}{2}}, \quad |\alpha| \gg 1. \tag{1.40}$$

The new ingredient of the proof of Theorems 1.2.4 and 1.2.5 consists in refined estimates for the linearized problem (S_α), while the nonlinear problem is handled exactly in the same manner as in the proof of Theorem 1.2.3. For (S_α) we observe that in polar coordinates the angular mode n of the streamfunction ψ satisfies the ODE in $r \in (1, \infty)$

$$\Big(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{n^2}{r^2} + i\alpha n\big(1 - \frac{1}{r^2}\big)\Big)\big(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{n^2}{r^2}\big)\psi_n = 0, \tag{1.41}$$

with the boundary condition $\psi_n(1) = \frac{\mathrm{d}\psi_n}{\mathrm{d}r}(1) = 0$ when $|n| \geq 1$. The thickness of the boundary layer originating from the fast rotation is determined by the balance between $\frac{\mathrm{d}^2}{\mathrm{d}r^2}$ and $i\alpha n(1 - \frac{1}{r^2}) \approx 2i\alpha n(r-1)$ near $r = 1$ as long as the dissipation $-\frac{n^2}{r^2} \approx -n^2$ is moderate. This implies that the thickness is $|2\alpha n|^{-\frac{1}{3}}$. Then the regime of n where the dissipation is relatively moderate is estimated from the condition $n^2 \ll \frac{\mathrm{d}^2}{\mathrm{d}r^2} \approx O(|\alpha n|^{\frac{2}{3}})$, which leads to $|n| \ll O(|\alpha|^{\frac{1}{2}})$. From this observation we employ the boundary layer analysis when the angular frequency n satisfies $1 \leq |n| \ll O(|\alpha|^{\frac{1}{2}})$, while we just apply an energy estimate again in the regime $|n| \geq O(|\alpha|^{\frac{1}{2}})$ where the boundary layer is no longer present.

The result in this chapter is based on a joint work with Isabelle Gallagher and Yasunori Maekawa, which corresponds to the paper [22] published in Mathematische Nachrichten.

1.2.3 Note on the stability of planar stationary flows in an exterior domain without symmetry (Chapter 4)

In Chapter 4 we consider the following perturbed Stokes equations for viscous incompressible flows in a two-dimensional exterior domain Ω with a smooth boundary.

$$\begin{cases} \partial_t v - \Delta v + V\cdot\nabla v + v\cdot\nabla V + \nabla q \,=\, 0\,, & t>0\,,\ x\in\Omega\,, \\ \operatorname{div} v \,=\, 0\,, & t\geq 0\,,\ x\in\Omega\,, \\ v|_{\partial\Omega} \,=\, 0\,, & t>0\,, \\ v|_{t=0} \,=\, v_0\,, & x\in\Omega\,. \end{cases} \tag{PS}$$

Here the given vector field $V = V(x) = (V_1(x), V_2(x))^\top$ is assumed to decay in the scale-critical order $V(x) = O(|x|^{-1})$ at spatial infinity. The exterior domain Ω is assumed to be contained by the domain exterior to the radius-$\frac{1}{2}$ disk $\{x\in\mathbb{R}^2 \mid |x| > \frac{1}{2}\}$.

The aim of Chapter 4 is to investigate the L^p-L^q estimates to the equations (PS), under a suitable condition on the vector field V. The equations (PS) have been studied as the linearization of the Navier-Stokes equations around a stationary solution V. The analysis of the two-dimensional problem as (PS) is, contrary to the three-dimensional case, quite complicated and there is no general result so far; see Borchers and Miyakawa [5] for the results in three dimensions. The difficulty arises from the unavailability of the Hardy inequality

$$\Big\|\frac{f}{|x|}\Big\|_{L^2(\Omega)} \leq C\|\nabla f\|_{L^2(\Omega)}\,, \qquad f\in \dot{W}^{1,2}_0(\Omega) \,=\, \overline{C^\infty_0(\Omega)}^{\|\nabla f\|_{L^2(\Omega)}}\,. \tag{1.42}$$

The validity of this bound is well known for three-dimensional exterior domains, and the three-dimensional results essentially rely on the inequality (1.42). One can recover the Hardy inequality in the two-dimensional case if the factor $|x|^{-1}$ in the left-hand side of (1.42) is replaced with a logarithmic correction $|x|^{-1}\log(e+|x|)^{-1}$, but this inequality has only a narrow application in our scale-critical framework. Another way to recover the inequality (1.42) is to impose the symmetry on both Ω and f, and such an inequality is applied in the analysis of (PS) for the case when V is symmetric. Yamazaki [59] proves the L^p-L^q estimates to (PS) with the symmetric Navier-Stokes flow $V(x) = O(|x|^{-1})$, under the symmetry conditions on both the domain and given data. We note that these estimates imply the asymptotic stability of V under symmetric initial L^2-perturbations.

An important remark is given by Russo [54] concerning the Hardy-type inequality in two-dimensional exterior domains without symmetry. Let us introduce the next scale-critical radial flow $W = W(x)$, which is called the flux carrier.

$$W(x) \,=\, \frac{x}{|x|^2}\,, \quad x\in\mathbb{R}^2\setminus\{0\}\,. \tag{1.43}$$

Then, from the existence of a potential to $W(x) = \nabla \log |x|$, one can show that the following Hardy-type inequality holds in the L^2-inner product $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$:

$$|\langle u \cdot \nabla u \,, W \rangle_{L^2(\Omega)}| \le C \|\nabla u\|_{L^2(\Omega)}^2 \,, \qquad u \in \dot{W}_{0,\sigma}^{1,2}(\Omega) \;=\; \overline{C_{0,\sigma}^{\infty}(\Omega)}^{\|\nabla u\|_{L^2(\Omega)}} \,. \tag{1.44}$$

By the energy method using (1.44), Guillod [27] proves the global L^2-stability of the flux carrier δW when the flux δ is small enough. On the other hand, the validity of the inequality (1.44) essentially depends on the potential property of W. Indeed, as is pointed out in [27], the bound (1.44) breaks down if W is replaced by the next rotating flow $U = U(x)$:

$$U(x) \;=\; \frac{x^{\perp}}{|x|^2} \,, \quad x \in \mathbb{R}^2 \setminus \{0\} \,. \tag{1.45}$$

Hence, if we consider the problem (PS) with $V = \alpha U$, $\alpha \in \mathbb{R} \setminus \{0\}$, the linearized term $\alpha(U \cdot \nabla v + v \cdot \nabla U)$ can no more be regarded as a perturbation from the Laplacian, and we cannot avoid the difficulty coming from the lack of the Hardy inequality. Maekawa [44] studies the stability of the flow αU in the exterior unit disk. The symmetry of the domain allows us to express the solution to the problem (PS) explicitly through the Dunford integral of the resolvent operator. Based on this representation formula, [44] obtains the L^p-L^q estimates to (PS) with $V = \alpha U$ for small but non-zero α, and shows the asymptotic L^2-stability of αU if α and initial perturbations are sufficiently small.

Our first motivation is to generalize the result in [44] to the case when the domain loses symmetry (and the second one is explained in Remark 4.1.2 (3) of Chapter 4). Let us prepare the assumptions on the domain Ω and the stationary flow V in (PS). We denote by $B_\rho(0)$ the two-dimensional disk of radius $\rho > 0$ centered at the origin.

Assumption 1.2.6 (1) *There is a positive constant $d \in (0, \frac{1}{4})$ such that the complement of the domain Ω satisfies*

$$\overline{B_{1-2d}(0)} \subset \Omega^{\mathrm{c}} \subset \overline{B_{1-d}(0)} \,. \tag{1.46}$$

(2) *Let the constants $\alpha \in (0,1)$ and $d \in (0, \frac{1}{4})$ in* (1.46) *be sufficiently small. Then the vector field V in* (PS) *satisfies* $\operatorname{div} V = 0$ *in Ω and the asymptotic behavior*

$$V(x) \;=\; \beta U(x) + R(x) \,, \quad x \in \Omega \,, \tag{1.47}$$

where $U(x)$ is the rotating flow in (1.45)*. The constant β and the remainder $R(x)$ are assumed to satisfy the following conditions with some $\gamma \in (\frac{1}{2}, 1)$ and $\kappa \in (0,1)$:*

$$\beta \;=\; \alpha + \tilde{\alpha}_d \,, \qquad |\tilde{\alpha}_d| \le C d \,, \qquad \beta \in (0,1) \,, \tag{1.48}$$

$$\sup_{x \in \Omega} |x|^{1+\gamma} |R(x)| \le C \beta^{\kappa} d \,, \tag{1.49}$$

where the constant C depends only on γ.

Remark 1.2.7 (1) Formally taking $d = 0$ in (1.46)–(1.49) we obtain the flow $V = \alpha U$ in the exterior disk $\Omega = \mathbb{R}^2 \setminus \overline{B_1(0)}$, which solves the two-dimensional stationary Navier-Stokes equations (SNS): $-\Delta u + u \cdot \nabla u + \nabla p = f$, $\operatorname{div} u = 0$ in Ω, $u = b$ on $\partial\Omega$, and $u \to 0$ as $|x| \to \infty$ with $f = 0$ and $b = \alpha x^{\perp}$. The vector field V in (1.47)–(1.49) describes the flow around αU created from a small perturbation to the exterior disk, and hence, one can naturally expect the existence of such solutions to (SNS) if f and $b - \alpha x^{\perp}$ are sufficiently small with respect to $0 < d \ll 1$. Indeed, imposing the symmetry on the domain perturbation in (1.46), we can construct the Navier-Stokes flow V satisfying at least (1.47) and (1.48) for small symmetric given data by the energy method and the recovered Hardy-inequality (1.42) thanks to the symmetry of the domain Ω and the remainder R.

(2) The novelty of our assumption is that we do not impose the symmetry either on the domain Ω and the flow V, and it is a crucial assumption for the stability analysis in [59] to resolve the difficulty related to the lack of the Hardy inequality. While one can realize the exterior disk case in [44] by putting $d = 0$ to (1.46)–(1.49) formally. In this sense, the assumption gives a generalization of the setting in [44] to non-symmetric domain cases.

Before stating the main result in Chapter 4, let us introduce some notations and basic facts related to the problem (PS). We denote by $L^2_\sigma(\Omega)$ the L^2-closure of $C^\infty_{0,\sigma}(\Omega)$. The orthogonal projection $\mathbb{P} : L^2(\Omega)^2 \to L^2_\sigma(\Omega)$ is called the Helmholtz projection. Then the Stokes operator $\mathbb{A}$ with the domain $D_{L^2}(\mathbb{A}) = L^2_\sigma(\Omega) \cap W^{1,2}_0(\Omega)^2 \cap W^{2,2}(\Omega)^2$ is defined as $\mathbb{A} = -\mathbb{P}\Delta$, and it is well known that the Stokes operator is nonnegative and self-adjoint in $L^2_\sigma(\Omega)$. Finally we define the perturbed Stokes operator $\mathbb{A}_V$ as

$$\begin{aligned} D_{L^2}(\mathbb{A}_V) &= D_{L^2}(\mathbb{A})\,, \\ \mathbb{A}_V v &= \mathbb{A} v + \mathbb{P}(V \cdot \nabla v + v \cdot \nabla V)\,. \end{aligned} \tag{1.50}$$

The perturbation theory for sectorial operators implies that $-\mathbb{A}_V$ generates a C_0-analytic semigroup in $L^2_\sigma(\Omega)$. We denote this semigroup by $e^{-t\mathbb{A}_V}$. Then our main result is stated as follows. Let d, β, and κ be the constants in Assumption 1.2.6.

Theorem 1.2.8 *There are positive constants β_* and μ_* such that if $\beta \in (0, \beta_*)$ and $d \in (0, \mu_* \beta^2)$. then the following statement holds. Let $q \in (1, 2]$. Then we have*

$$\|e^{-t\mathbb{A}_V} f\|_{L^2(\Omega)} \leq \frac{C}{\beta^2} t^{-\frac{1}{q}+\frac{1}{2}} \|f\|_{L^q(\Omega)}\,, \qquad t > 0\,, \tag{1.51}$$

$$\|\nabla e^{-t\mathbb{A}_V} f\|_{L^2(\Omega)} \leq \frac{C}{\beta^2} t^{-\frac{1}{q}} \|f\|_{L^q(\Omega)}\,, \qquad t > 0\,, \tag{1.52}$$

for $f \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$. Here the constant C is independent of β and depends on q.

As an application of Theorem 1.2.8, we can prove the local L^2-stability of V for the Navier-Stokes equations, as is stated in Theorem 4.1.4 in Chapter 4.

The proof of Theorem 1.2.8 relies on the resolvent estimate to the perturbed Stokes operator $\mathbb{A}_V$. Since the difference $\mathbb{A}_V - \mathbb{A}$ is relatively compact to $\mathbb{A}$ in $L^2_\sigma(\Omega)$, one can show that the spectrum of $-\mathbb{A}_V$ has the structure $\sigma(-\mathbb{A}_V) = (-\infty, 0] \cup \sigma_{\rm disc}(-\mathbb{A}_V)$ in $L^2_\sigma(\Omega)$, where $\sigma_{\rm disc}(-\mathbb{A}_V)$ denotes the set of discrete spectrum of $-\mathbb{A}_V$; see [44, Lemma 2.11 and Proposition 2.12]. By using the identity $v \cdot \nabla v = \frac{1}{2}\nabla |v|^2 + v^{\perp} {\rm rot}\, v$ with ${\rm rot}\, v = \partial_1 v_2 - \partial_2 v_1$ and ${\rm rot}\, U = 0$ in $x \in \Omega$, we can write the resolvent problem associated with (PS) as

$$\begin{cases} \lambda v - \Delta v + \beta U^{\perp} {\rm rot}\, v + {\rm div}\, (R \otimes v + v \otimes R) + \nabla q = f\,, & x \in \Omega\,, \\ {\rm div}\, v = 0\,, & x \in \Omega\,, \\ v|_{\partial\Omega} = 0\,. \end{cases} \tag{RS}$$

Here $\lambda \in \mathbb{C}$ is the resolvent parameter and we have used the conditions ${\rm div}\, v = {\rm div}\, R = 0$ to derive $R \cdot \nabla v + v \cdot \nabla R = {\rm div}\, (R \otimes v + v \otimes R)$. Hence, the proof of Theorem 1.2.8 is complete as soon as we show that there is a sector Σ included in the resolvent set $\rho(-\mathbb{A}_V)$, and that the following estimates to (RS) hold for $q \in (1,2]$ and $f \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$:

$$\begin{aligned} \|(\lambda + \mathbb{A}_V)^{-1} f\|_{L^2(\Omega)} &\le \frac{C}{\beta^2} |\lambda|^{-\frac{3}{2}+\frac{1}{q}} \|f\|_{L^q(\Omega)}\,, \quad \lambda \in \Sigma\,, \\ \|\nabla(\lambda + \mathbb{A}_V)^{-1} f\|_{L^2(\Omega)} &\le \frac{C}{\beta^2} |\lambda|^{-1+\frac{1}{q}} \|f\|_{L^q(\Omega)}\,, \quad \lambda \in \Sigma\,. \end{aligned} \tag{1.53}$$

Let us prepare the ingredients for the proof of the resolvent estimates (1.53). Our approach is based on the energy method to (RS), and thus one of the most important steps is to obtain the estimate for the term $|\langle \beta U^\perp \mathrm{rot}\, v\,, v\rangle_{L^2(\Omega)}|$ which enables us to close the energy computation. Again we note that the bound $|\langle \beta U^\perp \mathrm{rot}\, v\,, v\rangle_{L^2(\Omega)}| \le C\beta \|\nabla v\|^2_{L^2(\Omega)}$ is no longer available contrary to the three-dimensional cases.

Firstly let us examine the next inequality containing the parameter $T \gg 1$:

$$|\langle \beta U^\perp \mathrm{rot}\, v, v\rangle_{L^2(\Omega)}| \le \frac{\beta}{T} \|\nabla v\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + C\beta\Theta(T) \|\nabla v\|^2_{L^2(\Omega)}\,, \tag{1.54}$$

where the function $\Theta(T)$ satisfies $\Theta(T) \approx \log T$ if $T \gg 1$. This inequality leads to the closed energy computation for (RS), as long as the coefficient $C\beta\Theta(T)$ is small enough so that the second term in the right-hand side of (1.54) can be controlled by the dissipation from the Laplacian in (RS). However, this observation does not give the information about the spectrum of $-\mathbb{A}_V$ near the origin. More precisely, we cannot close the energy computation when the resolvent parameter λ is exponentially small with respect to β, that is, when $0 < |\lambda| \le O(e^{-\frac{1}{\beta}})$. We emphasize that this difficulty is essentially due to the unavailability of the Hardy inequality (1.42) in two-dimensional exterior domains.

To overcome the difficulty for the case $0 < |\lambda| \le O(e^{-\frac{1}{\beta}})$, we rely on the representation formula to the resolvent problem in the exterior unit disk established in [44]. Since the restriction $(v|_{\{|x|>1\}}, q|_{\{|x|>1\}})$ gives a unique solution to the next problem for (w, r):

$$\left\{ \begin{aligned} \lambda w - \Delta w + \beta U^\perp \mathrm{rot}\, w + \nabla r &= -\mathrm{div}\,(R \otimes v + v \otimes R) + f\,, \quad |x| > 1\,, \\ \mathrm{div}\, w &= 0\,, \quad |x| > 1\,, \\ w|_{\{|x|=1\}} &= v|_{\{|x|=1\}}\,, \end{aligned} \right. \tag{RS$^{\mathrm{ed}}$}$$

we can study the a priori estimates of $w = v|_{\{|x|>1\}}$ based on the solution formula to (RS$^{\mathrm{ed}}$). Then a detailed calculation shows that $|\langle \beta U^\perp \mathrm{rot}\, v, v\rangle_{L^2(\{|x|>1\})}|$ satisfies

$$\begin{aligned} &|\langle \beta U^\perp \mathrm{rot}\, v, v\rangle_{L^2(\{|x|>1\})}| \\ &\le \frac{C}{\beta^4} \big(\|R \otimes v + v \otimes R\|_{L^2(\Omega)} + \beta \sum_{|n|=1} \|\mathcal{P}_n v\|_{L^\infty(\{|x|=1\})} \big)^2 \\ &\quad + \frac{C}{\beta^4} |\lambda|^{-2+\frac{2}{q}} \|f\|^2_{L^q(\Omega)} + C\beta \|\nabla v\|^2_{L^2(\Omega)}\,, \end{aligned} \tag{1.55}$$

where $\mathcal{P}_n v$ denotes the Fourier n-mode of $v|_{\{|x|\ge 1\}}$; see (1.23) in Subsection 1.2.2 for the definition. Once we obtain (1.55) then the estimate of $|\langle \beta U^\perp \mathrm{rot}\, v, v\rangle_{L^2(\Omega)}|$ is derived by using the Poincaré inequality on the bounded domain $\Omega \setminus \{|x| \ge 1\}$. However, in closing the energy computation, we need to be careful about the β-singularity in the coefficients in (1.55). In fact, the first term in the right-hand side of (1.55) has to be controlled by the dissipation as

$$\big(\|R \otimes v + v \otimes R\|_{L^2(\Omega)} + \beta \sum_{|n|=1} \|\mathcal{P}_n v\|_{L^\infty(\{|x|=1\})}\big)^2 \leq \frac{C}{\beta^4}(\beta^\kappa d + \beta d^{\frac12})^2 \|\nabla v\|^2_{L^2(\Omega)}\,,$$

and then the smallness of $C(\beta^\kappa d + \beta d^{\frac12})^2 \beta^{-4} \ll 1$ is required in order to close the energy computation. This condition is achieved by imposing the smallness on the distance d between the domain Ω and the exterior unit disk, which is introduced in Assumption 1.2.6.

Next we pay close attention to the β-dependencies appearing in Theorem 1.2.8. If we consider the limit case $d = 0$ and $V = \alpha U$ in Assumption 1.2.6, then the term

$$\beta U^\perp \mathrm{rot}\, v + \mathrm{div}\,(R \otimes v + v \otimes R) \,=\, \alpha U^\perp \mathrm{rot}\, v$$

in (RS) has an oscillation effect on the solutions in the exterior disk $\Omega = \{|x| > 1\}$ at least when $\lambda = 0$. Indeed, for the solutions to (RS) with $\lambda = 0$, this effect leads to the faster spatial decay compared with the case $\alpha = 0$ (i.e. the Stokes equations case), and this observation is indeed an important step in Hillairet and Wittwer [32] to prove the existence of the Navier-Stokes flows around αU in the exterior disk when the rotation α is large, as explained in Subsection 1.1. However, contrary to the stationary problem, the situation becomes more complicated if we consider the nonstationary problem requiring the analysis of (RS) for nonzero $\lambda \in \mathbb{C} \setminus \{0\}$, since there is an interaction between the two oscillation effects due to the terms λv and $\alpha U^\perp \mathrm{rot}\, v$ in (RS). In fact, even in the exterior disk, a detailed analysis to the representation of the resolvent operator suggests the existence of a time-frequency domain, which we call the *nearly-resonance regime*, where the oscillation effect from $\alpha U^\perp \mathrm{rot}\, v$ is drastically weakened by the one from λv and the α-singularity appears in the operator norm of the resolvent. The existence of the nearly-resonance regime yields that the stability of the αU-type flows is sensitive under the perturbation of the domain. This is the reason why the distance d between the fluid domain Ω and the exterior disk is assumed to be small depending on $\beta = \alpha + \tilde{\alpha}_d$ in Theorem 1.2.8. Additionally, Lemma 4.5.6 in Chapter 4 implies that the nearly-resonance regime lies in the annulus $e^{-\frac{c}{\beta^2}} \leq |\lambda| \leq e^{-\frac{c'}{\beta}}$ in the complex plane. As far as the author knows, the existence of such time-frequency domain and the qualitative analysis seem to be new and have not been achieved before.

In the last section we discuss some future work. We proved the L^p-L^q estimates for the semigroup $e^{-t\mathbb{A}_V}$ in Theorem 1.2.8, however, they are singular in the small parameter β. Especially, these singularities lead to the restriction on the size of the initial data in the stability analysis. Our aim is to derive the semigroup estimates without the β-singularity by allowing *slower decays in time*. To make the problem simple, we take a formal limit $d = 0$ in Assumption 1.2.6. Then we obtain the rotating flow $\alpha U = \alpha \frac{x^\perp}{|x|^2}$, $\alpha \in (0,1)$, on the exterior disk $D = \{x \in \mathbb{R}^2 \mid |x| > 1\}$, and the perturbed Stokes equations (PS$_\alpha$) written as

$$\begin{cases} \partial_t v - \Delta v + \alpha(U \cdot \nabla v + v \cdot \nabla U) + \nabla q \,=\, 0\,, & t > 0\,,\ x \in D\,, \\ \mathrm{div}\, v \,=\, 0\,, & t \geq 0\,,\ x \in D\,, \\ v|_{\partial D} \,=\, 0\,, & t > 0\,, \\ v|_{t=0} \,=\, v_0\,, & x \in D\,. \end{cases} \tag{PS$_\alpha$}$$

We define the perturbed Stokes operator $\mathbb{A}_\alpha$ associated with the problem (PS$_\alpha$) by (1.50) with $\mathbb{A}_V$ and V respectively replaced with $\mathbb{A}_\alpha$ and αU. Again, the perturbation theory for sectorial operators leads to the generation of an analytic semigroup by $-\mathbb{A}_\alpha$ in $L^2_\sigma(D)$, which we denote as $e^{-t\mathbb{A}_\alpha}$. Our main result in the last section is the following exponentially large time estimates for the semigroup $e^{-t\mathbb{A}_\alpha}$:

Theorem 1.2.9 *There is a positive constant α_* such that if $\alpha \in (0, \alpha_*)$ then the following statement holds. Let $q \in (1, 2]$. Then we have*

$$\|e^{-t\mathbb{A}_\alpha} f\|_{L^2(D)} \le \begin{cases} Ct^{-\frac{1}{q}+\frac{1}{2}} \|f\|_{L^q(D)}\,, & t \in (0, e^{\frac{1}{6\alpha}}]\,, \\ C\alpha(\log t)^3 t^{-\frac{1}{q}+\frac{1}{2}} \|f\|_{L^q(D)}\,, & t \in (e^{\frac{1}{6\alpha}}, \infty)\,, \end{cases} \tag{1.56}$$

$$\|\nabla e^{-t\mathbb{A}_\alpha} f\|_{L^2(D)} \le \begin{cases} Ct^{-\frac{1}{q}} \|f\|_{L^q(D)}\,, & t \in (0, e^{\frac{1}{6\alpha}}]\,, \\ C\alpha^2(\log t)^{\frac{11}{2}} t^{-\frac{1}{q}} \|f\|_{L^q(D)}\,, & t \in (e^{\frac{1}{6\alpha}}, \infty)\,, \end{cases} \tag{1.57}$$

for $f \in L^2_\sigma(D) \cap L^q(D)^2$. Here the constant C is independent of α and depends on q.

Remark 1.2.10 Compared with the L^p-L^q estimate in Theorem 1.2.8, the new estimate in (1.56) or (1.57) is uniformly bounded in sufficiently small $\alpha \in (0, 1)$ for each fixed $t \in (0, \infty)$, while the bound in the right-hand side decays slower or even grows in time.

By applying Theorem 1.2.9, we can prove the nonlinear stability of αU for fast decaying initial data under a milder smallness condition compared with the result in the main sections.

The proof of Theorem 1.2.9 is carried out by resolving the α-singularity in (1.53) with $\mathbb{A}_V$, Ω, and β respectively replaced with $\mathbb{A}_\alpha$, D, and α. This resolution causes the appearance of the logarithm $|\log|\lambda||$ in the resolvent estimates, which finally leads to the logarithmic loss in the time decay estimates in Theorem 1.2.9; see the proof of Theorem 4.6.7 in Subsection 4.6.1 for the correspondence between the singularities $\frac{1}{\alpha}$ and $|\log|\lambda||$.

The result in this chapter basically corresponds to the submitted paper [29], while the work in the last section is in preparation.

Chapter 2

On stationary Navier-Stokes flows around a rotating obstacle in two-dimensions

Abstract We study the two-dimensional stationary Navier-Stokes equations describing the flows around a rotating obstacle. The unique existence of solutions and their asymptotic behavior at spatial infinity are established when the rotation speed of the obstacle and the given exterior force are sufficiently small.

2.1 Introduction

In this chapter we consider the two-dimensional Navier-Stokes equations for viscous incompressible flows around a rotating obstacle in two-dimensions:

$$\begin{cases} \partial_t v - \Delta v + v\cdot\nabla v + \nabla q \,=\, g\,, \quad \operatorname{div} v \,=\, 0\,, \qquad t>0\,,\ y\in\Omega(t)\,, \\ \qquad\qquad\qquad\qquad\quad v \,=\, \alpha y^{\perp}\,, \qquad t>0\,,\ y\in\partial\Omega(t)\,, \\ \qquad\qquad\qquad\qquad\quad v \,\to\, 0\,, \qquad t>0\,,\ |y|\to\infty\,. \end{cases} \tag{2.1}$$

Here $v = v(y,t) = (v_1(y,t), v_2(y,t))^\top$ and $q = q(y,t)$ are respectively the unknown velocity field and pressure field, and $g = g(y,t) = (g_1(y,t), g_2(y,t))^\top$ is a given external force. The time-dependent domain $\Omega(t)$ is defined as

$$\begin{aligned} \Omega(t) \,&=\, \left\{ y\in\mathbb{R}^2 \mid y = O(\alpha t)x\,,\ x\in\Omega \right\}, \\ O(\alpha t) \,&=\, \begin{pmatrix} \cos\alpha t & -\sin\alpha t \\ \sin\alpha t & \cos\alpha t \end{pmatrix}, \end{aligned} \tag{2.2}$$

where Ω is an exterior domain in $\mathbb{R}^2$ with a smooth compact boundary, while the real number $\alpha \neq 0$ represents the rotation speed of the obstacle $\Omega^{\mathrm{c}} = \mathbb{R}^2 \setminus \Omega$. We use the standard notation for derivatives: $\partial_t = \frac{\partial}{\partial t}$, $\partial_j = \frac{\partial}{\partial x_j}$, $\Delta = \sum_{j=1}^2 \partial_j^2$, $\operatorname{div} v = \sum_{j=1}^2 \partial_j v_j$, $v\cdot\nabla v = \sum_{j=1}^2 v_j \partial_j v$. The vector $x^\perp \in \mathbb{R}^2$ denotes the perpendicular of x: $x^\perp = (-x_2, x_1)^\top$. The system (2.1) describes the flow around the obstacle Ω^{c} which rotates with a constant angular velocity α, and the condition $v(t,y) = \alpha y^\perp$ on the boundary $\partial\Omega(t)$ represents the no-slip boundary condition. To remove the difficulty due to the time dependence of the fluid domain, it is more convenient to analyze the system (2.1) in the reference frame:

$$y = O(\alpha t)x\,, \quad u(x,t) = O(\alpha t)^\top v(y,t)\,, \quad p(x,t) = q(y,t)\,,$$
$$f(x,t) = O(\alpha t)^\top g(y,t)\,,$$

for $t \geq 0$ and $x \in \Omega$. Here $M^\top$ denotes the transpose of a matrix M. Then (2.1) is equivalent with the equations in the time-independent domain Ω:

$$\begin{cases} \partial_t u - \Delta u - \alpha(x^\perp \cdot \nabla u - u^\perp) + \nabla p = -u \cdot \nabla u + f\,, \quad \operatorname{div} u = 0\,, \quad t > 0\,,\ x \in \Omega\,, \\ u = \alpha x^\perp\,, \quad t > 0\,,\ x \in \partial\Omega\,, \\ u \to 0\,, \quad t > 0\,,\ |x| \to \infty\,. \end{cases}$$

In this chapter we are interested in the stationary solutions to this system. Thus we assume that f is independent of t and consider the next system

$$\begin{cases} -\Delta u - \alpha(x^\perp \cdot \nabla u - u^\perp) + \nabla p = -u \cdot \nabla u + f\,, \quad \operatorname{div} u = 0\,, \quad x \in \Omega\,, \\ u = \alpha x^\perp\,, \quad x \in \partial\Omega\,, \\ u \to 0\,, \quad |x| \to \infty\,. \end{cases} \tag{NS$_\alpha$}$$

To state our result let us introduce the function spaces used in this chapter. As usual, the class $C^\infty_{0,\sigma}(\Omega)$ is defined as the set of smooth divergence free vector fields with compact support in Ω, and the homogeneous space $\dot W^{1,2}_{0,\sigma}(\Omega)$ is the closure of $C^\infty_{0,\sigma}(\Omega)$ with respect to the norm $\|\nabla f\|_{L^2(\Omega)}$. For a fixed number $s \geq 0$ we also introduce the weighted L^∞ space $L^\infty_s(\Omega)$ and its subspace $L^\infty_{s,0}(\Omega)$ as follows.

$$\begin{aligned} L^\infty_s(\Omega) &= \big\{ f \in L^\infty(\Omega) \mid (1+|x|)^s f \in L^\infty(\Omega) \big\}\,, \\ L^\infty_{s,0}(\Omega) &= \big\{ f \in L^\infty_s(\Omega) \mid \lim_{R\to\infty} \operatorname{ess.sup}_{|x|\geq R} |x|^s |f(x)| = 0 \big\}\,. \end{aligned} \tag{2.3}$$

These are Banach spaces equipped with the natural norm

$$\|f\|_{L^\infty_s(\Omega)} = \operatorname{ess.sup}_{x\in\Omega} (1+|x|)^s |f(x)|\,,$$

and the set of functions with compact support is dense in $L^\infty_{s,0}(\Omega)$. Moreover, for any bounded sequence $\{f_n\}$ in $L^\infty_s(\Omega)$ (or $L^\infty_{s,0}(\Omega)$) with $\|f_n\|_{L^\infty_s(\Omega)} \leq M$ for some positive number M, there exists a subsequence $\{f_{n'}\}$ which converges in the weak-star topology in the sense that there is $f \in L^\infty_s(\Omega)$ (or $f \in L^\infty_{s,0}(\Omega)$, respectively) such that

$$\lim_{n'\to\infty} \int_\Omega f_{n'}(x)\phi(x)(1+|x|)^s \,\mathrm{d}x = \int_\Omega f(x)\phi(x)(1+|x|)^s \,\mathrm{d}x\,, \quad \text{for any } \phi \in L^1(\Omega)\,,$$

and $\|f\|_{L^\infty_s(\Omega)} \leq M$. We denote by $L^2_{\mathrm{loc}}(\overline{\Omega})$ the set of functions which belong to $L^2(\Omega\cap K)$ for any compact set $K \subset \mathbb{R}^2$, and $W^{k,2}_{\mathrm{loc}}(\overline{\Omega})$, $k \in \mathbb{N}$, is defined in a similar manner.

The main result of this chapter is stated as follows.

Theorem 2.1.1 *There exists $\epsilon = \epsilon(\Omega) > 0$ such that the following statement holds. Assume that $f \in L^2(\Omega)^2$ is of the form $f = \operatorname{div} F = (\partial_1 F_{11} + \partial_2 F_{12}, \partial_1 F_{21} + \partial_2 F_{22})^\top$ with some $F = (F_{ij})_{1\leq i,j\leq 2} \in L^\infty_2(\Omega)^{2\times 2}$ and $F_{12} - F_{21} \in L^1(\Omega)$. If $\alpha \neq 0$ and*

$$|\alpha|^{\frac12} \big|\log|\alpha|\big| + |\alpha|^{-\frac12} \big|\log|\alpha|\big| \big(\|f\|_{L^2(\Omega)} + \|F\|_{L^\infty_2(\Omega)} + \|F_{12} - F_{21}\|_{L^1(\Omega)} \big) < \epsilon\,, \tag{2.4}$$

then there exists a solution $(u, \nabla p) \in \big(W^{2,2}_{\mathrm{loc}}(\overline{\Omega}) \cap L^\infty_1(\Omega)\big)^2 \times L^2_{\mathrm{loc}}(\overline{\Omega})^2$ *to* (NS_α)*, which is unique in a suitable class of functions (see Theorem 2.4.1 for the precise description). If* $F \in L^\infty_{2,0}(\Omega)^{2\times 2}$ *in addition, then the solution* u *behaves as*

$$u(x) \,=\, \beta \frac{x^\perp}{4\pi |x|^2} + o(|x|^{-1})\,, \qquad |x| \to \infty\,, \tag{2.5}$$

where

$$\beta \,=\, \int_{\partial\Omega} y^\perp \cdot \big(T(u,p)\nu\big)\,\mathrm{d}\sigma_y + \lim_{\delta\to 0} \int_\Omega e^{-\delta |y|^2} y^\perp \cdot f \,\mathrm{d}y\,. \tag{2.6}$$

Here $T(u,p) \,=\, \nabla u + (\nabla u)^\top - p\,\mathbb{I}$*,* $\mathbb{I} = (\delta_{ij})_{1\le i,j\le 2}$*, denotes the Cauchy stress tensor, and* ν *is the outward unit normal vector to* $\partial\Omega$*.*

Remark 2.1.2 (i) The smallness condition on f and F in (2.4) can be slightly weakened with respect to the dependence on α; see Theorem 2.4.1 for details.
(ii) Both conditions $F \in L^\infty_2(\Omega)^{2\times 2}$ and $F_{12} - F_{21} \in L^1(\Omega)$ are critical in view of the Navier-Stokes equations scaling. Note that the L^1-summability of F is needed only for its antisymmetric part. These two conditions are not enough to ensure that u behaves like the circular flow $\beta \frac{x^\perp}{4\pi|x|^2}$ at spatial infinity, and the additional decay condition $F \in L^\infty_{2,0}(\Omega)^{2\times 2}$ as in Theorem 2.1.1 is required to achieve this asymptotic property.
(iii) The second term of the right-hand side of (2.6) is well-defined if $F \in L^\infty_{2,0}(\Omega)$ and $F_{12} - F_{21} \in L^1(\Omega)$. If F possesses an additional decay such as $L^\infty_{2+\gamma}(\Omega)$ with $\gamma \in (0,1)$ then the order $o(|x|^{-1})$ in (2.5) is replaced by $O(|x|^{-1-\gamma})$ at least when α and f are further small depending on γ. The precise description for this result is stated in Theorem 2.4.1.
(iv) The pressure p is determined uniquely up to a constant and belongs to $W^{1,2}_{\mathrm{loc}}(\overline{\Omega})$. Then the regularity $u \in W^{2,2}_{\mathrm{loc}}(\overline{\Omega})^2$ yields that the coefficient β in (2.6) is well-defined.
(v) In Theorem 2.1.1 we assume that the external force f is of divergence form. In fact, this is not an essential assumption, and it is possible to deal with the external force f satisfying

$$x^\perp \cdot f \in L^1(\Omega)\,, \qquad f \in L^\infty_3(\Omega)^2\,, \tag{2.7}$$

with the smallness in these norms. Moreover, the asymptotic expansion (2.5) is verified if $f \in L^\infty_{3,0}(\Omega)^2$ in addition. This is obtained by using the recent result by [30] in the whole space which solves the linearized problem for f satisfying (2.7). Although this result is not so trivial since the condition (2.7) is just in the scale-critical regime, we focus only on f of divergence form in this chapter, for the argument becomes shorter due to the fact that the nonlinear term is also written in the divergence form as $\mathrm{div}\,(u \otimes u)$.

As far as the author knows, Theorem 2.1.1 is the first general existence result of the Navier-Stokes flows around a rotating obstacle *in the two-dimensional case*. Before stating the main idea of the proof of Theorem 2.1.1, let us recall some known results on the mathematical analysis of flows around a rotating obstacle.

So far the mathematical results on this topic have been obtained mainly for the three-dimensional problem, as listed below. For the nonstationary problem the existence of global weak solutions is proved by Borchers [4], and the unique existence of time-local regular solutions is shown by Hishida [33] and Geissert, Heck, and Hieber [23], while the global strong solutions for small data are obtained by Galdi and Silvestre [20]. The spectrum of the linear operator related to this problem is studied by Farwig and Neustupa [12]; see also the linear analysis by Hishida [34]. The existence of stationary solutions to the associated system is proved in [4], Silvestre [55], Galdi [17], and Farwig and Hishida [9]. In particular, in [17] the stationary flows with the decay order $O(|x|^{-1})$ are obtained, while the work of [9] is based on the weak L^3 framework, which is another natural scale-critical space for the three-dimensional Navier-Stokes equations. Our Theorem 2.1.1 is considered as a two-dimensional counterpart of the result of [17]. For the asymptotic profiles of the stationary flows at spatial infinity are studied by Farwig and Hishida [10, 11] and Farwig, Galdi, and Kyed [8], where it is proved that they are described by the Landau solutions, stationary self-similar solutions to the Navier-Stokes equations in $\mathbb{R}^3 \setminus \{0\}$. The stability of the stationary solutions has been well studied in the three-dimensional case; the global L^2 stability is proved in [20], and the local L^3 stability is obtained by Hishida and Shibata [39].

All results mentioned above are in the three-dimensional case, while only a few results are known so far for the flow around a rotating obstacle in the two-dimensional case. Recently, an important progress has been made by Hishida [35], where the asymptotic behavior of the two-dimensional stationary Stokes flow around a rotating obstacle is investigated in details. The equations studied in [35] are written as

$$\begin{cases} -\Delta u - \alpha(x^\perp \cdot \nabla u - u^\perp) + \nabla p \,=\, f\,, \quad \operatorname{div} u \,=\, 0\,, \qquad x \in \Omega, \\ \qquad\qquad\qquad\qquad\qquad u \,=\, b\,, \qquad x \in \partial\Omega\,, \\ \qquad\qquad\qquad\qquad\qquad u \,\to\, 0\,, \qquad |x| \to \infty\,. \end{cases} \tag{S$_\alpha$}$$

Here $b = b(x)$ is a given smooth function on $\partial\Omega$. It is proved in [35] that if $\alpha \neq 0$ and the smooth external force f satisfies the decay conditions

$$\int_\Omega |x||f| \,\mathrm{d}x < \infty\,, \qquad f(x) \,=\, o\big(|x|^{-3}(\log|x|)^{-1}\big)\,, \qquad |x| \to \infty\,, \tag{2.8}$$

then the solution u to (S$_\alpha$) decaying at spatial infinity obeys the asymptotic expansion

$$u(x) \,=\, \frac{c_1 x^\perp - 2c_2 x}{4\pi|x|^2} + (1 + |\alpha|^{-1})\, o(|x|^{-1})\,, \qquad |x| \to \infty\,, \tag{2.9}$$

where

$$\begin{aligned} c_1 \,&=\, \int_{\partial\Omega} y^\perp \cdot \big(T(u,p) + \alpha\, b \otimes y^\perp\big)\nu \,\mathrm{d}\sigma_y + \int_\Omega y^\perp \cdot f \,\mathrm{d}y\,, \\ c_2 \,&=\, \int_{\partial\Omega} b \cdot \nu \,\mathrm{d}\sigma_y\,. \end{aligned} \tag{2.10}$$

The result of [35] leads to an important conclusion that the rotation of the obstacle resolves the Stokes paradox (see Chang and Finn [7] for the rigorous description of the Stokes paradox) as in the Oseen resolution. We recall that when the obstacle is translating with a constant velocity $u_\infty \in \mathbb{R}^2 \setminus \{0\}$ the Navier-Stokes flows have been constructed by Finn and Smith [14, 15] for small but nonzero u_∞, through the analysis of the Oseen linearization; see also Galdi [19]. The resolution of the Stokes paradox for (S$_\alpha$) is due to the fact that the rotation removes the logarithmic singularity of the associated fundamental solution, which has been well known for the Oseen problem where the resolution occurs by the translation.

For a two-dimensional exterior problem related with ours, the reader is referred to a work by Hillairet and Wittwer [32], where the stationary problem of (2.1) is discussed when $\Omega(t) = \{y \in \mathbb{R}^2 \mid |y| > 1\}$ and the boundary condition is given as $v = \alpha y^\perp + b$ with a smooth and time-independent b. We note that the flow $\alpha \frac{y^\perp}{|y|^2}$ exactly solves this problem when $b = 0$. When α is large enough and b is sufficiently small the stationary solutions are constructed in [32] around the explicit solution $\tilde{\alpha} \frac{y^\perp}{|y|^2}$, where $\tilde{\alpha}$ is a number close to α. Although the problem discussed in [32] is different from ours due to the time-independent given data b in the original frame (2.1), the solutions obtained in [32] share a common property with the ones in Theorem 2.1.1 in view of their asymptotics at spatial infinity.

It is well known that the existence of stationary Navier-Stokes flows in two-dimensional exterior domains (hence, formally $\alpha = 0$ in (NS$_\alpha$)) is an open problem in general. Partial results related to this problem have been obtained by Galdi [18], Russo [53], Yamazaki [58], and Pileckas and Russo [52], where the solutions are constructed under some symmetry conditions on both domains and given data. In particular, the two-dimensional Navier-Stokes flows decaying in the scale-critical order $O(|x|^{-1})$ are obtained in [58] in this category. The uniqueness is also available again under some symmetry conditions, see Nakatsuka [48].

The stability of the stationary solutions obtained in [32, 58] or in Theorem 2.1.1 is a highly challenging issue due to their spatial decay in the scale-critical order, and it is still an open question in general. The difficulty is brought from the fact that the Hardy inequality $\|\frac{f}{|x|}\|_{L^2(\Omega)} \le C \|\nabla f\|_{L^2(\Omega)}$, $f \in \dot{W}_0^{1,2}(\Omega)$, does not hold when Ω is an exterior domain in $\mathbb{R}^2$. We will discuss the stability problem more in detail in Chapter 4.

Finally, let us state the key idea for the proof of Theorem 2.1.1. Our approach is motivated by the linear analysis developed in [35], where (2.10) is obtained through the detailed analysis of the fundamental solution associated to the system (S$_\alpha$) in $\mathbb{R}^2$. The expansion (2.9) strongly indicates that the similar asymptotics is valid also for the Navier-Stokes flow, since the leading profile in (2.9) is a stationary self-similar solution to the Navier-Stokes equations in $\mathbb{R}^2 \setminus \{0\}$. Thus our strategy for the proof of Theorem 2.1.1 can be summarized as follows; we derive at the same time the unique existence of solutions and their asymptotic behavior, under the smallness condition on the given data (α, f) in (NS$_\alpha$). The solution in the form $u = \beta \frac{x^\perp}{|x|^2} + w$ is constructed through the Banach fixed point theorem, where both the coefficient β and the remainder term w are sufficiently small corresponding to the size of (α, f). However, it is far from trivial to justify this idea directly from the results of [35], especially to ensure the smallness of (β, w) in the iteration scheme. Indeed, there are at least two difficulties for this procedure: (I) the condition (2.8) is slightly restrictive to handle the nonlinear term $u \cdot \nabla u$ in the scale-critical framework, and more seriously, (II) the singularity in (2.9) for small $|\alpha|$ may prevent us from closing the nonlinear estimates. In fact, the smooth flows subject to the system (NS$_\alpha$) are naturally pointwise bounded above by $|\alpha|$ near the boundary due to the boundary condition $u = \alpha x^\perp$.

In resolving the difficulty (I), the structure of the nonlinear term $u \cdot \nabla u = \nabla \cdot (u \otimes u)$ is essential. Indeed, the symmetry of the tensor $u \otimes u$ leads to a crucial cancellation in the coefficient "$\int_\Omega y^\perp \cdot (u \cdot \nabla u) \, \mathrm{d}y$", which removes a possible singularity caused by the scale-critical decay of the flow $u = O(|x|^{-1})$. To overcome the difficulty (II), we revisit the argument of [35] analyzing the fundamental solution to (S$_\alpha$) in $\mathbb{R}^2$ and modify the singularity of α appearing in the estimates of the remainder term; see Theorem 2.3.1, Lemma 2.3.3, and Theorem 2.3.8. Applying these improved estimates, the nonlinear problem (NS$_\alpha$) is solved by the standard Banach fixed point theorem. However, the argument becomes quite complicated since we have to control two kinds of norms: the one bounds the local quantity, while the other one controls the spatial decay. This machinery is needed since the flow in a far field region ($|x| \gg 1$) exhibits a different dependence on $|\alpha|$ from the flow in a finite fluid region, and in principle, the problem becomes more singular at $|x| \gg 1$ as $|\alpha|$ is decreasing. In order to close the nonlinear estimates it is important to distinguish these two dependences on $|\alpha|$ and to estimate their interaction through the nonlinearity carefully.

This chapter is organized as follows. In Section 2.2 the basic results on the oscillatory integrals are collected, which are used to establish the pointwise estimates of the fundamental solution to (S_α) with a milder singularity on small $|\alpha|$. In Section 2.3 the linearized problem (S_α) with $b = 0$ is studied in details. Subsection 2.3.1 is devoted to the analysis in $\mathbb{R}^2$, while the exterior problem is discussed in Subsection 2.3.2. Finally the nonlinear problem (NS_α) is solved in Section 2.4 by the strategy explained as above.

2.2 Preliminaries

In this section we collect the results of the oscillatory integrals used in Section 2.3.1.

Lemma 2.2.1 *Let $\alpha \in \mathbb{R} \setminus \{0\}$ and let $m, r > 0$. Then we have*

$$\Big|\int_0^\infty e^{i\alpha t} e^{-\frac{r^2}{t}} \frac{\mathrm{d}t}{t^m}\Big| + \Big|\int_0^\infty e^{i\alpha t} \int_t^\infty e^{-\frac{r^2}{s}} \frac{\mathrm{d}s}{s^{m+1}}\, \mathrm{d}t\Big| \le C \min\Big\{\frac{1}{|\alpha| r^{2m}}, \frac{1}{|\alpha|^{\frac{1}{m+1}} r^{\frac{2m^2}{m+1}}}\Big\}, \tag{2.11}$$

where the constant $C = C(m)$ is independent of r and α. Moreover, for $m > 1$ we have

$$\int_0^\infty e^{-\frac{r^2}{t}} \frac{\mathrm{d}t}{t^m} = \frac{\Gamma(m-1)}{r^{2(m-1)}}, \qquad \int_0^\infty \int_t^\infty e^{-\frac{r^2}{s}} \frac{\mathrm{d}s}{s^{m+1}}\, \mathrm{d}t = \frac{\Gamma(m-1)}{r^{2(m-1)}}, \tag{2.12}$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Proof: The proof of (2.12) is a direct computation, and we omit the details. To show (2.11) let us take a constant $l = l(r, \alpha) > 0$ to be determined later and split the integral as

$$\int_0^\infty e^{i\alpha t} e^{-\frac{r^2}{t}} \frac{\mathrm{d}t}{t^m} = \int_0^l e^{i\alpha t} e^{-\frac{r^2}{t}} \frac{\mathrm{d}t}{t^m} + \int_l^\infty e^{i\alpha t} e^{-\frac{r^2}{t}} \frac{\mathrm{d}t}{t^m}.$$

The first term is estimated without using the effect from oscillation:

$$\Big|\int_0^l e^{i\alpha t} e^{-\frac{r^2}{t}} \frac{\mathrm{d}t}{t^m}\Big| \le \frac{1}{r^{2m}} \int_0^l e^{-\frac{r^2}{t}} \Big(\frac{r^2}{t}\Big)^m \mathrm{d}t \le \frac{Cl}{r^{2m}}.$$

For the second term we use the oscillation effect to obtain

$$\begin{aligned}\int_l^\infty e^{i\alpha t} e^{-\frac{r^2}{t}} \frac{\mathrm{d}t}{t^m} &= \frac{1}{i\alpha} \int_l^\infty \frac{\mathrm{d}}{\mathrm{d}t}\Big[e^{i\alpha t}\Big] \frac{e^{-\frac{r^2}{t}}}{t^m}\, \mathrm{d}t \\ &= \frac{1}{i\alpha}\Big[e^{i\alpha t} \frac{e^{-\frac{r^2}{t}}}{t^m}\Big]_{t=l}^{t=\infty} - \frac{1}{i\alpha} \int_l^\infty e^{i\alpha t} \Big(\frac{r^2 e^{-\frac{r^2}{t}}}{t^{m+2}} - \frac{m e^{-\frac{r^2}{t}}}{t^{m+1}}\Big)\, \mathrm{d}t,\end{aligned}$$

which yields

$$\Big|\int_l^\infty e^{i\alpha t} e^{-\frac{r^2}{t}} \frac{\mathrm{d}t}{t^m}\Big| \le \frac{1}{|\alpha|}\Big(\frac{e^{-\frac{r^2}{l}}}{l^m} + \frac{1}{r^{2(m+1)}} \int_l^\infty \big(\frac{r^2}{t} + m\big)\big(\frac{r^2}{t}\big)^{m+1} e^{-\frac{r^2}{t}}\, \mathrm{d}t\Big). \tag{2.13}$$

By taking the limit of $l = 0$ we observe that the left-hand side of (2.13) is then bounded from above by $\frac{C}{|\alpha| r^{2m}}$ thanks to (2.12). On the other hand, the right-hand side of (2.13) is also bounded from above by $\frac{C}{|\alpha| l^m}$. Then by taking $l = r^{\frac{2m}{m+1}} |\alpha|^{-\frac{1}{m+1}}$ we see that

$$\Big| \int_0^\infty e^{i\alpha t} e^{-\frac{r^2}{t}} \frac{\mathrm{d}t}{t^m} \Big| \le \frac{C}{|\alpha|^{\frac{1}{m+1}} r^{\frac{2m^2}{m+1}}} .$$

The estimate of the integral

$$\int_0^\infty e^{i\alpha t} \int_t^\infty e^{-\frac{r^2}{s}} \frac{\mathrm{d}s}{s^{m+1}} \, \mathrm{d}t$$

is obtained in the same manner, and the details are omitted here. The proof is complete. □

Lemma 2.2.2 *Let $m > 1$. Then we have*

$$\begin{aligned} &\int_0^\infty |e^{-\frac{|O(\alpha t)x-y|^2}{4t}} - e^{-\frac{|x|^2}{4t}}| \frac{\mathrm{d}t}{t^m} + \int_0^\infty \int_t^\infty |e^{-\frac{|O(\alpha t)x-y|^2}{4s}} - e^{-\frac{|x|^2}{4s}}| \frac{\mathrm{d}s}{s^{m+1}} \, \mathrm{d}t \\ &\le C \frac{|y|}{|x|^{2m-1}} , \qquad |x| > 2|y| , \end{aligned} \tag{2.14}$$

and

$$\Big| \int_0^\infty e^{i\alpha t} e^{-\frac{|x|^2}{4t}} \frac{\mathrm{d}t}{t^m} \Big| \le C \min\big\{ \frac{1}{|\alpha||x|^{2m}}, \frac{1}{|x|^{2(m-1)}} \big\} , \qquad |x| > 0 . \tag{2.15}$$

Moreover, for $m > 1$ we have

$$\Big| \int_0^\infty e^{i\alpha t} \int_t^\infty e^{-\frac{|x|^2}{4s}} \frac{\mathrm{d}s}{s^{m+1}} \, \mathrm{d}t \Big| \le C \min\big\{ \frac{1}{|\alpha||x|^{2m}}, \frac{1}{|x|^{2(m-1)}} \big\} , \qquad |x| > 0 . \tag{2.16}$$

Here $C = C(m)$ is independent of x, y, and α.

Proof: By using the Taylor formula with respect to y around $y = 0$, we see that

$$e^{-\frac{|O(\alpha t)x-y|^2}{4t}} = e^{-\frac{|x|^2}{4t}} + \frac{\langle O(\alpha t)x, y\rangle}{2t} e^{-\frac{|x|^2}{4t}} + \frac{\langle y, Qy\rangle}{8t^2} e^{-\frac{|O(\alpha t)x-\theta y|^2}{4t}} . \tag{2.17}$$

Here the matrix $Q = Q(x,y,t)$ is defined by $Q(x,y,t) = \big(O(\alpha t)x - \theta y\big) \otimes \big(O(\alpha t)x - \theta y\big) - 2t\mathbb{I}$ with some constant $\theta = \theta(\alpha,t,x,y) \in (0,1)$, and $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^2$; $\langle x, y\rangle = x \cdot y$. By using the condition

$$|O(\alpha t)x - \theta y| \ge |x| - |y| > \frac{|x|}{2} , \qquad |x| > 2|y| ,$$

from Lemma 2.2.1 we have

$$\begin{aligned} &\int_0^\infty |e^{-\frac{|O(\alpha t)x-y|^2}{4t}} - e^{-\frac{|x|^2}{4t}}| \frac{\mathrm{d}t}{t^m} \\ &\le C\Big(|x||y| \int_0^\infty e^{-\frac{|x|^2}{4t}} \frac{\mathrm{d}t}{t^{m+1}} + (|x|^2|y|^2 + |x||y|^3 + |y|^4) \int_0^\infty e^{-\frac{|x|^2}{16t}} \frac{\mathrm{d}t}{t^{m+2}} \Big) \\ &\le \frac{C|y|}{|x|^{2m-1}} , \qquad |x| > 2|y| . \end{aligned}$$

In the similar manner we have from Lemma 2.2.1,

$$\int_0^\infty \int_t^\infty \big|e^{-\frac{|O(\alpha t)x-y|^2}{4s}} - e^{-\frac{|x|^2}{4s}}\big| \frac{\mathrm{d}s}{s^{m+1}} \,\mathrm{d}t \le \frac{C|y|}{|x|^{2m-1}}\,, \qquad |x| > 2|y|\,.$$

The proof of (2.14) is complete. Since we have $m > 1$, the estimates (2.15) and (2.16) are immediate consequences of (2.11) and (2.12). This completes the proof. □

2.3 Stokes equations with a rotation effect

This section is devoted to the analysis of the linearized problem (S_α) with $b = 0$, which is already introduced in the previous section.

2.3.1 Linear estimate in the whole plane

In this subsection let us consider the linear problem in the whole plane for $\alpha \in \mathbb{R} \setminus \{0\}$:

$$-\Delta u - \alpha(x^\perp \cdot \nabla u - u^\perp) + \nabla p \,=\, f\,, \qquad \mathrm{div}\, u \,=\, 0\,, \qquad x \in \mathbb{R}^2\,. \qquad (\mathrm{S}_{\alpha,\mathbb{R}^2})$$

Our main interest is the estimate of solutions that are represented in terms of the fundamental solution defined by (2.18) below. We will see that such solutions decay at spatial infinity for a suitable class of f thanks to the effect from the rotation; see also Remark 2.3.2 about the uniqueness for solutions to ($\mathrm{S}_{\alpha,\mathbb{R}^2}$). The couple (u, p) is said to be a weak solution to ($\mathrm{S}_{\alpha,\mathbb{R}^2}$) if $(u,p) \in L^{q_1}(\mathbb{R}^2)^2 \times L^{q_2}(\mathbb{R}^2)$ for some $q_1 \in [2,\infty)$ and $q_2 \in [1,\infty)$, and (i) $\mathrm{div}\, u = 0$ in the sense of distributions, and (ii) (u, p) satisfies

$$\int_{\mathbb{R}^2} u \cdot \mathcal{L}_{-\alpha}\phi \,\mathrm{d}x - \int_{\mathbb{R}^2} p \,\mathrm{div}\, \phi \,\mathrm{d}x \,=\, \int_{\mathbb{R}^2} f \cdot \phi \,\mathrm{d}x\,, \quad \text{for all } \ \phi \in \mathcal{S}(\mathbb{R}^2)^2\,,$$

where the operator $\mathcal{L}_\alpha$ is defined as

$$\mathcal{L}_\alpha u \,=\, -\Delta u - \alpha(x^\perp \cdot \nabla u - u^\perp)\,.$$

The fundamental solution to ($\mathrm{S}_{\alpha,\mathbb{R}^2}$) plays a central role in this chapter, which is defined as

$$\Gamma_\alpha(x,y) \,=\, \int_0^\infty O(\alpha t)^\top K(O(\alpha t)x - y, t) \,\mathrm{d}t\,, \tag{2.18}$$

where

$$K(x,t) \,=\, G(x,t)\mathbb{I} + H(x,t)\,, \qquad H(x,t) \,=\, \int_t^\infty \nabla^2 G(x,s) \,\mathrm{d}s\,,$$

and $G(x,t)$ is the two-dimensional Gauss kernel

$$G(x,t) \,=\, \frac{1}{4\pi t} e^{-\frac{|x|^2}{4t}}\,.$$

The next theorem is the main result of this subsection, which extends the result in [35] to our setting. For $f \in L^2(\mathbb{R}^2)^2$ and $F = (F_{ij})_{1\le,i,j\le 2} \in L^2(\mathbb{R}^2)^{2\times 2}$ we formally set

$$\begin{aligned} c[f] \,&=\, \lim_{\epsilon\to 0} \int_{\mathbb{R}^2} e^{-\epsilon|y|^2} y^\perp \cdot f(y) \,\mathrm{d}y\,, \\ \tilde{c}[F] \,&=\, \lim_{\epsilon\to 0} \int_{\mathbb{R}^2} e^{-\epsilon|y|^2} \big(F_{12}(y) - F_{21}(y)\big) \,\mathrm{d}y\,. \end{aligned} \tag{2.19}$$

Note that if $f \in L^2(\mathbb{R}^2)^2$ is of the form $f = \operatorname{div} F$ with some $F \in L^1(\mathbb{R}^2)^{2\times 2}$, then $c[f] = \tilde{c}[F]$ holds. Indeed, from the integration by parts we see that

$$c[f] \;=\; \tilde{c}[F] + \lim_{\epsilon\to 0} 2\int_{\mathbb{R}^2} e^{-\epsilon|y|^2}\epsilon y^{\perp}\cdot\big(F(y)y\big)\,\mathrm{d}y\,.$$

Then the Lebesgue dominated convergence theorem implies $c[f] = \tilde{c}[F]$. Moreover, if F is symmetric then $\tilde{c}[F] = 0$. In the following $B_R \subset \mathbb{R}^2$ denotes the open disk of radius $R > 0$ centered at the origin, and its complement is denoted by $B_R^{\mathrm{c}} = \{x \in \mathbb{R}^2 \mid |x| \geq R\}$.

Theorem 2.3.1 *Let $\alpha \in \mathbb{R}\setminus\{0\}$. We formally set*

$$L[f](x) \;=\; \lim_{\epsilon\to 0}\int_{\mathbb{R}^2} e^{-\epsilon|y|^2}\Gamma_\alpha(x,y)f(y)\,\mathrm{d}y\,. \tag{2.20}$$

Then the following statements hold.
(i) *Let $\gamma \in [0,1)$. Suppose that $f \in L^2(\mathbb{R}^2)^2$ satisfies* $\operatorname{supp} f \subset B_R$ *for some $R \geq 1$. Then $u = L[f]$ is a weak solution to* $(\mathrm{S}_{\alpha,\mathbb{R}^2})$ *and is written as*

$$u(x) \;=\; c[f]\frac{x^{\perp}}{4\pi|x|^2} + \mathcal{R}[f](x)\,, \qquad x \neq 0\,, \tag{2.21}$$

where $\mathcal{R}[f]$ satisfies

$$\|\mathcal{R}[f]\|_{L^\infty_{1+\gamma}(B^{\mathrm{c}}_{2R})} \leq C_1\big(|\alpha|^{-\frac{1+\gamma}{2}}\|f\|_{L^1(B_R)} + \||y|^{1+\gamma}f\|_{L^1(B_R)}\big)\,. \tag{2.22}$$

Here C_1 is a numerical constant, and is independent of γ, α, R, and f.
(ii) *Let $\gamma \in [0,1)$. Suppose that $f \in L^2(\mathbb{R}^2)^2$ is of the form $f = \operatorname{div} F$ with some $F \in L^\infty_{2+\gamma}(\mathbb{R}^2)^{2\times 2}$, and in addition that $\tilde{c}[F]$ in* (2.19) *converges when $\gamma = 0$. Then $u = L[f]$ is a weak solution to* $(\mathrm{S}_{\alpha,\mathbb{R}^2})$ *and is written as*

$$u(x) \;=\; \tilde{c}[F]\frac{x^{\perp}}{4\pi|x|^2} + \mathcal{R}[f](x)\,, \qquad x \neq 0\,, \tag{2.23}$$

where $\mathcal{R}[f]$ satisfies for $R \geq 1$,

$$\begin{aligned}\|\mathcal{R}[f]\|_{L^\infty_{1+\gamma}(B^{\mathrm{c}}_{2R})} \leq C_2\Bigg(&\|F\|_{L^\infty_{2+\gamma}(B^{\mathrm{c}}_{R})} + \sup_{|x|\geq 2R}|x|^{-1+\gamma}\|yF\|_{L^1(B_{\frac{|x|}{2}})}\\ &+ \sup_{|x|\geq 2R}\min\big\{\frac{1}{|\alpha||x|^{2-\gamma}}, |x|^{\gamma}\big\}\|F\|_{L^1(B_{\frac{|x|}{2}})}\\ &+ \sup_{|x|\geq 2R}|x|^{\gamma}\,\big|\lim_{\epsilon\to 0}\int_{2|y|\geq|x|} e^{-\epsilon|y|^2}\big(F_{12}(y) - F_{21}(y)\big)\,\mathrm{d}y\big|\Bigg)\,.\end{aligned} \tag{2.24}$$

Here C_2 is a numerical constant, and is independent of γ, α, R, and f.

Remark 2.3.2 Under the assumptions of (i) or (ii) in Theorem 2.3.1 it is not difficult to see that $L[f]$ belongs to $W^{2,2}_{\mathrm{loc}}(\mathbb{R}^2)$, and thus, $L[f]$ is bounded in $\mathbb{R}^2$ by the Sobolev embedding in B_2 and the estimates stated in Theorem 2.3.1 for $|x| \geq 1$ (by taking $R = 1$). Set

$$p(x) \;=\; \int_{\mathbb{R}^2}\frac{x-y}{2\pi|x-y|^2}f(y)\,\mathrm{d}y\,. \tag{2.25}$$

Then, ∇p belongs to $L^2(\mathbb{R}^2)^2$ under the assumptions of (i) or (ii) in Theorem 2.3.1 by the Calderón-Zygmund inequality, and as is shown in [35, Proposition 3.2], the pair $(L[f], \nabla p)$ satisfies $(\mathrm{S}_{\alpha,\mathbb{R}^2})$ in the sense of distributions. Thanks to the uniqueness result stated in [35, Lemma 3.5], if f satisfies one of the assumptions in Theorem 2.3.1, and if $(v, q) \in \mathcal{S}'(\mathbb{R}^2)^2 \times \mathcal{S}'(\mathbb{R}^2)$ is a solution to $(\mathrm{S}_{\alpha,\mathbb{R}^2})$ in the sense of distributions, then (v, q) has a representation as $v = L[f] + P_1$ and $q = p + P_2$ with some polynomials P_1 and P_2. Hence, by the definition stated above, any weak solution (u, p) to $(\mathrm{S}_{\alpha,\mathbb{R}^2})$ is represented as $u = L[f]$ and p is given by (2.25), if the condition (i) or (ii) on f in Theorem 2.3.1 is assumed.

We note that in (ii) of Theorem 2.3.1 the coefficient $\tilde{c}[F]$ is always well-defined when $\gamma > 0$. The asymptotic expansion (2.21) for (i) is firstly established by [35, Proposition 3.2]. Indeed, for (i) it is shown in [35, Proposition 3.2] that $\mathcal{R}[f]$ decays at infinity as $O(|x|^{-2})$, while the singularity $|\alpha|^{-1}$ appears in the coefficient of the estimates there. The novelty of Theorem 2.3.1 are (2.22) and (2.24), where both the consistency in the weighted L^∞ spaces and the milder singularity on α for small $|\alpha|$ are essential to solve the nonlinear problem in Section 2.4. On the other hand, as in [35], the key step to prove Theorem 2.3.1 is the expansion and the pointwise estimate of the fundamental solution $\Gamma_\alpha(x, y)$, which are stated in Lemma 2.3.3 below. The fundamental solution $\Gamma_\alpha(x, y)$ is studied in details in [35, Proposition 3.1] and we will revisit the argument developed by [35] in the proof.

Lemma 2.3.3 *Set*

$$L(x, y) = \frac{x^\perp \otimes y^\perp}{4\pi |x|^2} . \tag{2.26}$$

Then for $m = 0, 1$ the kernel $\Gamma_\alpha(x, y)$ satisfies

$$\begin{aligned} &|\nabla_y^m \big(\Gamma_\alpha(x, y) - L(x, y)\big)| \\ &\le C\bigg(\delta_{0m} \min\big\{\frac{1}{|\alpha||x|^2}, \frac{1}{|\alpha|^{\frac12}|x|}\big\} + |x|^{1-m} \min\big\{\frac{1}{|\alpha||x|^3}, \frac{1}{|x|}\big\} + \frac{|y|^{2-m}}{|x|^2}\bigg), \\ &\qquad\qquad |x| > 2|y| . \end{aligned} \tag{2.27}$$

Here δ_{0m} is the Kronecker delta and the constant C is independent of x, y, and α.

Remark 2.3.4 The case $m = 0$ of (2.27) is already obtained in [35, Proposition 3.1] but with $|\alpha|^{-1}$ dependence of the coefficients in the estimate. The case $m = 1$ is not stated explicitly in [35], although it can be handled in the similar spirit as in the case $m = 0$. Hence, in this sense Lemma 2.3.3 is not a completely new result, and is an improvement of [35, Proposition 3.1] with respect to the singularity on small $|\alpha| > 0$.

Proof of Lemma 2.3.3: In principle, our proof of Lemma 2.3.3 will proceed along the line of [35, Proposition 3.1]. In fact, the only key difference for the case $m = 0$ in out proof is the application of Lemmas 2.2.1 and 2.2.2 in suitable parts. While in the proof for the case $m = 1$, the inequality (2.14) will be essentially used in addition.

Following the argument of [35, Section 3], we decompose $\Gamma_\alpha(x,y)$ and define $\Gamma^0_\alpha(x,y)$, $\Gamma^{11}_\alpha(x,y)$, and $\Gamma^{12}_\alpha(x,y)$ as

$$
\begin{aligned}
&\Gamma_\alpha(x,y)\\
&= \Gamma^0_\alpha(x,y)+\Gamma^{11}_\alpha(x,y)+\Gamma^{12}_\alpha(x,y)\\
&= \int_0^\infty O(\alpha t)^\top G(O(\alpha t)x-y,t)\,\mathrm{d}t\\
&\quad+\int_0^\infty O(\alpha t)^\top (O(\alpha t)x-y)\otimes(O(\alpha t)x-y)\int_t^\infty G(O(\alpha t)x-y,s)\frac{\mathrm{d}s}{4s^2}\,\mathrm{d}t\\
&\quad-\int_0^\infty O(\alpha t)^\top \int_t^\infty G(O(\alpha t)x-y,s)\frac{\mathrm{d}s}{2s}\,\mathrm{d}t\,.
\end{aligned}
\tag{2.28}
$$

We also decompose $L(x,y)$ and define $L^0(x,y)$, $L^{111}(x,y)$, $L^{112}(x,y)$, and $L^{122}(x,y)$ as

$$
\begin{aligned}
L(x,y) &= L^0(x,y)+L^{111}(x,y)+L^{112}(x,y)+L^{12}(x,y)\\
&= \frac{x\otimes y+x^\perp\otimes y^\perp}{4\pi|x|^2}+\frac{-3(x\otimes y)+x^\perp\otimes y^\perp}{8\pi|x|^2}+\frac{x\otimes y}{4\pi|x|^2}-\frac{x\otimes y+x^\perp\otimes y^\perp}{8\pi|x|^2}\,.
\end{aligned}
\tag{2.29}
$$

Then, by Lemma 2.2.1 the following representations hold:

$$
\begin{aligned}
L^0(x,y) &= \int_0^\infty G(x,t)\frac{\mathrm{d}t}{4t}\begin{pmatrix} x\cdot y & x^\perp\cdot y\\ -x^\perp\cdot y & x\cdot y\end{pmatrix}\,,\\
L^{111}(x,y) &= \int_0^\infty\int_t^\infty G(x,s)\frac{\mathrm{d}s}{4s^2}\,\mathrm{d}t\left(\frac{-3(x\otimes y)+(x^\perp\otimes y^\perp)}{2}\right),\\
L^{112}(x,y) &= \int_0^\infty\int_t^\infty G(x,s)\frac{\mathrm{d}s}{16s^3}\,\mathrm{d}t\,|x|^2(x\otimes y)\,,\\
L^{12}(x,y) &= -\int_0^\infty\int_t^\infty G(x,s)\frac{\mathrm{d}s}{8s^2}\,\mathrm{d}t\begin{pmatrix} x\cdot y & x^\perp\cdot y\\ -x^\perp\cdot y & x\cdot y\end{pmatrix}\,,
\end{aligned}
\tag{2.30}
$$

where we have used the equality

$$
x\otimes y+x^\perp\otimes y^\perp = \begin{pmatrix} x\cdot y & x^\perp\cdot y\\ -x^\perp\cdot y & x\cdot y\end{pmatrix}\,.
$$

To prove (2.27) we observe that

$$
\begin{aligned}
&|\nabla^m_y\big(\Gamma_\alpha(x,y)-L(x,y)\big)|\\
&\le |\nabla^m_y\big(\Gamma^0_\alpha(x,y)-L^0(x,y)\big)|+|\nabla^m_y\big(\Gamma^{11}_\alpha(x,y)-L^{111}(x,y)-L^{112}(x,y)\big)|\\
&\quad+|\nabla^m_y\big(\Gamma^{12}_\alpha(x,y)-L^{12}(x,y)\big)|\,.
\end{aligned}
$$

Let us estimate each term in the right-hand side of the above inequality. The key idea is to use the Taylor formula for $G(O(\alpha t)x-y,t')$ around $y=0$ as follows.

$$
G(O(\alpha t)x-y,t') = G(x,t')+\frac{\langle O(\alpha t)x,y\rangle}{2t'}G(x,t')+\frac{\langle y,Qy\rangle}{8t'^2}G(O(\alpha t)x-\theta y,t')\ ,
\tag{2.31}
$$

where

$$
Q = Q(x,\theta y,\alpha t,t') = (O(\alpha t)x-\theta y)\otimes(O(\alpha t)x-\theta y)-2t'\mathbb{I}\,,
$$

and $\theta = \theta(\alpha, t', x, y) \in (0,1)$. To estimate $\Gamma^0_\alpha(x,y) - L^0(x,y)$ we use the identity

$$O(\alpha t)^\top \langle O(\alpha t)x, y\rangle = \frac{1}{2}\begin{pmatrix} x\cdot y & x^\perp \cdot y \\ -x^\perp\cdot y & x\cdot y\end{pmatrix} + \frac{\cos 2\alpha t}{2}\begin{pmatrix} x\cdot y & -x^\perp\cdot y \\ x^\perp \cdot y & x\cdot y\end{pmatrix} + \frac{\sin 2\alpha t}{2}\begin{pmatrix} x^\perp\cdot y & x\cdot y \\ -x\cdot y & x^\perp\cdot y\end{pmatrix}. \tag{2.32}$$

Let $|x| > 2|y|$. Then we have from (2.31) and (2.32),

$$\begin{aligned}
|\Gamma^0_\alpha(x,y) - L^0(x,y)| &= \Big| \int_0^\infty O(\alpha t)^\top G(x,t)\,\mathrm{d}t \\
&\quad + \int_0^\infty \frac{1}{2t}\bigg(O(\alpha t)^\top \langle O(\alpha t)x, y\rangle - \frac12 \begin{pmatrix} x\cdot y & x^\perp\cdot y \\ -x^\perp\cdot y & x\cdot y\end{pmatrix}\bigg) G(x,t)\,\mathrm{d}t \\
&\quad + \int_0^\infty O(\alpha t)^\top \frac{\langle y, Qy\rangle}{8t^2} G(O(\alpha t)x - \theta y, t)\,\mathrm{d}t\Big| \\
&\le \Big| \int_0^\infty O(\alpha t)^\top G(x,t)\,\mathrm{d}t\Big| + C|x||y| \min\big\{\frac{1}{|\alpha||x|^4}, \frac{1}{|x|^2}\big\} \\
&\quad + C|y|^2 \int_0^\infty \big\{(|x|^2 + |x||y| + |y|^2)t^{-3} + t^{-2}\big\} e^{-\frac{|x|^2}{16t}}\,\mathrm{d}t\,.
\end{aligned} \tag{2.33}$$

Here we have used (2.15) for the second term and used the condition $|x| > 2|y|$ for the third term to achieve the last line. Clearly the last term in the right-hand side of (2.33) is bounded from above by $C\frac{|y|^2}{|x|^2}$ for $|x| > 2|y|$, while in virtue of (2.11) the first term is estimated as

$$\Big| \int_0^\infty O(\alpha t)^\top G(x,t)\,\mathrm{d}t\Big| \le C \min\big\{\frac{1}{|\alpha||x|^2}, \frac{1}{|\alpha|^{\frac12}|x|}\big\}, \qquad |x| > 0\,. \tag{2.34}$$

Thus we have arrived at

$$\begin{aligned}
&|\Gamma^0_\alpha(x,y) - L^0(x,y)| \\
&\le C\bigg(\min\big\{\frac{1}{|\alpha||x|^2}, \frac{1}{|\alpha|^{\frac12}|x|}\big\} + |y|\min\big\{\frac{1}{|\alpha||x|^3}, \frac{1}{|x|}\big\} + \frac{|y|^2}{|x|^2}\bigg), \qquad |x| > 2|y|\,.
\end{aligned} \tag{2.35}$$

Next we consider the derivative estimate for $\Gamma^0_\alpha(x,y) - L^0(x,y)$. Let us go back to the definition of $\Gamma^0_\alpha(x,y)$ in (2.28). Then $\partial_{y_k}\big(\Gamma^0_\alpha(x,y) - L^0(x,y)\big)$ is computed as

$$\begin{aligned}
&\big|\partial_{y_k}(\Gamma^0_\alpha(x,y) - L^0(x,y))\big| \\
&= \Big| \int_0^\infty \bigg(\frac{O(\alpha t)^\top (O(\alpha t)x - y)_k}{2t} G(O(\alpha t)x - y, t) - \frac{1}{4t}\partial_{y_k}\begin{pmatrix} x\cdot y & x^\perp\cdot y \\ -x^\perp\cdot y & x\cdot y\end{pmatrix} G(x,t)\bigg)\,\mathrm{d}t\Big| \\
&\le \Big| \int_0^\infty \frac{O(\alpha t)^\top (O(\alpha t)x - y)_k}{2t} \Big(G(O(\alpha t)x - y, t) - G(x,t)\Big)\,\mathrm{d}t\Big| \\
&\quad + \Big| \int_0^\infty \bigg(O(\alpha t)^\top (O(\alpha t)x - y)_k - \frac12 \partial_{y_k}\begin{pmatrix} x\cdot y & x^\perp\cdot y \\ -x^\perp\cdot y & x\cdot y\end{pmatrix}\bigg) G(x,t)\frac{\mathrm{d}t}{2t}\Big|\,.
\end{aligned} \tag{2.36}$$

By applying (2.14) the first term is bounded from above by $C\frac{(|x|+|y|)|y|}{|x|^3}$. To estimate the second term we observe that

$$
\begin{aligned}
&O(\alpha t)^\top (O(\alpha t)x - y)_k - \frac{1}{2}\partial_{y_k} \begin{pmatrix} x\cdot y & x^\perp \cdot y \\ -x^\perp \cdot y & x \cdot y \end{pmatrix} \\
&= \begin{cases} \dfrac{\cos 2\alpha t}{2}\begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix} + \dfrac{\sin 2\alpha t}{2}\begin{pmatrix} -x_2 & x_1 \\ -x_1 & -x_2 \end{pmatrix} - y_1 O(\alpha t)^\top \,, & \text{if } k = 1\,, \\ \dfrac{\cos 2\alpha t}{2}\begin{pmatrix} x_2 & -x_1 \\ x_1 & x_2 \end{pmatrix} + \dfrac{\sin 2\alpha t}{2}\begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix} - y_2 O(\alpha t)^\top \,, & \text{if } k = 2\,, \end{cases}
\end{aligned} \tag{2.37}
$$

Then, by using (2.15) the second term in the right-hand side of (2.36) is bounded from above by $C(|x|+|y|)\min\{\frac{1}{|\alpha||x|^4}, \frac{1}{|x|^2}\}$. Hence we have shown that

$$
\left|\partial_{y_k}(\Gamma^0_\alpha(x,y) - L^0(x,y))\right| \le C\bigg(\frac{|y|}{|x|^2} + \min\Big\{\frac{1}{|\alpha||x|^3}, \frac{1}{|x|}\Big\}\bigg)\,, \qquad |x| > 2|y|\,. \tag{2.38}
$$

Exactly in the same way we obtain for $m = 0, 1$ and $|x| > 2|y|$,

$$
\begin{aligned}
&|\nabla^m_y(\Gamma^{12}_\alpha(x,y) - L^{12}(x,y))| \\
&\le C\bigg(\delta_{0m}\min\Big\{\frac{1}{|\alpha||x|^2}, \frac{1}{|\alpha|^{\frac12}|x|}\Big\} + |y|^{1-m}\min\Big\{\frac{1}{|\alpha||x|^3}, \frac{1}{|x|}\Big\} + \frac{|y|^{2-m}}{|x|^2}\bigg)\,.
\end{aligned} \tag{2.39}
$$

Next we estimate the term $|\Gamma^{11}_\alpha(x,y) - L^{111}(x,y) - L^{112}(x,y)|$. By the Taylor expansion stated in (2.31), we decompose $\Gamma^{11}_\alpha(x,y)$ and define $\Gamma^{11k}_\alpha(x,y)$, $k = 1, 2, 3$, as

$$
\begin{aligned}
&\Gamma^{11}_\alpha(x,y) \\
&= \Gamma^{111}_\alpha(x,y) + \Gamma^{112}_\alpha(x,y) + \Gamma^{113}_\alpha(x,y) \\
&= \int_0^\infty O(\alpha t)^\top (O(\alpha t)x - y)\otimes(O(\alpha t)x - y)\int_t^\infty G(x,s)\frac{\mathrm{d}s}{4s^2}\,\mathrm{d}t \\
&\quad + \int_0^\infty O(\alpha t)^\top (O(\alpha t)x - y)\otimes(O(\alpha t)x - y)\int_t^\infty \langle O(\alpha t)x, y\rangle G(x,s)\frac{\mathrm{d}s}{8s^3}\,\mathrm{d}t \\
&\quad + \int_0^\infty O(\alpha t)^\top (O(\alpha t)x - y)\otimes(O(\alpha t)x - y)\int_t^\infty \langle y, Qy\rangle G(O(\alpha t)x - \theta y, s)\frac{\mathrm{d}s}{32s^4}\,\mathrm{d}t\,.
\end{aligned}
$$

For the last term $\Gamma^{113}_\alpha(x,y)$ it is straightforward to see from (2.12) that, for $|x| > 2|y|$,

$$
\begin{aligned}
|\Gamma^{113}_\alpha(x,y)| &\le C|y|^2(|x|+|y|)^2\int_0^\infty\int_t^\infty (|x|^2+|y|^2+s)e^{-\frac{|x|^2}{16s}}\frac{\mathrm{d}s}{s^5}\,\mathrm{d}t \\
&\le C\frac{|y|^2}{|x|^2}\,.
\end{aligned} \tag{2.40}
$$

To estimate the first two terms we observe

$$
\begin{aligned}
&O(\alpha t)^\top (O(\alpha t)x - y)\otimes(O(\alpha t)x - y) \\
&= A_0 + (\cos\alpha t)A_1 + (\sin\alpha t)A_2 + \frac{\cos 2\alpha t}{2}A_3 + \frac{\sin 2\alpha t}{2}A_4\,,
\end{aligned} \tag{2.41}
$$

where

$$A_0(x,y) = \frac{-3(x\otimes y) + (x^\perp \otimes y^\perp)}{2}\,, \qquad A_1(x,y) = \begin{pmatrix} x_1^2+y_1^2 & x_1x_2+y_1y_2 \\ x_1x_2+y_1y_2 & x_2^2+y_2^2 \end{pmatrix}\,,$$

$$A_2(x,y) = \begin{pmatrix} -x_1x_2+y_1y_2 & x_1^2+y_2^2 \\ -(x_2^2+y_1^2) & x_1x_2-y_1y_2 \end{pmatrix}\,, \qquad A_3(x,y) = \begin{pmatrix} -x\cdot y & x^\perp\cdot y \\ -x^\perp\cdot y & -x\cdot y \end{pmatrix}\,,$$

$$A_4(x,y) = \begin{pmatrix} -x^\perp\cdot y & -x\cdot y \\ x\cdot y & -x^\perp\cdot y \end{pmatrix}\,.$$

Then, by using (2.41) and by applying (2.11) the term $\Gamma_\alpha^{111}(x,y)$ is estimated as

$$\begin{aligned} &\big|\Gamma_\alpha^{111}(x,y) - L^{111}(x,y)\big| \\ &= \Big|\int_0^\infty\int_t^\infty \Big((\cos\alpha t)A_1 + (\sin\alpha t)A_2 + \frac{\cos 2\alpha t}{2}A_3 + \frac{\sin 2\alpha t}{2}A_4\Big)G(x,s)\frac{\mathrm{d}s}{4s^2}\,\mathrm{d}t\Big| \\ &\le |x|\min\big\{\frac{1}{|\alpha||x|^3},\frac{1}{|x|}\big\}\,, \qquad |x|>2|y|\,. \end{aligned} \tag{2.42}$$

Next we see

$$\begin{aligned} &\langle O(\alpha t)x, y\rangle\, O(\alpha t)^\top (O(\alpha t)x-y)\otimes(O(\alpha t)x-y) \\ &= \frac{|x|^2}{2}x\otimes y + (\cos 2\alpha t)B_1(x,y) + (\sin 2\alpha t)B_2(x,y) + B_3(x,y,\alpha t)\,, \end{aligned} \tag{2.43}$$

where each component of the matrices B_1 and B_2 is a fourth order polynomial of x, y written as a suitable sum of the terms $x_1^{l_1}x_2^{l_2}y_1^{k_1}y_2^{k_2}$ with $l_1+l_2=3$ and $k_1+k_2=1$, while B_3 is estimated as $|B_3|\le C|x|^2|y|^2$ for $|x|>2|y|$. Thus we have from (2.43) and (2.11),

$$\begin{aligned} &\big|\Gamma_\alpha^{112}(x,y) - L^{112}(x,y)\big| \\ &\le \Big|\int_0^\infty\int_t^\infty \Big((\cos 2\alpha t)B_1(x,y) + (\sin 2\alpha t)B_2(x,y)\Big)G(x,s)\frac{\mathrm{d}s}{8s^3}\,\mathrm{d}t\Big| \\ &\quad + C|x|^2|y|^2\int_0^\infty\int_t^\infty G(x,s)\frac{\mathrm{d}s}{s^3}\,\mathrm{d}t \\ &\le C\Big(|x|\min\big\{\frac{1}{|\alpha||x|^3},\frac{1}{|x|}\big\} + \frac{|y|^2}{|x|^2}\Big)\,, \qquad |x|>2|y|\,. \end{aligned} \tag{2.44}$$

Summing up (2.40), (2.42), and (2.44), we obtain

$$\begin{aligned} &\big|\Gamma_\alpha^{11}(x,y) - L^{111}(x,y) - L^{112}(x,y)\big| \\ &\le C\Big(\min\big\{\frac{1}{|\alpha||x|^2},\frac{1}{|\alpha|^{\frac12}|x|}\big\} + |x|\min\big\{\frac{1}{|\alpha||x|^3},\frac{1}{|x|}\big\} + \frac{|y|^2}{|x|^2}\Big)\,, \quad |x|>2|y|\,. \end{aligned} \tag{2.45}$$

To estimate the derivatives in y of $\Gamma_\alpha^{11}(x,y)$ we recall the definition of $\Gamma_\alpha^{11}(x,y)$ in (2.28) and use (2.41), which leads to the representation

$$\begin{aligned} &\Gamma_\alpha^{11}(x,y) \\ &= \int_0^\infty\int_t^\infty A_0\, G(O(\alpha t)x-y,s)\frac{\mathrm{d}s}{4s^2}\,\mathrm{d}t \\ &+ \int_0^\infty\int_t^\infty \Big((\cos\alpha t)A_1 + (\sin\alpha t)A_2 + \frac{\cos 2\alpha t}{2}A_3 + \frac{\sin 2\alpha t}{2}A_4\Big)G(O(\alpha t)x-y,s)\frac{\mathrm{d}s}{4s^2}\,\mathrm{d}t \\ &= \tilde\Gamma_\alpha^{111}(x,y) + \tilde\Gamma_\alpha^{112}(x,y)\,. \end{aligned} \tag{2.46}$$

From the expression of $L^{111}(x,y)$ in (2.30), we have for $|x|>2|y|$,

$$
\begin{aligned}
&\big|\partial_{y_k}\big(\tilde\Gamma^{111}_{\alpha}(x,y)-L^{111}(x,y)\big)\big|\\
&=\bigg|\int_0^\infty\int_t^\infty \big(\partial_{y_k}A_0\big)\Big(G(O(\alpha t)x-y,s)-G(x,s)\Big)\frac{\mathrm{d}s}{4s^2}\,\mathrm{d}t\\
&\quad+\int_0^\infty\int_t^\infty (O(\alpha t)x-y)_k\,A_0\,\Big(G(O(\alpha t)x-y,s)-G(x,s)\Big)\frac{\mathrm{d}s}{8s^3}\,\mathrm{d}t\\
&\quad+\int_0^\infty\int_t^\infty (O(\alpha t)x-y)_k\,A_0\,G(x,s)\frac{\mathrm{d}s}{8s^3}\,\mathrm{d}t\bigg|\\
&\le C\bigg(\frac{|x||y|}{|x|^3}+\frac{(|x|^2|y|+|x||y|^2)|y|}{|x|^5}+\frac{(|x|^2|y|+|x||y|^2)}{|x|^4}\bigg)\le C\frac{|y|}{|x|^2}\,.
\end{aligned}
\tag{2.47}
$$

Here we have used (2.14). Next we estimate the derivatives of $\tilde\Gamma^{112}_{\alpha}(x,y)$:

$$
\begin{aligned}
&\partial_{y_k}\tilde\Gamma^{112}_{\alpha}(x,y)\\
&=\int_0^\infty\int_t^\infty\bigg((\cos\alpha t)\partial_{y_k}A_1+(\sin\alpha t)\partial_{y_k}A_2+\frac{\cos 2\alpha t}{2}\partial_{y_k}A_3+\frac{\sin 2\alpha t}{2}\partial_{y_k}A_4\bigg)\\
&\qquad\times G(O(\alpha t)x-y,s)\frac{\mathrm{d}s}{4s^2}\,\mathrm{d}t\\
&\quad+\int_0^\infty\int_t^\infty (O(\alpha t)x-y)_k\bigg((\cos\alpha t)A_1+(\sin\alpha t)A_2+\frac{\cos 2\alpha t}{2}A_3+\frac{\sin 2\alpha t}{2}A_4\bigg)\\
&\qquad\times G(O(\alpha t)x-y,s)\frac{\mathrm{d}s}{8s^3}\,\mathrm{d}t\\
&=I_k(x,y)+II_k(x,y)\,.
\end{aligned}
\tag{2.48}
$$

To estimate $I_k(x,y)$ we observe that

$$
\begin{aligned}
&\bigg|\int_0^\infty\int_t^\infty\Big((\cos\alpha t)\partial_{y_k}A_1+(\sin\alpha t)\partial_{y_k}A_2\Big)G(O(\alpha t)x-y,s)\frac{\mathrm{d}s}{4s^2}\,\mathrm{d}t\bigg|\\
&\le C|y|\int_0^\infty\int_t^\infty e^{-\frac{|x|^2}{16s}}\frac{\mathrm{d}s}{s^3}\,\mathrm{d}t\;\le\;C\frac{|y|}{|x|^2}\,,\qquad |x|>2|y|\,,
\end{aligned}
\tag{2.49}
$$

and that

$$
\begin{aligned}
&\bigg|\int_0^\infty\int_t^\infty\Big(\frac{\cos 2\alpha t}{2}\partial_{y_k}A_3+\frac{\sin 2\alpha t}{2}\partial_{y_k}A_4\Big)G(O(\alpha t)x-y,s)\frac{\mathrm{d}s}{4s^2}\,\mathrm{d}t\bigg|\\
&\le\bigg|\int_0^\infty\int_t^\infty\Big(\frac{\cos 2\alpha t}{2}\partial_{y_k}A_3+\frac{\sin 2\alpha t}{2}\partial_{y_k}A_4\Big)\Big(G(O(\alpha t)x-y,s)-G(x,s)\Big)\frac{\mathrm{d}s}{4s^2}\,\mathrm{d}t\bigg|\\
&\quad+\bigg|\int_0^\infty\int_t^\infty\Big(\frac{\cos 2\alpha t}{2}\partial_{y_k}A_3+\frac{\sin 2\alpha t}{2}\partial_{y_k}A_4\Big)G(x,s)\frac{\mathrm{d}s}{4s^2}\,\mathrm{d}t\bigg|\\
&\le C\frac{|y|}{|x|^2}+C\min\Big\{\frac{1}{|\alpha||x|^3},\frac{1}{|x|}\Big\}\,,\qquad |x|>2|y|\,.
\end{aligned}
\tag{2.50}
$$

Here we have used (2.14) for the first term and (2.16) for the second term to derive the last line. It remains to estimate $II_k(x,y)$ in (2.48). We consider the case $k=1$ only, for the case $k=2$ is obtained in the same way. A direct computation yields the following identity:

$$
\begin{aligned}
&(O(\alpha t)x-y)_1\big((\cos\alpha t)A_1+(\sin\alpha t)A_2\big)\\
&=\frac{|x|^2}{2}\begin{pmatrix}x_1 & 0\\ x_2 & 0\end{pmatrix}+(\cos 2\alpha t)D_1(x,y)+(\sin 2\alpha t)D_2(x,y)+D_3(x,y,\alpha t)\,.
\end{aligned}
\tag{2.51}
$$

Here D_1 and D_2 are the matrices whose components are suitable sums of the third order polynomials of the form $x_1^{l_1}x_2^{l_2}y_1^{k_1}y_2^{k_2}$ with $l_1+l_2\geq 1$, while $D_3(x,y,\alpha t)$ is estimated as $|D_3|\leq C|x|^2|y|$ for $|x|>2|y|$. Recalling the expression of $L^{112}(x,y)$ in (2.30), we have

$$\begin{aligned}
&\Big|\int_0^\infty\int_t^\infty (O(\alpha t)x-y)_1\big((\cos\alpha t)A_1+(\sin\alpha t)A_2\big)G(O(\alpha t)x-y,s)\frac{\mathrm{d}s}{8s^3}\,\mathrm{d}t-\partial_{y_1}L^{112}(x,y)\Big|\\
&=\Big|\int_0^\infty\int_t^\infty\Big((\cos 2\alpha t)D_1+(\sin 2\alpha t)D_2+D_3\Big)G(O(\alpha t)x-y,s)\frac{\mathrm{d}s}{8s^3}\,\mathrm{d}t\Big|\\
&\leq\Big|\int_0^\infty\int_t^\infty\Big((\cos 2\alpha t)D_1+(\sin 2\alpha t)D_2+D_3\Big)\Big(G(O(\alpha t)x-y,s)-G(x,s)\Big)\frac{\mathrm{d}s}{8s^3}\,\mathrm{d}t\Big|\\
&\quad+\Big|\int_0^\infty\int_t^\infty\Big((\cos 2\alpha t)D_1+(\sin 2\alpha t)D_2+D_3\Big)G(x,s)\frac{\mathrm{d}s}{8s^3}\,\mathrm{d}t\Big|\\
&\leq C\frac{|y|}{|x|^2}+C\min\big\{\frac{1}{|\alpha||x|^3},\frac{1}{|x|}\big\}\,,\qquad |x|>2|y|\,. \qquad (2.52)
\end{aligned}$$

Here, we have again applied (2.14) for the first term and (2.16) for the second term to derive the last line. Finally we have

$$\begin{aligned}
&\Big|\int_0^\infty\int_t^\infty (O(\alpha t)x-y)_1\Big(\frac{\cos 2\alpha t}{2}A_3+\frac{\sin 2\alpha t}{2}A_4\Big)G(O(\alpha t)x-y,s)\frac{\mathrm{d}s}{8s^3}\,\mathrm{d}t\Big|\\
&\leq C(|x|+|y|)|x||y|\int_0^\infty\int_t^\infty e^{-\frac{|x|^2}{16s}}\frac{\mathrm{d}s}{s^4}\,\mathrm{d}t\;\leq\;C\frac{|y|}{|x|^2}\,,\qquad |x|>2|y|\,. \qquad (2.53)
\end{aligned}$$

Collecting (2.49), (2.50), (2.52), and (2.53), we have shown that

$$\big|\partial_{y_1}\big(\tilde\Gamma_\alpha^{112}(x,y)-L^{112}(x,y)\big)\big|\leq C\bigg(\frac{|y|}{|x|^2}+\min\big\{\frac{1}{|\alpha||x|^3},\frac{1}{|x|}\big\}\bigg)\,,\quad |x|>2|y|\,. \qquad (2.54)$$

The estimate of $\partial_{y_2}\big(\tilde\Gamma_\alpha^{112}(x,y)-L^{112}(x,y)\big)$ is obtained in the similar manner. Thus, from (2.47) and (2.54) we have obtained the estimates of the derivatives in y for $\Gamma_\alpha^{11}(x,y)$. The proof of Lemma 2.3.3 is complete. □

Proof of Theorem 2.3.1: The assertion that $u=L[f]$ is a weak solution to ($\mathrm{S}_{\alpha,\mathbb{R}^2}$) (whose definitions are stated in the beginning of this subsection) follows from a similar argument as in [35, Proposition 3.2]. Hence we omit its details and focus on the estimates of u here.
(i) Let $\gamma\in[0,1)$. Suppose that $\operatorname{supp} f\subset B_R$ for some $R\geq 1$. Note that $\frac{(y^\perp\cdot f(y))x^\perp}{4\pi|x|^2}=L(x,y)f(y)$ holds. Let $|x|\geq 2R$. Then we have from Lemma 2.3.3 with $m=0$,

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^2}\Gamma_\alpha(x,y)f\,\mathrm{d}y-c[f]\frac{x^\perp}{4\pi|x|^2}\Big|\\
&=\Big|\int_{|y|\leq R}\big(\Gamma_\alpha(x,y)-L(x,y)\big)f(y)\,\mathrm{d}y\Big|\\
&\leq C\int_{|y|\leq R}\bigg(\min\big\{\frac{1}{|\alpha||x|^2},\frac{1}{|\alpha|^{\frac12}|x|}\big\}+|x|\min\big\{\frac{1}{|\alpha||x|^3},\frac{1}{|x|}\big\}+\frac{|y|^2}{|x|^2}\bigg)|f(y)|\,\mathrm{d}y\,,
\end{aligned}$$

which implies $L[f](x) = c[f]\frac{x^\perp}{4\pi|x|^2} + \mathcal{R}[f](x)$ with

$$
\begin{aligned}
|x|^{1+\gamma}\,|\mathcal{R}[f](x)| \le C\Big(&\min\big\{\frac{1}{|\alpha||x|^{1-\gamma}}, \frac{|x|^\gamma}{|\alpha|^{\frac12}}\big\}\|f\|_{L^1(B_R)} \\
&+ \min\big\{\frac{1}{|\alpha||x|^{1-\gamma}}, |x|^{1+\gamma}\big\}\|f\|_{L^1(B_R)} + \||y|^{1+\gamma} f\|_{L^1(B_R)}\Big)\,.
\end{aligned}
\tag{2.55}
$$

Here C is independent of x, R, α, γ, and f. Then we use the inequality for $\gamma \in [0,1)$,

$$
\min\big\{\frac{1}{|\alpha||x|^{1-\gamma}}, \frac{|x|^\gamma}{|\alpha|^{\frac12}}\big\} \le |\alpha|^{-\frac{1+\gamma}{2}}\,, \qquad \min\big\{\frac{1}{|\alpha||x|^{1-\gamma}}, |x|^{1+\gamma}\big\} \le |\alpha|^{-\frac{1+\gamma}{2}}\,, \tag{2.56}
$$

which leads to (2.22).

(ii) Let $\gamma \in [0,1)$ and write $\Gamma_\alpha(x,y) = \big(\Gamma_\alpha(x,y)_{ij}\big)_{1\le i,j\le 2}$ and $L(x,y) = (L(x,y)_{ij})_{1\le i,j\le 2}$. From the integration by parts we see for $k=1,2$ and $f = (\sum_{l=1,2}\partial_l F_{1l}, \sum_{l=1,2}\partial_l F_{2l})^\top$,

$$
\begin{aligned}
&\int_{\mathbb{R}^2} e^{-\epsilon|y|^2}(\Gamma_\alpha(x,y)f)_k \,\mathrm{d}y \;=\; \sum_{j=1,2}\int_{\mathbb{R}^2} e^{-\epsilon|y|^2}\Gamma_\alpha(x,y)_{kj} f_j \,\mathrm{d}y \\
&= -\sum_{j=1,2}\sum_{l=1,2}\int_{\mathbb{R}^2} e^{-\epsilon|y|^2}\partial_{y_l}\Gamma_\alpha(x,y)_{kj}F_{jl}\,\mathrm{d}y + 2\epsilon\sum_{j=1,2}\sum_{l=1,2}\int_{\mathbb{R}^2} e^{-\epsilon|y|^2} y_l\Gamma_\alpha(x,y)_{kj}F_{jl}\,\mathrm{d}y \\
&= -\sum_{j=1,2}\sum_{l=1,2}\int_{\mathbb{R}^2} e^{-\epsilon|y|^2}\partial_{y_l}\big(\Gamma_\alpha(x,y)_{kj} - L(x,y)_{kj}\big)F_{jl}\,\mathrm{d}y \\
&\quad -\sum_{j=1,2}\sum_{l=1,2}\int_{\mathbb{R}^2} e^{-\epsilon|y|^2}\partial_{y_l}L(x,y)_{kj}\,F_{jl}\,\mathrm{d}y + 2\epsilon\int_{\mathbb{R}^2} e^{-\epsilon|y|^2}(\Gamma_\alpha(x,y)\,F\,y)_k\,\mathrm{d}y\,.
\end{aligned}
$$

Note that

$$
\big(-\sum_{j=1,2}\sum_{l=1,2}\partial_{y_l}L(x,y)_{1j}\,F_{jl}\,,\; -\sum_{j=1,2}\sum_{l=1,2}\partial_{y_l}L(x,y)_{2j}\,F_{jl}\big)^\top \;=\; (F_{12}-F_{21})\frac{x^\perp}{4\pi|x|^2}
$$

by the definition of $L(x,y)$. Moreover, we have $|\Gamma_\alpha(x,y)| \le \frac{C(\alpha,|x|)}{|y|}$ for $|y| > 2|x|$ by [35, Proposition 3.1], and $\int_{|y|\le 2|x|}|\Gamma_\alpha(x,y)|\,\mathrm{d}y \le C'(\alpha,|x|) < \infty$ by [35, Lemma 3.3], which implies

$$
\lim_{\epsilon\to 0}\epsilon\int_{\mathbb{R}^2} e^{-\epsilon|y|^2}\Gamma_\alpha(x,y)\,F\,y\,\mathrm{d}y \;=\; 0
$$

for $F \in L^\infty_{2+\gamma}(\mathbb{R}^2)^{2\times 2}$. For simplicity we use the next notations:

$$
\begin{aligned}
\nabla_y\Gamma_\alpha(x,y)\,F \;&=\; \big(\sum_{j=1,2}\sum_{l=1,2}\partial_{y_l}\Gamma_\alpha(x,y)_{1j}\,F_{jl}\,,\; \sum_{j=1,2}\sum_{l=1,2}\partial_{y_l}\Gamma_\alpha(x,y)_{2j}\,F_{jl}\big)^\top\,, \\
\nabla_y L(x,y)\,F \;&=\; \big(\sum_{j=1,2}\sum_{l=1,2}\partial_{y_l}L(x,y)_{1j}\,F_{jl}\,,\; \sum_{j=1,2}\sum_{l=1,2}\partial_{y_l}L(x,y)_{2j}\,F_{jl}\big)^\top\,.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
L[f](x) \;&=\; -\int_{\mathbb{R}^2}\nabla_y\Gamma_\alpha(x,y)\,F(y)\,\mathrm{d}y \\
&= -\int_{|y|<\frac{|x|}{2}}\nabla_y\big(\Gamma_\alpha(x,y) - L(x,y)\big)F(y)\,\mathrm{d}y - \int_{|y|\ge\frac{|x|}{2}}\nabla_y\Gamma_\alpha(x,y)\,F(y)\,\mathrm{d}y \\
&\quad -\lim_{\epsilon\to 0}\int_{|y|\ge\frac{|x|}{2}} e^{-\epsilon|y|^2}\big(F_{12}(y) - F_{21}(y)\big)\,\mathrm{d}y\frac{x^\perp}{4\pi|x|^2} + \tilde{c}[F]\frac{x^\perp}{4\pi|x|^2}\,.
\end{aligned}
\tag{2.57}
$$

The sum of the first three terms of the right-hand side of this equality is denoted by $\mathcal{R}[f]$. To estimate $\mathcal{R}[f]$ we firstly observe from Lemma 2.3.3,

$$\begin{aligned}&\Big|\int_{|y|<\frac{|x|}{2}} \nabla_y\big(\Gamma_\alpha(x,y)-L(x,y)\big)F(y)\,\mathrm{d}y\Big|\\&\le C\bigg(\frac{1}{|x|^2}\int_{|y|<\frac{|x|}{2}}|y\,F(y)|\,\mathrm{d}y+\min\big\{\frac{1}{|\alpha||x|^3},\frac{1}{|x|}\big\}\int_{|y|<\frac{|x|}{2}}|F(y)|\,\mathrm{d}y\bigg)\,,\qquad x\neq 0\,.\end{aligned}\tag{2.58}$$

Next we have from the direct calculation

$$|(\nabla_x K)(x,t)|\le C\big(t^{-\frac32}e^{-\frac{|x|^2}{16t}}+\int_t^\infty s^{-\frac52}e^{-\frac{|x|^2}{16s}}\,\mathrm{d}s\big)\,,$$

which implies

$$\int_0^\infty |(\nabla K)(O(\alpha t)x,t)|\,\mathrm{d}t\le\frac{C}{|x|}\,,\qquad x\neq 0\,.$$

Then by the transformation of the variables $y=O(\alpha t)z$ we have

$$\begin{aligned}&\Big|\int_{|y|\ge\frac{|x|}{2}}\nabla_y\Gamma_\alpha(x,y)F(y)\,\mathrm{d}y\Big|\\&\le\int_{|y|\ge\frac{|x|}{2}}\bigg(\int_0^\infty|(\nabla K)(O(\alpha t)x-y,t)|\,\mathrm{d}t\bigg)|F(y)|\,\mathrm{d}y\\&\le\|F\|_{L^\infty_{2+\gamma}(B^{\mathrm{c}}_{\frac{|x|}{2}})}\int_{|z|\ge\frac{|x|}{2}}\bigg(\int_0^\infty|(\nabla K)(O(\alpha t)(x-z),t)|\,\mathrm{d}t\bigg)|z|^{-2-\gamma}\,\mathrm{d}z\\&\le C\|F\|_{L^\infty_{2+\gamma}(B^{\mathrm{c}}_{\frac{|x|}{2}})}\int_{|z|\ge\frac{|x|}{2}}|x-z|^{-1}|z|^{-2-\gamma}\,\mathrm{d}z\\&\le\frac{C}{|x|^{1+\gamma}}\|F\|_{L^\infty_{2+\gamma}(B^{\mathrm{c}}_{\frac{|x|}{2}})}\,.\end{aligned}\tag{2.59}$$

Here C is independent of x and $\gamma\in[0,1)$. Collecting (2.57), (2.58), and (2.59), we obtain (2.23) and (2.24). The proof of Theorem 2.3.1 is complete. □

Based on the results of Theorem 2.3.1 we study the exterior problem (S_α) in the next subsection, where its asymptotics profile is represented as a solution to ($\mathrm{S}_{\alpha,\mathbb{R}^2}$) by a cut-off technique. However, the existence of solutions to (S_α) decaying at spatial infinity has to be proved carefully. As in [35], for the exterior problem, a way to construct decaying solutions is to consider first a regularized system and to take the limit; see the proof of Theorem 2.3.8 for details. In this procedure we need to treat the following system in the whole space:

$$\begin{cases}\lambda u_\lambda-\Delta u_\lambda-\alpha(x^\perp\cdot\nabla u_\lambda-u_\lambda^\perp)+\nabla p_\lambda\;=\;f\,,\quad \operatorname{div}u_\lambda\;=\;0\,,\qquad x\in\mathbb{R}^2\,,\\ \qquad\qquad\qquad\qquad\qquad\qquad u_\lambda\;\to\;0\,,\qquad |x|\to\infty\,,\end{cases}\tag{$\mathrm{S}^\lambda_{\alpha,\mathbb{R}^2}$}$$

where λ is a small positive number. Let us introduce the integral kernel $\Gamma^\lambda_\alpha(x,y)$ as

$$\Gamma^\lambda_\alpha(x,y)\;=\;\int_0^\infty e^{-\lambda t}O(\alpha t)^\top K(O(\alpha t)x-y,t)\,\mathrm{d}t\,,\qquad x\neq y\,.\tag{2.60}$$

In virtue of the positive λ, the integral in (2.60) converges absolutely for $x \neq y$. Furthermore, the velocity u_λ defined by

$$u_\lambda(x) \;=\; \int_{\mathbb{R}^2} \Gamma_\alpha^\lambda(x,y) f(y)\,\mathrm{d}y\,, \qquad f \in L^2(\mathbb{R}^2)^2\,, \tag{2.61}$$

satisfies ($\mathrm{S}^\lambda_{\alpha,\mathbb{R}^2}$) in the sense of distributions with a suitable pressure ∇p_λ. The next lemma will be used in the proof of Theorem 2.3.8.

Lemma 2.3.5 *Let $\alpha \in \mathbb{R} \setminus \{0\}$ and $\gamma \in [0,1)$. Suppose that $f \in L^2(\mathbb{R}^2)^2$ is of the form $f = \operatorname{div} F$ with some $F \in L^\infty_{2+\gamma}(\mathbb{R}^2)^{2\times 2}$. Then for any $\theta \in (0,1)$ and $R \geq 1$, the velocity u_λ defined by* (2.61) *satisfies*

$$\|u_\lambda\|_{L^\infty_\theta(B^c_{2R})} \leq C\big(\|F\|_{L^\infty_{2+\gamma}(B^c_R)} + \|F\|_{L^1(B_R)}\big)\,. \tag{2.62}$$

Here the constant C is independent of λ and γ, and depends only on θ and R.

Proof: In the same way as in the proof of Lemma 2.3.3, we define $L^\lambda = L^\lambda(x,y)$ by

$$L^\lambda(x,y) \;=\; L^{\lambda,0}(x,y) + L^{\lambda,111}(x,y) + L^{\lambda,112}(x,y) + L^{\lambda,12}(x,y)\,,$$

where

$$\begin{aligned}
L^{\lambda,0}(x,y) \;&=\; \int_0^\infty e^{-\lambda t} G(x,t)\frac{\mathrm{d}t}{4t}\begin{pmatrix} x\cdot y & x^\perp \cdot y \\ -x^\perp\cdot y & x\cdot y\end{pmatrix}\,,\\
L^{\lambda,111}(x,y) \;&=\; \int_0^\infty\int_t^\infty e^{-\lambda t} G(x,s)\frac{\mathrm{d}s}{4s^2}\,\mathrm{d}t\left(\frac{-3(x\otimes y) + (x^\perp \otimes y^\perp)}{2}\right)\,,\\
L^{\lambda,112}(x,y) \;&=\; \int_0^\infty\int_t^\infty e^{-\lambda t} G(x,s)\frac{\mathrm{d}s}{16s^3}\,\mathrm{d}t\,|x|^2(x\otimes y)\,,\\
L^{\lambda,12}(x,y) \;&=\; -\int_0^\infty\int_t^\infty e^{-\lambda t} G(x,s)\frac{\mathrm{d}s}{8s^2}\,\mathrm{d}t\begin{pmatrix} x\cdot y & x^\perp \cdot y \\ -x^\perp\cdot y & x\cdot y\end{pmatrix}\,.
\end{aligned}$$

Then we have

$$\begin{aligned}
|\nabla_y L^\lambda(x,y)| &\leq C|x|\bigg(\int_0^\infty e^{-\frac{|x|^2}{4t}}\frac{\mathrm{d}t}{t^2} + \int_0^\infty\int_t^\infty e^{-\frac{|x|^2}{4s}}\frac{\mathrm{d}s}{s^3}\,\mathrm{d}t + |x|^2\int_0^\infty\int_t^\infty e^{-\frac{|x|^2}{4s}}\frac{\mathrm{d}s}{s^4}\,\mathrm{d}t\bigg)\\
&\leq \frac{C}{|x|}\,, \qquad |x| > 0\,,
\end{aligned} \tag{2.63}$$

where the constant C is independent of α and λ. By integration by parts we rewrite u_λ as

$$\begin{aligned}
u_\lambda(x) \;&=\; -\int_{\mathbb{R}^2} \nabla_y \Gamma^\lambda_\alpha(x,y)\,F(y)\,\mathrm{d}y\\
&=\; -\int_{|y|<\frac{|x|}{2}} \nabla_y\big(\Gamma^\lambda_\alpha(x,y) - L^\lambda(x,y)\big)F(y)\,\mathrm{d}y - \int_{|y|\geq\frac{|x|}{2}} \nabla_y \Gamma^\lambda_\alpha(x,y)F(y)\,\mathrm{d}y\\
&\quad - \int_{|y|<\frac{|x|}{2}} \nabla_y L^\lambda(x,y)F(y)\,\mathrm{d}y\,.
\end{aligned} \tag{2.64}$$

Then, proceeding as in the proof of Lemma 2.3.3, we obtain

$$|\nabla_y\big(\Gamma^\lambda_\alpha(x,y) - L^\lambda(x,y)\big)| \leq C\bigg(\frac{|y|}{|x|^2} + \min\Big\{\frac{1}{|\alpha||x|^3}, \frac{1}{|x|}\Big\}\bigg)\,, \qquad |x| > 2|y|\,, \tag{2.65}$$

where C is independent of x, y, α, and λ. Then we have

$$\begin{aligned}\Big|\int_{|y|<\frac{|x|}{2}} \nabla_y\big(\Gamma_\alpha^\lambda(x,y)-L^\lambda(x,y)\big)F(y)\,\mathrm{d}y\Big| &\le \frac{C}{|x|}\|F\|_{L^1(B_{\frac{|x|}{2}})}\\ &\le \frac{C\log(2+|x|)}{|x|}\|F\|_{L^\infty_{2+\gamma}(\mathbb{R}^2)}\,,\qquad |x|>1\,,\end{aligned} \tag{2.66}$$

where the constant C is independent of λ and γ. The second term in the right-hand side of (2.64) is also estimated as in the proof of Lemma 2.3.3, resulting the estimate

$$\Big|\int_{|y|\ge\frac{|x|}{2}} \nabla_y\Gamma_\alpha^\lambda(x,y)F(y)\,\mathrm{d}y\Big| \le \frac{C}{|x|^{1+\gamma}}\|F\|_{L^\infty_{2+\gamma}(B^c_{\frac{|x|}{2}})}\,. \tag{2.67}$$

For the last term in the right-hand side of (2.64) it is straightforward from (2.63) to see

$$\Big|\int_{|y|<\frac{|x|}{2}} \nabla_y L^\lambda(x,y)F\,\mathrm{d}y\Big| \le \frac{C\log(2+|x|)}{|x|}\|F\|_{L^\infty_{2+\gamma}(\mathbb{R}^2)}\,,\qquad |x|>1\,. \tag{2.68}$$

Collecting (2.66), (2.67), and (2.68), we obtain (2.62). This completes the proof. □

2.3.2 Linear estimate in the exterior domain

In this subsection we study the asymptotic estimates for solutions to the Stokes system in the exterior domain

$$\begin{cases} -\Delta u-\alpha(x^\perp\cdot\nabla u-u^\perp)+\nabla p \,=\, f\,, \quad \operatorname{div} u \,=\, 0\,, \qquad x\in\Omega\,,\\ \qquad\qquad\qquad\qquad u \,=\, 0\,, \qquad x\in\partial\Omega\,,\\ \qquad\qquad\qquad\qquad u \,\to\, 0\,, \qquad |x|\to\infty\,, \end{cases} \tag{S_α}$$

where $\alpha\in\mathbb{R}\setminus\{0\}$ is a given constant. In the following, we fix a positive number $R_0\ge 1$ large enough so that $\mathbb{R}^2\setminus\Omega\subset B_{R_0}$ holds. We also fix a radial cut-off function $\varphi\in C_0^\infty(\mathbb{R}^2)$ such that $\varphi(x)=1$ for $|x|\le R_0$ and $\varphi(x)=0$ for $|x|\ge 2R_0$. As in the previous subsection, for $f\in L^2(\Omega)^2$ and $F\in L^2(\Omega)^{2\times 2}$ we formally set

$$\begin{aligned} c_\Omega[f] \,&=\, \lim_{\epsilon\to 0}\int_\Omega e^{-\epsilon|y|^2}y^\perp\cdot f(y)\,\mathrm{d}y\,,\\ \tilde c_\Omega[F] \,&=\, \lim_{\epsilon\to 0}\int_\Omega e^{-\epsilon|y|^2}\big(F_{12}(y)-F_{21}(y)\big)\,\mathrm{d}y\,. \end{aligned} \tag{2.69}$$

These are well-defined at least when $f=\operatorname{div}F$ with $F\in L^\infty_{2+\gamma}(\Omega)^{2\times 2}$ for some $\gamma>0$, and $c_\Omega[f]=\tilde c_\Omega[F]$ holds in this case if the generalized traces $\nu\cdot(x_2\vec{F_1})$, $\nu\cdot(x_1\vec{F_2})$ on $\partial\Omega$ are zero in addition. Here we have set $F=(\vec{F_1},\vec{F_2})^\top$. Note that the coefficient $\tilde c_\Omega[F]$ is well-defined under the condition $F_{12}-F_{21}\in L^1(\Omega)$. In general, we have the following.

Lemma 2.3.6 *Let $f\in L^2(\Omega)^2$ be of the form $f=\operatorname{div}F=(\sum_{j=1,2}\partial_j F_{1j},\sum_{j=1,2}\partial_j F_{2j})^\top$ for some $F\in L^\infty_{2,0}(\Omega)^{2\times 2}$ and $F_{12}-F_{21}\in L^1(\Omega)$. Then both $c_\Omega[f]$ and $\tilde c_\Omega[F]$ converge.*

Proof: It is trivial that $\tilde{c}_\Omega[F]$ converges. Let $\varphi \in C_0^\infty(\mathbb{R}^2)$ be a cut-off function introduced at the beginning of this subsection. The convergence of $c_\Omega[f]$ easily follows from the integration by parts:

$$\begin{aligned} c_\Omega[f] \;&=\; \int_\Omega y^\perp \cdot f\varphi \,\mathrm{d}y + \lim_{\epsilon\to 0}\int_\Omega e^{-\epsilon|y|^2}(1-\varphi)y^\perp\cdot f\,\mathrm{d}y \\ &=\; \int_\Omega y^\perp \cdot f\varphi\,\mathrm{d}y + \tilde{c}_\Omega[F] - \int_\Omega (F_{12}-F_{21})\varphi\,\mathrm{d}y \\ &\quad + \int_\Omega y^\perp\cdot F\nabla\varphi\,\mathrm{d}y + \lim_{\epsilon\to 0} 2\int_\Omega e^{-\epsilon|y|^2}\epsilon y^\perp\cdot(Fy)(1-\varphi)\,\mathrm{d}y\,. \end{aligned} \tag{2.70}$$

The last term in the right-hand side of (2.70) vanishes in virtue of the decay $|F(x)| = o(|x|^{-2})$ as $|x|\to\infty$. In fact, by extending F to the whole space by zero we have

$$\begin{aligned} \Big|\int_\Omega e^{-\epsilon|y|^2}\epsilon y\cdot\big(F(y)y^\perp\big)(1-\varphi)\,\mathrm{d}y\Big| &\le \int_{\mathbb{R}^2} e^{-\epsilon|y|^2}\epsilon|y|^2|F(y)|\,\mathrm{d}y \\ &= \int_{\mathbb{R}^2} e^{-|z|^2}\big(\frac{|z|}{\epsilon^{\frac12}}\big)^2\,\big|F\big(\frac{z}{\epsilon^{\frac12}}\big)\big|\,\mathrm{d}z\,, \end{aligned}$$

where we have used the transformation of the variables $y = \epsilon^{-\frac12}z$. Then the Lebesgue dominated convergence theorem implies the right-hand side of the above inequality goes to zero as $\epsilon\to 0$. In particular, we have

$$c_\Omega[f] \;=\; \tilde{c}_\Omega[F] + \int_\Omega \big\{\big(y^\perp\cdot f - F_{12} + F_{21}\big)\varphi + y^\perp\cdot F\nabla\varphi\big\}\,\mathrm{d}y\,. \tag{2.71}$$

The proof is complete. □

Let us denote by $T(u,p)$ the stress tensor, which is defined as

$$T(u,p) \;=\; Du - p\mathbb{I}\,,\qquad Du = \nabla u + (\nabla u)^\top\,,\qquad \mathbb{I} = (\delta_{jk})_{1\le j,k\le 2}\,. \tag{2.72}$$

The next lemma is a counterpart of [35, Theorem 2.1] in our functional setting. We denote by Ω_r the truncated domain defined as $\Omega_r = \{x\in\Omega \mid |x| < r\}$ for $r > 0$.

Lemma 2.3.7 *Let $\alpha\in\mathbb{R}\setminus\{0\}$ and $\gamma\in[0,1)$. Assume that $f\in L^2(\Omega)^2$ is of the form $f = \operatorname{div} F$ with some $F\in L^\infty_{2+\gamma}(\Omega)^{2\times 2}$, and that $\tilde{c}_\Omega[F]$ converges when $\gamma = 0$. Suppose that $(u,\nabla p)\in W^{2,2}_{\mathrm{loc}}(\overline{\Omega})^2\times L^2_{\mathrm{loc}}(\overline{\Omega})^2$ is a solution to the system* (S_α) *satisfying $\|\nabla u\|_{L^2(\Omega)} < \infty$ and $\lim\limits_{|x|\to\infty}|u(x)| = 0$. Then u is represented as*

$$u(x) \;=\beta\frac{x^\perp}{4\pi|x|^2} + \mathcal{R}(x)\,,\qquad x\in\Omega\setminus\{0\}\,, \tag{2.73}$$

where

$$\begin{aligned} \beta \;&=\; \int_{\partial\Omega} y^\perp\cdot\big(T(u,p)\nu\big)\,\mathrm{d}\sigma_y \;+\; b_\Omega[f]\,, \\ b_\Omega[f] \;&=\; \tilde{c}_\Omega[F] + \int_\Omega\big\{\big(y^\perp\cdot f - F_{12} + F_{21}\big)\varphi + y^\perp\cdot F\nabla\varphi\big\}\,\mathrm{d}y\,, \end{aligned} \tag{2.74}$$

while $\mathcal{R}$ satisfies

$$
\begin{aligned}
\|\mathcal{R}\|_{L^\infty_{1+\gamma}(B^c_{4R_0})} \le C\Big(&\|F\|_{L^\infty_{2+\gamma}(B^c_{2R_0})} + \sup_{|x|\ge 4R_0} |x|^{-1+\gamma}\|yF\|_{L^1(\Omega_{\frac{|x|}{2}})} \\
&+ \sup_{|x|\ge 4R_0} \min\big\{\frac{1}{|\alpha||x|^{2-\gamma}}, |x|^\gamma\big\}\|F\|_{L^1(\Omega_{\frac{|x|}{2}})} \\
&+ \sup_{|x|\ge 4R_0} |x|^\gamma \big|\lim_{\epsilon\to 0}\int_{2|y|\ge|x|} e^{-\epsilon|y|^2}(F_{12}-F_{21})\,\mathrm{d}y\big|\Big) \\
&+ C\big(|\alpha|^{-\frac{1+\gamma}{2}} + |\alpha|^{-\frac12}+1\big)\big(\|F\|_{L^2(\Omega_{2R_0})} + (1+|\alpha|)\|\nabla u\|_{L^2(\Omega_{2R_0})}\big)\,.
\end{aligned}
\tag{2.75}
$$

Here the constant C is independent of γ, α, and F. The coefficient $b_\Omega[f]$ coincides with $c_\Omega[f]$ when F belongs in addition to $L^\infty_{2,0}(\Omega)^{2\times 2}$.

Proof: We may assume that $\int_{\Omega_{2R_0}} p\,\mathrm{d}x = 0$. Let $\varphi \in C_0^\infty(\mathbb{R}^2)$ be a cut-off function introduced at the beginning of this subsection. We introduce the Bogovskii operator $\mathbb{B}$ in the closed annulus $A = \{x\in\mathbb{R}^2 \mid R_0 \le |x| \le 2R_0\}$, and set

$$
v \,=\, (1-\varphi)u + \mathbb{B}[\nabla\varphi\cdot u]\,, \qquad q \,=\, (1-\varphi)p\,.
$$

Note that $\mathbb{B}[\nabla\varphi\cdot u]$ satisfies

$$
\operatorname{supp}\mathbb{B}[\nabla\varphi\cdot u] \subset A\,, \qquad \operatorname{div}\mathbb{B}[\nabla\varphi\cdot u] \,=\, \nabla\varphi\cdot u\,, \tag{2.76}
$$

and the estimates

$$
\|\mathbb{B}[\nabla\varphi\cdot u]\|_{W^{m+1,2}(\Omega)} \le C\|\nabla\varphi\cdot u\|_{W^{m,2}(\Omega)}\,, \qquad m=0,1\,. \tag{2.77}
$$

See, e.g. Borchers and Sohr [6]. Then $(v,\nabla q)$ satisfies

$$
-\Delta v - \alpha(x^\perp\cdot\nabla v - v^\perp) + \nabla q \,=\, \operatorname{div}\mathcal{F} + g\,, \quad \operatorname{div} v \,=\, 0\,, \qquad x\in\mathbb{R}^2\,, \tag{2.78}
$$

where $\mathcal{F}$ and g are the functions on $\mathbb{R}^2$ given by

$$
\begin{aligned}
\mathcal{F} \,&=\, (1-\varphi)F - \nabla\mathbb{B}[\nabla\varphi\cdot u]\,,\\
g \,&=\, F\cdot\nabla\varphi + 2\nabla\varphi\cdot\nabla u + (\Delta\varphi + \alpha x^\perp\cdot\nabla\varphi)u\\
&\quad - \alpha\big(x^\perp\nabla\mathbb{B}[\nabla\varphi\cdot u] - \mathbb{B}[\nabla\varphi\cdot u]^\perp\big) - (\nabla\varphi)p\,.
\end{aligned}
$$

Note that $\operatorname{supp} g \subset A$ by (2.76). Recalling the uniqueness result in Remark 2.3.2, we find

$$
\begin{aligned}
u(x) \,=\, v(x) \,&=\, L[\operatorname{div}\mathcal{F}] + L[g]\\
&=\, \big(\tilde{c}[\mathcal{F}] + c[g]\big)\frac{x^\perp}{4\pi|x|^2} + \mathcal{R}(x)\,, \qquad |x|\ge 4R_0\,,
\end{aligned}
\tag{2.79}
$$

where $\tilde{c}[\mathcal{F}]$ and $c[g]$ are defined in (2.19). Recalling that $R_0 \geq 1$, we see from Theorem 2.3.1 that $\mathcal{R}(x)$ satisfies

$$\begin{aligned}
\|\mathcal{R}\|_{L^\infty_{1+\gamma}(B^c_{4R_0})} &\leq C\big(\|\mathcal{R}[\operatorname{div}\mathcal{F}]\|_{L^\infty_{1+\gamma}(B^c_{4R_0})} + \|\mathcal{R}[g]\|_{L^\infty_{1+\gamma}(B^c_{4R_0})}\big) \\
&\leq C\Bigg(\|F\|_{L^\infty_{2+\gamma}(B^c_{2R_0})} + \sup_{|x|\geq 4R_0} |x|^{-1+\gamma}\|yF\|_{L^1(\{2R_0\leq|y|\leq\frac{|x|}{2}\})} \\
&\quad + \sup_{|x|\geq 4R_0} \min\big\{\frac{1}{|\alpha||x|^{2-\gamma}}, |x|^\gamma\big\}\|F\|_{L^1(\{2R_0\leq|y|\leq\frac{|x|}{2}\})} \\
&\quad + \sup_{|x|\geq 4R_0} |x|^\gamma \big|\lim_{\epsilon\to 0}\int_{2|y|\geq|x|} e^{-\epsilon|y|^2}(F_{12}-F_{21})\,\mathrm{d}y\big| \\
&\quad + \big(\sup_{|x|\geq 4R_0} \min\big\{\frac{1}{|\alpha||x|^{2-\gamma}}, |x|^\gamma\big\} + 1\big)\|\mathcal{F}\|_{L^1(B_{2R_0})}\Bigg) \\
&\quad + C(|\alpha|^{-\frac{1+\gamma}{2}} + 1)\|g\|_{L^1(B_{2R_0})}\,.
\end{aligned}$$

Here C depends only on R_0. It is easy to see

$$\|\mathcal{F}\|_{L^1(B_{2R_0})} \leq C\big(\|F\|_{L^2(\Omega_{2R_0})} + \|\nabla u\|_{L^2(\Omega_{2R_0})}\big)$$

by applying (2.77) and the Poincaré inequality. Similarly, the function g is estimated as

$$\|g\|_{L^1(B_{2R_0})} \leq C\big(\|F\|_{L^2(\Omega_{2R_0})} + (1+|\alpha|)\|\nabla u\|_{L^2(\Omega_{2R_0})} + \|p\|_{L^2(\Omega_{2R_0})}\big)\,.$$

In order to estimate the pressure term let us recall the condition $\int_{\Omega_{2R_0}} p\,\mathrm{d}x = 0$, which yields from ($S_\alpha$),

$$\begin{aligned}
\|p\|_{L^2(\Omega_{2R_0})} \leq C\|\nabla p\|_{H^{-1}(\Omega_{2R_0})} &= C\|\operatorname{div}[F + \nabla u + \alpha(u\otimes x^\perp - x^\perp\otimes u)]\|_{H^{-1}(\Omega_{2R_0})} \\
&\leq C\big(\|F\|_{L^2(\Omega_{2R_0})} + (1+|\alpha|)\|\nabla u\|_{L^2(\Omega_{2R_0})}\big)\,,
\end{aligned}$$

where $H^{-1}(\Omega_{2R_0})$ is the dual of $W^{1,2}_0(\Omega_{2R_0})$. Collecting these estimates, we obtain (2.75).

Finally let us determine the coefficient β in (2.73). In view of (2.79) it suffices to compute $\tilde{c}[\mathcal{F}] + c[g]$. We follow the argument in the proof of [35, Theorem 2.1]. Fix $N \geq 2R_0$ and let $\phi_N \in C^\infty_0(\mathbb{R}^2)$ be a radial cut-off function such that $\phi_N(x) = 1$ for $|x| \leq N$ and $\phi_N(x) = 0$ for $|x| \geq 2N$. Then we have

$$\begin{aligned}
\tilde{c}[\mathcal{F}] + c[g] &= \lim_{\epsilon\to 0}\int_{\mathbb{R}^2} e^{-\epsilon|y|^2}(F_{12}-F_{21})(1-\phi_N)\,\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^2}(\mathcal{F}_{12}-\mathcal{F}_{21})\phi_N\,\mathrm{d}y + \int_{\mathbb{R}^2} y^\perp\cdot g\phi_N\,\mathrm{d}y \\
&= \tilde{c}_\Omega[F] - \int_\Omega (F_{12}-F_{21})\phi_N\,\mathrm{d}y + \int_{\mathbb{R}^2}(\mathcal{F}_{12}-\mathcal{F}_{21})\phi_N\,\mathrm{d}y + \int_{\mathbb{R}^2} y^\perp\cdot g\phi_N\,\mathrm{d}y\,.
\end{aligned} \tag{2.80}$$

We set $S(v,q)(x) = T(v,q)(x) + \alpha(v\otimes x^\perp - x^\perp\otimes v)$. Since $\operatorname{div}\mathcal{F} + g = -\operatorname{div} S(v,q) = (-\sum_{j=1,2}\partial_j S_{1j}(v,q), -\sum_{j=1,2}\partial_j S_{2j}(v,q))^\top$ in $\mathbb{R}^2$, the integration by parts and the symmetry of $T(v,q)$ yield

$$
\begin{aligned}
\int_{\mathbb{R}^2} y^\perp \cdot g\phi_N \,\mathrm{d}y \;&=\; -\int_{\mathbb{R}^2} \phi_N y^\perp \cdot \operatorname{div} S(v,q)\,\mathrm{d}y - \int_{\mathbb{R}^2} \phi_N y^\perp \cdot \operatorname{div} \mathcal{F}\,\mathrm{d}y \\
&=\; 2\int_{\mathbb{R}^2} \phi_N y \cdot v\,\mathrm{d}y + \int_{\mathbb{R}^2} y^\perp \cdot S(v,q)\nabla\phi_N\,\mathrm{d}y \\
&\quad - \int_{\mathbb{R}^2} (\mathcal{F}_{12} - \mathcal{F}_{21})\phi_N\,\mathrm{d}y + \int_{\mathbb{R}^2} y^\perp \cdot \mathcal{F}\nabla\phi_N\,\mathrm{d}y \\
&=\; \int_{\mathbb{R}^2} y^\perp \cdot S(v,q)\nabla\phi_N\,\mathrm{d}y \\
&\quad - \int_{\mathbb{R}^2} (\mathcal{F}_{12} - \mathcal{F}_{21})\phi_N\,\mathrm{d}y + \int_{\mathbb{R}^2} y^\perp \cdot \mathcal{F}\nabla\phi_N\,\mathrm{d}y\,.
\end{aligned}
\tag{2.81}
$$

Here we have used the fact that ϕ_N is radial, and thus, $y\phi_N(y) = \nabla_y\big(\int_{|y|}^\infty r\tilde{\phi}_N(r)\,\mathrm{d}r\big)$, where $\tilde{\phi}_N(r)$ is such that $\tilde{\phi}_N(|y|) = \phi_N(y)$. Since $S(v,q) = S(u,p)$ for $|x| \geq 2R_0$ and $-\operatorname{div} S(u,p) = f$ in Ω, again from the integration parts we have

$$
\begin{aligned}
&\int_{\mathbb{R}^2} y^\perp \cdot S(v,q)\nabla\phi_N\,\mathrm{d}y \\
&= \int_{\Omega} y^\perp \cdot S(u,p)\nabla\phi_N\,\mathrm{d}y \\
&= \int_{\partial\Omega} y^\perp \cdot S(u,p)\nu\,\mathrm{d}\sigma_y - 2\int_{\Omega} \phi_N y \cdot u\,\mathrm{d}y + \int_{\Omega} \phi_N y^\perp \cdot f\,\mathrm{d}y \\
&= \int_{\partial\Omega} y^\perp \cdot T(u,p)\nu\,\mathrm{d}\sigma_y + \int_{\Omega} \phi_N y^\perp \cdot f\,\mathrm{d}y\,.
\end{aligned}
\tag{2.82}
$$

Here we have used the boundary condition $u = 0$ on $\partial\Omega$ and also the radial symmetry of ϕ_N. By taking the cut-off function φ above, and using the relation $\varphi\phi_N = \varphi$, we then compute the second term in the above as

$$
\begin{aligned}
\int_{\Omega} \phi_N y^\perp \cdot f\,\mathrm{d}y \;&=\; \int_{\Omega} \varphi y^\perp \cdot f\,\mathrm{d}y + \int_{\Omega} \phi_N(1-\varphi) y^\perp \cdot f\,\mathrm{d}y \\
&=\; \int_{\Omega} \varphi y^\perp \cdot f\,\mathrm{d}y + \int_{\Omega} (F_{12} - F_{21})\phi_N\,\mathrm{d}y - \int_{\Omega} (F_{12} - F_{21})\varphi\,\mathrm{d}y \\
&\quad - \int_{\Omega} y^\perp \cdot F\nabla\phi_N\,\mathrm{d}y + \int_{\Omega} y^\perp \cdot F\nabla\varphi\,\mathrm{d}y\,.
\end{aligned}
\tag{2.83}
$$

Collecting (2.80)–(2.83) and using $\mathcal{F} = F$ for $|x| \geq 2R_0$, we obtain

$$
\begin{aligned}
\tilde{c}[F] + c[g] \;&=\; \int_{\partial\Omega} y^\perp \cdot T(u,p)\nu\,\mathrm{d}\sigma_y \\
&\quad + \tilde{c}_\Omega[F] + \int_{\Omega} \big\{(y^\perp \cdot f - F_{12} + F_{21})\varphi + y^\perp \cdot F\nabla\varphi\big\}\,\mathrm{d}y\,,
\end{aligned}
\tag{2.84}
$$

as desired. When $F \in L^\infty_{2,0}(\Omega)^{2\times 2}$ the coefficient $b_\Omega[f]$ coincides with $c_\Omega[f]$ in virtue of (2.71). The proof is complete. □

Let us recall that $R_0 \geq 1$ is taken so that $\mathbb{R}^2 \setminus \Omega \subset B_{R_0}$. Let $\varphi \in C_0^\infty(\Omega)$ be a radial cut-off function such that $\varphi(x) = 1$ for $|x| \leq R_0$ and $\varphi(x) = 0$ for $|x| \geq 2R_0$. Then we set

$$
V(x) \;=\; \big(1 - \varphi(x)\big)\frac{x^\perp}{4\pi|x|^2}\,.
\tag{2.85}
$$

Note that V is a radial circular flow satisfying $\operatorname{div} V = 0$, which describes the asymptotic behavior of solutions to the Stokes system ($\mathrm{S}_{\alpha,\mathbb{R}^2}$) as is shown in Theorem 2.3.1. The main result of this section is stated as follows.

Theorem 2.3.8 *Let $\alpha \in \mathbb{R} \setminus \{0\}$ and $\gamma \in [0, 1)$. Suppose that $f \in L^2(\Omega)^2$ is of the form $f = \operatorname{div} F$ with $F \in L^\infty_{2+\gamma}(\Omega)^{2\times 2}$. Assume in addition that $\tilde{c}_\Omega[F]$ converges when $\gamma = 0$. Then there exists a unique solution $(u, \nabla p) \in W^{2,2}_{\mathrm{loc}}(\overline{\Omega})^2 \times L^2_{\mathrm{loc}}(\overline{\Omega})$ to (S_α) satisfying $\lim_{|x|\to\infty} |u(x)| = 0$ and*

$$\|\nabla u\|_{L^2(\Omega)} \le \|F\|_{L^2(\Omega)}\,, \tag{2.86}$$

$$\|p\|_{L^2(\Omega_{6R_0})} \le C(1 + |\alpha|)\|F\|_{L^2(\Omega)}\,, \tag{2.87}$$

$$\|\nabla^2 u\|_{L^2(\Omega_{kR_0})} + \|\nabla p\|_{L^2(\Omega_{kR_0})} \le C(1 + |\alpha|)\big(\|F\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega_{(k+1)R_0})}\big)\,, \quad 2 \le k \le 5\,. \tag{2.88}$$

Moreover, the velocity u is written as

$$u(x) \;=\; \beta V(x) + \mathcal{R}_\Omega[f](x)\,, \qquad x \in \Omega\,, \tag{2.89}$$

where $\beta \in \mathbb{R}$ is given by

$$\begin{aligned} \beta \;&=\; \int_{\partial\Omega} y^\perp \cdot \big(T(u,p)\nu\big)\,\mathrm{d}\sigma_y + b_\Omega[f]\,,\\ b_\Omega[f] \;&=\; \tilde{c}_\Omega[F] + \int_\Omega \big\{\big(y^\perp \cdot f - F_{12} + F_{21}\big)\varphi + y^\perp \cdot F\nabla\varphi\big\}\,\mathrm{d}y\,, \end{aligned} \tag{2.90}$$

while $\mathcal{R}_\Omega[f]$ satisfies

$$\begin{aligned} \|\mathcal{R}_\Omega[f]\|_{L^\infty_{1+\gamma}(B^{\mathrm{c}}_{4R_0})} \le C\bigg(&\|F\|_{L^\infty_{2+\gamma}(B^{\mathrm{c}}_{2R_0})} + \sup_{|x|\ge 4R_0} |x|^{-1+\gamma}\|yF\|_{L^1(\Omega_{\frac{|x|}{2}})}\\ &+ \sup_{|x|\ge 4R_0} \min\big\{\frac{1}{|\alpha||x|^{2-\gamma}}, |x|^\gamma\big\}\|F\|_{L^1(\Omega_{\frac{|x|}{2}})}\\ &+ \sup_{|x|\ge 4R_0} |x|^\gamma \Big|\lim_{\epsilon\to 0}\int_{2|y|\ge|x|} e^{-\epsilon|y|^2}(F_{12} - F_{21})\,\mathrm{d}y\Big|\bigg)\\ &+ C\big(|\alpha|^{-\frac{1+\gamma}{2}} + |\alpha|^{-\frac12} + 1\big)(1 + |\alpha|)\|F\|_{L^2(\Omega)}\,. \end{aligned} \tag{2.91}$$

Here the constant C is independent of γ, α, and F. If $F \in L^\infty_{2,0}(\Omega)^{2\times 2}$ then the coefficient $b_\Omega[f]$ coincides with $c_\Omega[f]$.

Proof: We follow the argument of [35, Theorem 2.2]. Since the argument is quite parallel to it, we only give the outline here. (Uniqueness) Let $(u, \nabla p)$, $(u', \nabla p') \in W^{2,2}_{\mathrm{loc}}(\overline{\Omega})^2 \times L^2_{\mathrm{loc}}(\overline{\Omega})^2$ be solutions to (S_α) with the same f such that $\|\nabla u\|_{L^2(\Omega)}$ and $\|\nabla u'\|_{L^2(\Omega)}$ are finite and $|u(x)| + |u'(x)| \to 0$ as $|x| \to \infty$. Then the difference $(v, \nabla q) = (u - u', \nabla(p - p')) \in W^{2,2}_{\mathrm{loc}}(\overline{\Omega})^2 \times L^2_{\mathrm{loc}}(\overline{\Omega})^2$ solves (S_α) with $f = 0$ and satisfies $\|\nabla v\|_{L^2(\Omega)} < \infty$ as well as $|v(x)| \to 0$ as $|x| \to \infty$. Moreover, the standard elliptic regularity of the Stokes operator implies that $(v, \nabla q)$ is smooth in Ω. Then we can apply [35, Theorem 2.1, (2.8)], which gives $\int_\Omega |Dv|^2\,\mathrm{d}x = 0$. Hence v is the rigid motion, but the condition $v = 0$ on the boundary leads to $v = 0$ in Ω. Then we obtain $\nabla q = 0$ from the equation. The proof of the uniqueness is complete. (Existence) Firstly we consider the regularized system

$$\begin{cases} \lambda u_\lambda - \Delta u_\lambda - \alpha(x^\perp \cdot \nabla u_\lambda - u_\lambda^\perp) + \nabla p_\lambda \,=\, f\,, \quad \operatorname{div} u_\lambda \,=\, 0\,, \qquad x \in \Omega\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad u_\lambda \,=\, 0\,, \qquad x \in \partial\Omega\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad u_\lambda \,\to\, 0\,, \qquad |x| \to \infty\,. \end{cases} \tag{$\mathrm{S}_\alpha^\lambda$}$$

Here λ is a small positive number. For ($\mathrm{S}_\alpha^\lambda$) one can show the existence of the solution $(u_\lambda, \nabla p_\lambda)$ satisfying $\int_{\Omega_{2R_0}} p_\lambda \,\mathrm{d}x = 0$ and the energy estimate

$$\lambda \|u_\lambda\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\nabla u_\lambda\|_{L^2(\Omega)}^2 \le \frac{1}{2}\|F\|_{L^2(\Omega)}^2\,. \tag{2.92}$$

Moreover, the assumption $f \in L^2(\Omega)^2$ and the elliptic regularity for the Stokes operator imply the regularity $u_\lambda \in W^{2,2}_{\mathrm{loc}}(\overline{\Omega})^2$, $\nabla p_\lambda \in L^2_{\mathrm{loc}}(\overline{\Omega})^2$, where in virtue of (2.92) each seminorm of $W^{2,2}_{\mathrm{loc}}(\overline{\Omega})$ can be bounded uniformly in $\lambda \in (0,1)$. Indeed, since (u_λ, p_λ) solves the Stokes system with the source term $f + \alpha(x^\perp \cdot \nabla u_\lambda - u_\lambda^\perp)$, for any bounded subdomain $\omega \subset \Omega$, there exists $\rho > 0$ with $\omega \subset \Omega_\rho$ such that

$$\|u_\lambda\|_{W^{2,2}(\omega)} \le C(\|f\|_{L^2(\Omega)} + \|\nabla u_\lambda\|_{L^2(\Omega)} + \|u_\lambda\|_{L^2(\Omega_\rho)})\,,$$

where the constant C depends on Ω, R_0, ω, and ρ; see [56, page 117, Theorem 1.5.1] for the proof. From (2.92) and the Poincaré inequality $\|u_\lambda\|_{L^2(\Omega_\rho)} \le C_\rho \|\nabla u_\lambda\|_{L^2(\Omega)}$ with C_ρ depending only on Ω and ρ, we obtain the bound of u_λ in $W^{2,2}(\omega)$ which is independent of λ. Let us recall that $R_0 \ge 1$ is taken so that $\mathbb{R}^2 \setminus \Omega \subset B_{R_0}$ and $\varphi \in C_0^\infty(\mathbb{R}^2)$ is a radial cut-off function such that $\varphi(x) = 1$ for $|x| \le R_0$ and $\varphi(x) = 0$ for $|x| \ge 2R_0$. As in the proof of Lemma 2.3.7, we introduce the Bogovskii operator $\mathbb{B}$ in the closed annulus $A = \{x \in \mathbb{R}^2 \mid R_0 \le |x| \le 2R_0\}$, and set

$$v_\lambda = (1-\varphi)u_\lambda + \mathbb{B}[\nabla\varphi \cdot u_\lambda]\,, \qquad q_\lambda = (1-\varphi)p_\lambda\,.$$

Recall that $\mathbb{B}[\nabla\varphi \cdot u_\lambda]$ satisfies

$$\operatorname{supp} \mathbb{B}[\nabla\varphi \cdot u_\lambda] \subset A\,, \qquad \operatorname{div} \mathbb{B}[\nabla\varphi \cdot u_\lambda] \,=\, \nabla\varphi \cdot u_\lambda\,, \tag{2.93}$$

$$\|\mathbb{B}[\nabla\varphi \cdot u_\lambda]\|_{W^{m+1,2}(\Omega)} \le C\|\nabla\varphi \cdot u_\lambda\|_{W^{m,2}(\Omega)}\,, \qquad m = 0, 1\,. \tag{2.94}$$

Then $(v_\lambda, \nabla q_\lambda)$ satisfies

$$\begin{cases} \lambda v_\lambda - \Delta v_\lambda - \alpha(x^\perp \cdot \nabla v_\lambda - v_\lambda^\perp) + \nabla q_\lambda \,=\, \operatorname{div} F_\lambda + g_\lambda\,, \quad \operatorname{div} u_\lambda \,=\, 0\,, \qquad x \in \mathbb{R}^2\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad v_\lambda \,\to\, 0\,, \qquad |x| \to \infty\,, \end{cases} \tag{2.95}$$

where

$$\begin{aligned} F_\lambda \,&=\, (1-\varphi)F - \nabla\mathbb{B}[\nabla\varphi \cdot u_\lambda]\,, \\ g_\lambda \,&=\, F \cdot \nabla\varphi + \lambda\mathbb{B}[\nabla\varphi \cdot u_\lambda] + 2\nabla\varphi \cdot \nabla u_\lambda + (\Delta\varphi + \alpha x^\perp \cdot \nabla\varphi)u_\lambda \\ &\quad - \alpha\big(x^\perp \nabla\mathbb{B}[\nabla\varphi \cdot u_\lambda] - \mathbb{B}[\nabla\varphi \cdot u_\lambda]^\perp\big) - (\nabla\varphi)p_\lambda\,. \end{aligned}$$

Note that $\operatorname{supp} g_\lambda \subset A$ due to (2.93). Let $\Gamma_\alpha^\lambda(x,y)$ be the function defined in (2.60). Then, as is shown in [35] (see also Remark 2.3.2), the velocity v_λ is written as

$$\begin{aligned} v_\lambda(x) \,&=\, \int_{\mathbb{R}^2} \Gamma_\alpha^\lambda(x,y) \operatorname{div} F_\lambda(y)\,\mathrm{d}y + \int_{\mathbb{R}^2} \Gamma_\alpha^\lambda(x,y) g_\lambda(y)\,\mathrm{d}y \\ &=\, w_\lambda(x) + r_\lambda(x)\,. \end{aligned} \tag{2.96}$$

Since $g_\lambda = 0$ for $|x| \geq 2R_0$, we have from [35, Proposition 3.3],

$$\begin{aligned}\|r_\lambda\|_{L^\infty_1(B^c_{4R_0})} &\leq C_\alpha \int_\Omega (1+|y|)|g_\lambda(y)|\,\mathrm{d}y \\ &\leq C_\alpha \|g_\lambda\|_{L^2(\Omega)} \\ &\leq C_\alpha\big(\|F\|_{L^2(\Omega_{2R_0})} + (1+|\alpha|)\|\nabla u_\lambda\|_{L^2(\Omega_{2R_0})} + \|p_\lambda\|_{L^2(\Omega_{2R_0})}\big)\,. \quad (2.97)\end{aligned}$$

Since $\int_{\Omega_{2R_0}} p_\lambda \,\mathrm{d}x = 0$ we have from ($\mathrm{S}^\lambda_\alpha$),

$$\|p_\lambda\|_{L^2(\Omega_{2R_0})} \leq C\|\nabla p_\lambda\|_{H^{-1}(\Omega_{2R_0})} \leq C\big(\|F\|_{L^2(\Omega_{2R_0})} + (1+|\alpha|)\|\nabla u_\lambda\|_{L^2(\Omega_{2R_0})}\big)$$

Combining this estimate with (2.92) and (2.97), we obtain

$$\|r_\lambda\|_{L^\infty_1(B^c_{4R_0})} \leq C_\alpha \|F\|_{L^2(\Omega)}\,. \quad (2.98)$$

Here C_α depends only on α and R_0, but is independent of $\lambda \in (0,1)$. As for w_λ, from Lemma 2.3.5, there is $0 < \theta < 1$ such that

$$\begin{aligned}\|w_\lambda\|_{L^\infty_\theta(B^c_{4R_0})} &\leq C\big(\|F\|_{L^\infty_{2+\gamma}(B^c_{2R_0})} + \|F_\lambda\|_{L^1(B_{2R_0})}\big) \\ &\leq C\big(\|F\|_{L^\infty_{2+\gamma}(B^c_{2R_0})} + \|F\|_{L^2(\Omega)}\big)\,. \quad (2.99)\end{aligned}$$

Collecting (2.92), (2.98), (2.99), and $u_\lambda \in W^{2,2}_{\mathrm{loc}}(\overline{\Omega})^2$ with its uniform bound on $\lambda \in (0,1)$, we have a uniform estimate in $\lambda \in (0,1)$:

$$\|u_\lambda\|_{L^\infty_\theta(\Omega)} \leq C_\alpha\big(\|F\|_{L^\infty_{2+\gamma}(\Omega)} + \|F\|_{L^2(\Omega)}\big)\,, \quad (2.100)$$

where the Sobolev embedding $W^{2,2}(\Omega_{5R_0}) \hookrightarrow L^\infty(\Omega_{5R_0})$ has been applied. Thus, there are a subsequence, denoted again by $(u_\lambda, \nabla p_\lambda)$, and $(u, \nabla p) \in W^{2,2}_{\mathrm{loc}}(\overline{\Omega})^2 \times L^2_{\mathrm{loc}}(\overline{\Omega})^2$, such that $u_\lambda \rightharpoonup^* u$ in $L^\infty_\theta(\Omega)^2$, $\nabla u_\lambda \rightharpoonup \nabla u$ in $L^2(\Omega)^{2\times 2}$, and $p_\lambda \rightharpoonup p$ in $W^{1,2}_{\mathrm{loc}}(\overline{\Omega})$. It is easy to see that $(u, \nabla p)$ satisfies (S_α) in the sense of distributions (note that each term of (S_α) makes sense at least as a function in $L^2_{\mathrm{loc}}(\overline{\Omega})$). The proof of the existence is complete.

(Estimates) We note that the solution $(u, \nabla p)$ obtained in the existence proof above satisfies $\|\nabla u\|_{L^2(\Omega)} \leq \|F\|_{L^2(\Omega)}$ by (2.92). Thus (2.86) holds. Since the pressure p is uniquely determined up to a constant, we may assume $\int_{\Omega_{6R_0}} p \,\mathrm{d}x = 0$. Then we have from ($\mathrm{S}_\alpha$),

$$\begin{aligned}\|p\|_{L^2(\Omega_{6R_0})} \leq C\|\nabla p\|_{H^{-1}(\Omega_{6R_0})} &\leq C\big(\|F\|_{L^2(\Omega_{6R_0})} + (1+|\alpha|)\|\nabla u\|_{L^2(\Omega_{6R_0})}\big) \\ &\leq C(1+|\alpha|)\|F\|_{L^2(\Omega)}\,.\end{aligned}$$

Here C depends only on R_0. This proves (2.87). The local estimates (2.88) follow from a standard cut-off argument and elliptic estimates for the Stokes system in bounded domains, together with the estimates (2.86) and (2.87). Since the argument is rather standard, we omit the details. The expansion (2.89) with (2.90) and the estimate (2.91) follow from Lemma 2.3.7 and (2.86). Note that the constant vector u_∞ in (2.73) must be zero, for the solution u constructed here decays as $|x| \to \infty$. The proof of Theorem 2.3.8 is complete. □

Remark 2.3.9 Let $R_0 \geq 1$ be as in Theorem 2.3.8 and let $\gamma \in [0,1)$. Then we have for $|x| \geq 4R_0$,

$$\begin{aligned}\|yF\|_{L^1(\Omega_{\frac{|x|}{2}})} &\leq \frac{C}{1-\gamma}|x|^{1-\gamma}\|F\|_{L^\infty_{2+\gamma}(\Omega)}\,, \\ \|F\|_{L^1(\Omega_{\frac{|x|}{2}})} &\leq C\|F\|_{L^\infty_{2+\gamma}(\Omega)}\log|x|\,.\end{aligned}$$

Here C is independent of γ and F. Since

$$\min\Big\{\frac{1}{|\alpha||x|^{2-\gamma}}, |x|^{\gamma}\Big\}\log|x| \le |\alpha|^{-\frac{\gamma}{2}}\big|\log|\alpha|\big|\,, \qquad |\alpha|>0\,,$$

we have for $\gamma\in[0,1)$ and $0<|\alpha|<1$, by using (2.91),

$$\begin{aligned}\|\mathcal{R}_\Omega[f]\|_{L^\infty_{1+\gamma}(B^{\rm c}_{4R_0})} \le \frac{C}{1-\gamma}\bigg(&|\alpha|^{-\frac{\gamma}{2}}\big|\log|\alpha|\big|\,\|F\|_{L^\infty_{2+\gamma}(\Omega)} + |\alpha|^{-\frac{1+\gamma}{2}}\|F\|_{L^2(\Omega)}\\ &+\sup_{|x|\ge 4R_0}|x|^{\gamma}\,\Big|\lim_{\epsilon\to 0}\int_{2|y|\ge|x|} e^{-\epsilon|y|^2}(F_{12}-F_{21})\,{\rm d}y\Big|\bigg)\,.\end{aligned} \tag{2.101}$$

Here C is independent of $0<|\alpha|<1$, $\gamma\in[0,1)$, and F. The estimate (2.101) plays a central role to solve the Navier-Stokes equations for small $|\alpha|$ in the next section. We note that $\tilde{c}_\Omega[F]$ and the last term in the right-hand side of (2.101) do not converge in general if $F\in L^\infty_2(\Omega)^{2\times 2}$. In solving the nonlinear problem, especially for the case $\gamma=0$, it is crucial that we only need the decay of the component $F_{12}-F_{21}$, which always vanishes when F is symmetric.

2.4 Solvability of nonlinear problem

Based on the linear analysis in the previous sections the following Navier-Stokes equations are studied in this section:

$$\begin{cases} -\Delta u-\alpha(x^\perp\cdot\nabla u-u^\perp)+\nabla p \,=\, -u\cdot\nabla u+f\,, \quad {\rm div}\, u\,=\,0\,, \qquad x\in\Omega\,,\\ \qquad\qquad\qquad\qquad\qquad\quad u\,=\,\alpha x^\perp\,, \qquad x\in\partial\Omega\,,\\ \qquad\qquad\qquad\qquad\qquad\quad u\,\to\,0\,, \qquad |x|\to\infty\,,\end{cases} \tag{NS$_\alpha$}$$

Our aim is to prove, under some conditions on f, the unique existence of solutions $(u,\nabla p)$ to (NS$_\alpha$) satisfying the asymptotic behavior

$$u(x)\,=\,\beta V(x)+o(|x|^{-1}) \qquad \text{as } |x|\to\infty$$

for some $\beta\in\mathbb{R}$, where V is a radial circular flow defined by (2.85) and coincides with $\frac{x^\perp}{4\pi|x|^2}$ for $|x|\gg 1$. As in the previous sections we fix a positive number $R_0\ge 1$ large enough so that $\mathbb{R}^2\setminus\Omega\subset B_{R_0}$, and let $\varphi\in C_0^\infty(\mathbb{R}^2)$ be a radial cut-off function satisfying $\varphi(x)=1$ for $|x|\le R_0$, $\varphi(x)=0$ for $|x|\ge 2R_0$. Set

$$U(x)\,=\,\varphi(x)x^\perp\,, \tag{2.102}$$

which is a radial circular flow supported in the ball B_{2R_0}. We also introduce the function space X_γ, $\gamma\ge 0$, as

$$X_\gamma\,=\,\mathbb{R}\times\big(\dot W^{1,2}_{0,\sigma}(\Omega)\cap L^\infty_{1+\gamma}(\Omega)^2\big)\,, \tag{2.103}$$

which is the Banach space under the norm for $(\beta,w)\in X_\gamma$:

$$\|(\beta,w)\|_{X_\gamma}\,=\,|\beta|+\|\nabla w\|_{L^2(\Omega)}+\|w\|_{L^\infty_{1+\gamma}(\Omega)}\,. \tag{2.104}$$

We sketch the proof that X_γ is complete. It suffices to show the completeness of the space $\dot W^{1,2}_{0,\sigma}(\Omega)\cap L^\infty_{1+\gamma}(\Omega)^2$. Suppose that $\{w^{(n)}\}\subset \dot W^{1,2}_{0,\sigma}(\Omega)\cap L^\infty_{1+\gamma}(\Omega)^2$ is a Cauchy sequence.

Then there exist $u \in \dot{W}^{1,2}_{0,\sigma}(\Omega)^2$ and $v \in L^\infty_{1+\gamma}(\Omega)^2$ such that $\|\nabla(w^{(n)} - u)\|_{L^2(\Omega)} \to 0$ and $\|w^{(n)} - v\|_{L^\infty_{1+\gamma}(\Omega)} \to 0$ as $n \to \infty$. What we need to show is $u = v$. To show this, set $f = u - v$. Note that the fact $u, w^{(n)} \in \dot{W}^{1,2}_0(\Omega)^2$ implies $u = w^{(n)} = 0$ on $\partial\Omega$. Then, for any $\phi \in W^{1,2}(\Omega)$ with compact support, the integration by parts yields for $j, k = 1, 2$,

$$
\begin{aligned}
\int_\Omega f_j \partial_k \phi \,\mathrm{d}x &= \int_\Omega (u_j - v_j)\partial_k \phi \,\mathrm{d}x \\
&= -\int_\Omega \phi \partial_k u_j \,\mathrm{d}x - \int_\Omega v_j \partial_k \phi \,\mathrm{d}x \\
&= -\lim_{n\to\infty} \Big(\int_\Omega \phi \partial_k w_j^{(n)} \,\mathrm{d}x + \int_\Omega w_j^{(n)} \partial_k \phi \,\mathrm{d}x \Big) \;=\; 0\,.
\end{aligned}
$$

Since we can take an arbitrary $\phi \in C_0^\infty(\Omega)$ we first conclude that f_j is a constant in Ω, denoted by c_j. Next we have for $\varphi \in W^{1,2}(\Omega)^2$ such that $\operatorname{supp}\varphi$ is compact,

$$
c_j \int_{\partial\Omega} \varphi \cdot \nu \,\mathrm{d}\sigma_x \;=\; \int_\Omega c_j \operatorname{div}\varphi \,\mathrm{d}x \;=\; \int_\Omega f_j \operatorname{div}\varphi \,\mathrm{d}x \;=\; 0\,,
$$

where the result of the above computation is used. This implies $c_j = 0$ since we can choose φ so that $\int_{\partial\Omega} \varphi \cdot \nu \,\mathrm{d}\sigma_x \neq 0$. Thus we obtain $u = v$, and hence, X_γ is complete.

Let us recall that for $f \in L^2(\Omega)^2$ of the form $f = \operatorname{div} F = (\sum_{j=1,2} \partial_j F_{1j}, \sum_{j=1,2} \partial_j F_{2j})^\top$ with some $F \in L^2(\Omega)^{2\times 2}$ satisfying $F_{12} - F_{21} \in L^1(\Omega)$ the coefficients $\tilde{c}_\Omega[F]$ and $b_\Omega[f]$ in (2.69) and (2.90) are well-defined. The main results of this section are Theorems 2.4.1, 2.4.3 below. Let us start from the next theorem.

Theorem 2.4.1 *Let $\gamma \in [0, 1)$. There exists a positive constant $\epsilon = \epsilon(\Omega, \gamma)$ such that the following statement holds. Suppose that $f \in L^2(\Omega)^2$ is of the form $f = \operatorname{div} F$ with some $F \in L^\infty_{2+\gamma}(\Omega)^{2\times 2}$, and in addition that $F_{12} - F_{21} \in L^1(\Omega)$ when $\gamma = 0$. If $\alpha \neq 0$ and*

$$
\begin{aligned}
|\alpha|^{\frac{1-\gamma}{2}} \big|\log|\alpha|\big| + |\alpha|^{-\frac{\gamma}{2}} \big|\log|\alpha|\big| \bigg(& |\alpha|^{-\frac12} \big(|b_\Omega[f]| + \|F\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega_{6R_0})} \big) \\
& + \|F_{12} - F_{21}\|_{L^1(\Omega)} + \big|\log|\alpha|\big| \, \|F\|_{L^\infty_2(\Omega)} \bigg) < \epsilon\,,
\end{aligned}
\tag{2.105}
$$

then there exists a unique solution $(u, \nabla p) \in W^{2,2}_{\mathrm{loc}}(\overline{\Omega})^2 \times L^2_{\mathrm{loc}}(\overline{\Omega})^2$ to (NS$_\alpha$) *satisfying*

$$
\|\nabla u\|_{L^2(\Omega)} \le \frac{\|F\|_{L^2(\Omega)} + C_2|\alpha|}{\sqrt{1 - C_1|\alpha|}}\,,
\tag{2.106}
$$

and enjoying the expression $u = \alpha U + \beta V + w$ with U and V defined by (2.102) *and* (2.85)*, respectively, and*

$$
\beta \;=\; \int_{\partial\Omega} y^\perp \cdot \big(T(u, p)\nu\big) \,\mathrm{d}\sigma_y \;+\; b_\Omega[f]\,,
\tag{2.107}
$$

while

$$
\begin{aligned}
\|w\|_{L^\infty_1(\Omega)} \le C_3 \bigg(& |\alpha|^{-\frac12} \big(|\alpha| + |b_\Omega[f]| + \|F\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega_{6R_0})} \big) \\
& + \big|\log|\alpha|\big| \, \|F\|_{L^\infty_2(\Omega)} + \|F_{12} - F_{21}\|_{L^1(\Omega)} \bigg)\,,
\end{aligned}
\tag{2.108}
$$

and if $\gamma \in (0,1)$,

$$\|w\|_{L^\infty_{1+\gamma}(\Omega)} \le C_3\Big(|\alpha|^{-\frac{1+\gamma}{2}}\big(|\alpha||\log|\alpha|| + |b_\Omega[f]| + \|F\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega_{6R_0})}\big) + \big(|\alpha|^{-\frac{\gamma}{2}}\big|\log|\alpha|\big| + \frac{1}{\gamma}\big)\|F\|_{L^\infty_{2+\gamma}(\Omega)}\Big)\,. \tag{2.109}$$

Here ϵ, C_1, C_2, *and* C_3 *depend only on* Ω *and* γ, *and are taken uniformly with respect to* γ *in each compact subset of* $[0,1)$.

Remark 2.4.2 (i) A careful analysis implies that β in Theorem 2.4.1 is estimated as

$$|\beta| \le C_4\big(|\alpha| + |b_\Omega[f]| + \|F\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega_{6R_0})}\big)\,, \tag{2.110}$$

where C_4 depends only on Ω. But we do not go into details in this chapter.
(ii) In Theorem 2.4.1 when $\gamma = 0$ the term w decays with the order $O(|x|^{-1})$ and there is no reason why βV provides a leading term of the asymptotic behavior of u at $|x| \to \infty$. To achieve this asymptotics we need the additional decay of F such as $F \in L^\infty_{2,0}(\Omega)^{2\times 2}$; see Theorem 2.4.3 below.

Proof of Theorem 2.4.1: In the following argument we will freely use the condition $0 < |\alpha| < e^{-1}$. We look for the solution to (NS$_\alpha$) of the form

$$u = \alpha U + v\,, \qquad v = \beta V + w\,, \qquad (\beta, w) \in X_\gamma\,. \tag{2.111}$$

We need to determine β and w. Inserting (2.111) into (NS$_\alpha$), we see that v is the solution to the system

$$\begin{cases} -\Delta v - \alpha(x^\perp\cdot\nabla v - v^\perp) + \nabla q = \operatorname{div} G_\alpha(\beta,w) + \operatorname{div} H_\alpha(F)\,, & x\in\Omega\,,\\ \operatorname{div} v = 0\,, & x\in\Omega\,,\\ v = 0\,, & x\in\partial\Omega\,.\\ v \to 0\,, & |x|\to\infty\,. \end{cases} \tag{NS$'_\alpha$}$$

Here

$$\begin{aligned} q &= p + P\,,\\ G_\alpha(\beta,w) &= -\alpha(U\otimes w + w\otimes U) - \beta(V\otimes w + w\otimes V) - w\otimes w\,,\\ H_\alpha(F) &= \alpha\nabla U + F\,, \end{aligned}$$

and we may assume that $\int_{\Omega_{6R_0}} q\,\mathrm{d}x = 0$. Note that we have used the relations $x^\perp\cdot\nabla U - U^\perp = 0$, and the radial scalar function $P = P(|x|)$ is taken so that $\nabla P = \operatorname{div}[(\alpha U + \beta V)\otimes(\alpha U + \beta V)]$. Both of these follow from the direct calculation. The proof of the unique existence below relies on the standard Banach fixed point argument in a suitable class of functions. To this end we introduce the closed convex set $\mathcal{B}_{\vec\delta,\gamma}$ in X_0:

$$\mathcal{B}_{\vec\delta,\gamma} = \mathcal{B}_{(\delta_1,\delta_2,\delta_3),\gamma} = \big\{(\beta,w)\in X_0 \mid \ |\beta| + \|\nabla w\|_{L^2(\Omega)} + \|w\|_{L^\infty(\Omega_{5R_0})} \le \delta_1\,, \ \|w\|_{L^\infty_1(\Omega)} \le \delta_2\,, \ \|w\|_{L^\infty_{1+\gamma}(\Omega)} \le \delta_3\big\}\,. \tag{2.112}$$

Here we have set $\vec{\delta} = (\delta_1, \delta_2, \delta_3)$, and the positive numbers $\delta_1, \delta_2, \delta_3$ with $\delta_2 \le \delta_3$ will be suitably determined later. We note that the following inclusion always holds for $\delta_2 \le \delta_3$.

$$\mathcal{B}_{(\delta_1,\delta_2,\delta_3),\gamma} \subset \mathcal{B}_{(\delta_1,\delta_2,\delta_2),0}\,. \tag{2.113}$$

For any $\omega = (\beta, w) \in \mathcal{B}_{\vec{\delta},\gamma}$, let $(u_\omega, \nabla q_\omega)$ be the unique solution in Theorem 2.3.8 to the linear system

$$\begin{cases} -\Delta u_\omega - \alpha(x^\perp \cdot \nabla u_\omega - u_\omega^\perp) + \nabla q_\omega \;=\; \operatorname{div} G_\alpha(\beta, w) + \operatorname{div} H_\alpha(F)\,, \qquad x \in \Omega\,, \\ \operatorname{div} u_\omega \;=\; 0\,, \qquad x \in \Omega, \\ u_\omega \;=\; 0\,, \qquad x \in \partial\Omega\,, \\ u_\omega \;\to\; 0\,, \qquad |x| \to \infty\,, \end{cases}$$

Our aim is to show the unique existence of $(\beta, w) \in \mathcal{B}_{\vec{\delta},\gamma}$ such that $u_\omega = u_{(\beta,w)} = \beta V + w$ for suitably chosen and sufficiently small $0 < \delta_1 \le \delta_2 < e^{-2}$ and $\delta_2 \le \delta_3$. We remark that the value δ_3 need not to be small when γ is positive. Let us start from the estimates for $G_\alpha(\beta, w)$. Firstly we estimate its L^2 norm as

$$\begin{aligned} &\|G_\alpha(\beta, w)\|_{L^2(\Omega)} \\ &\le C\Big(|\alpha|\,\|\nabla w\|_{L^2(\Omega)} + |\beta|\,\|w\|_{L^\infty_1(\Omega)} + \|w\|_{L^\infty_1(\Omega)}\|\nabla w\|_{L^2(\Omega)}|\log\|\nabla w\|_{L^2(\Omega)}|\Big)\,. \end{aligned} \tag{2.114}$$

Here, for the nonlinear term, we have used (2.161) and the smallness of δ_1 and δ_2 to obtain

$$\begin{aligned} \|w \otimes w\|_{L^2(\Omega)} &\le C\|w\|_{L^\infty_1(\Omega)}\|(1+|x|)^{-1}w\|_{L^2(\Omega)} \\ &\le C\|w\|_{L^\infty_1(\Omega)}\|\nabla w\|_{L^2(\Omega)}|\log\|\nabla w\|_{L^2(\Omega)}|\,. \end{aligned}$$

On the other hand, it is not difficult to see that

$$\|G_\alpha(\beta, w)\|_{L^\infty_{2+\gamma'}(\Omega)} \le C\big(|\alpha| + |\beta| + \|w\|_{L^\infty_1(\Omega)}\big)\|w\|_{L^\infty_{1+\gamma'}(\Omega)}\,, \quad 0 \le \gamma' \le \gamma\,, \tag{2.115}$$

$$\|\operatorname{div} G_\alpha(\beta, w)\|_{L^2(\Omega_{6R_0})} \le C\big(|\alpha| + |\beta| + \|w\|_{L^\infty(\Omega_{6R_0})}\big)\|\nabla w\|_{L^2(\Omega)}\,, \tag{2.116}$$

and

$$\|H_\alpha(F)\|_{L^2(\Omega)} \le C\big(|\alpha| + \|F\|_{L^2(\Omega)}\big)\,, \tag{2.117}$$

$$\|H_\alpha(F)\|_{L^\infty_{2+\gamma'}(\Omega)} \le C\big(|\alpha| + \|F\|_{L^\infty_{2+\gamma'}(\Omega)}\big)\,, \qquad 0 \le \gamma' \le \gamma\,, \tag{2.118}$$

$$\|\operatorname{div} H_\alpha(F)\|_{L^2(\Omega_{6R_0})} \le C\big(|\alpha| + \|f\|_{L^2(\Omega_{6R_0})}\big)\,. \tag{2.119}$$

Then we can apply the result of Theorem 2.3.8. To simplify the notation we set

$$\begin{aligned} M(\alpha, \beta, F, w) \;=\; &(|\alpha| + |\beta|)\,\|\nabla w\|_{L^2(\Omega)} + |\beta|\,\|w\|_{L^\infty_1(\Omega)} \\ &+ \|w\|_{L^\infty_1(\Omega)}\|\nabla w\|_{L^2(\Omega)}|\log\|\nabla w\|_{L^2(\Omega)}| \;+ |\alpha| + \|F\|_{L^2(\Omega)}\,. \end{aligned} \tag{2.120}$$

From (2.86), (2.114), and (2.117), we have

$$\|\nabla u_{(\beta,w)}\|_{L^2(\Omega)} \le CM(\alpha, \beta, F, w)\,. \tag{2.121}$$

Moreover, by the Sobolev embedding $W^{2,2}(\Omega_{5R_0}) \hookrightarrow L^\infty(\Omega_{5R_0})$ and (2.86) - (2.88) combined with (2.114), (2.116), (2.117), (2.119), and $\|w\|_{L^\infty(\Omega_{6R_0})} \le \|w\|_{L^\infty_1(\Omega)}$, we have

$$\begin{aligned} &\|u_{(\beta,w)}\|_{L^\infty(\Omega_{5R_0})} + \|u_{(\beta,w)}\|_{W^{2,2}(\Omega_{5R_0})} + \|q_{(\beta,w)}\|_{W^{1,2}(\Omega_{5R_0})} \\ &\le C\big(M(\alpha,\beta,F,w) + \|f\|_{L^2(\Omega_{6R_0})}\big)\,. \end{aligned} \tag{2.122}$$

Set $\tilde{F} = G_\alpha(\beta,w) + H_\alpha(F)$ and $\tilde{f} = \operatorname{div}\tilde{F}$. By Theorem 2.3.8, the velocity $u_\omega = u_{(\beta,w)}$ is written as

$$u_\omega \;=\; \psi[\omega]V + R[\omega]\,,$$

where $R[\omega]$ belongs to $L^\infty_{1+\gamma}(\Omega)^2$ and $\psi[\omega]$ is given by

$$\begin{aligned} \psi[\omega] \;&=\; \int_{\partial\Omega} y^\perp \cdot T(u_\omega,q_\omega)\nu \;\mathrm{d}\sigma_y + b_\Omega[\tilde{f}]\,, \\ b_\Omega[\tilde{f}] \;&=\; \tilde{c}_\Omega[\tilde{F}] + \int_\Omega \big\{(y^\perp\cdot\tilde{f} - \tilde{F}_{12} + \tilde{F}_{21})\varphi + y^\perp\cdot\tilde{F}\nabla\varphi\big\}\,\mathrm{d}y\,. \end{aligned} \tag{2.123}$$

We observe that $\tilde{c}_\Omega[G_\alpha(\beta,w)] = 0$ and

$$\int_\Omega \big\{\big(y^\perp\cdot\operatorname{div}G_\alpha(\beta,w) - G_\alpha(\beta,w)_{12} + G_\alpha(\beta,w)_{21}\big)\varphi + y^\perp\cdot(G_\alpha(\beta,w)\nabla\varphi)\big\}\,\mathrm{d}y \;=\; 0\,.$$

Here we have used the facts that $G_\alpha(\beta,w)$ is symmetric and its trace on the boundary is zero. This implies $b_\Omega[\operatorname{div}G_\alpha(\beta,w)] = 0$. Moreover, we have

$$b_\Omega[\Delta U] \;=\; c_\Omega[\Delta U] \;=\; 0$$

in virtue of the computation

$$\begin{aligned} \int_\Omega y^\perp\cdot\Delta U\,\mathrm{d}y \;=\; \int_\Omega y\cdot\nabla\mathrm{rot}\,U\,\mathrm{d}y \;&=\; \int_{\partial\Omega} y\cdot\nu\,(\mathrm{rot}\,U)\,\mathrm{d}\sigma_y - 2\int_\Omega \mathrm{rot}\,U\,\mathrm{d}y \\ &=\; \int_{\partial\Omega} y\cdot\nu\,(\mathrm{rot}\,U)\,\mathrm{d}\sigma_y - 2\int_{\partial\Omega}\nu^\perp\cdot U\,\mathrm{d}\sigma_y \\ &=\; 2\int_{\partial\Omega} y\cdot\nu\;\mathrm{d}\sigma_y - 2\int_{\partial\Omega}\nu^\perp\cdot y^\perp\,\mathrm{d}\sigma_y \;=\; 0\,. \end{aligned}$$

Here $\mathrm{rot}\,U = \partial_1 U_2 - \partial_2 U_1$ and we have used the identity $U(x) = x^\perp$ near $\partial\Omega$. Hence, (2.123) is in fact written as

$$\psi[\omega] \;=\; \int_{\partial\Omega} y^\perp\cdot T(u_\omega,q_\omega)\nu\;\mathrm{d}\sigma_y + b_\Omega[f]\,. \tag{2.124}$$

Now let us define the mapping $\Phi : \mathcal{B}_{\vec{\delta},\gamma} \to X_0$ as

$$\Phi[\omega] \;=\; (\psi[\omega], R[\omega])\,, \qquad \psi[\omega] \text{ is given by (2.124)}\,, \quad R[\omega] = u_\omega - \psi[\omega]V\,. \tag{2.125}$$

Recalling the inclusion (2.113), our aim is to show
(i) Φ is a mapping from $\mathcal{B}_{\vec{\delta},\gamma}$ into $\mathcal{B}_{\vec{\delta},\gamma}$, and
(ii) Φ is a contraction on $\mathcal{B}_{\vec{\delta},0}$ in the topology of X_0. i.e., there is $\tau \in (0,1)$ such that $\|\Phi(\omega_1) - \Phi(\omega_2)\|_{X_0} \le \tau\|\omega_1 - \omega_2\|_{X_0}$ for any $\omega_1,\omega_2 \in \mathcal{B}_{\vec{\delta},0}$.

The properties (i) and (ii) imply the existence of the fixed point of Φ in $\mathcal{B}_{\vec{\delta},\gamma}$ even for the case $\gamma > 0$. Indeed, note that the sequence $\{\omega^{(n)}\}_{n=0}^{\infty} = \{(\beta^{(n)}, w^{(n)}\}_{n=0}^{\infty}$ defined by $\omega^{(0)} = \Phi(0)$ and $\omega^{(n)} = \Phi(\omega^{(n-1)})$ for $n = 1, \ldots$ is a Cauchy sequence in X_0 and each $\omega^{(n)}$ belongs to $\mathcal{B}_{\vec{\delta},\gamma}$, which is not difficult to see from (i) and (ii). Then the limit $\omega = (\beta, w)$ of $\{\omega^{(n)}\}_{n=0}^{\infty}$ in X_0 also belongs to $\mathcal{B}_{\vec{\delta},\gamma}$ since $\mathcal{B}_{\vec{\delta},\gamma}$ is a closed subset in X_0 by the definition.

To prove (i) let us estimate $\psi[\omega]$ based on the representation (2.124). By the trace theorem we have

$$|\int_{\partial\Omega} y^{\perp} \cdot T(u_\omega, q_\omega)\nu \, \mathrm{d}\sigma_y| \le C\big(\|\nabla u_\omega\|_{W^{1,2}(\Omega_{5R_0})} + \|q_\omega\|_{W^{1,2}(\Omega_{5R_0})}\big)\,,$$

Hence we have from (2.122),

$$|\psi[\omega]| \le C\big(M(\alpha, \beta, F, w) + |b_\Omega[f]| + \|f\|_{L^2(\Omega_{6R_0})}\big)\,. \tag{2.126}$$

Next let us estimate $R[\omega]$. Firstly we observe from (2.122), (2.121), and (2.126) that

$$\begin{aligned} \|R[\omega]\|_{L^\infty(\Omega_{5R_0})} + \|\nabla R[\omega]\|_{L^2(\Omega)} &= \|u_\omega - \psi[\omega]V\|_{L^\infty(\Omega_{5R_0})} + \|\nabla(u_\omega - \psi[\omega]V)\|_{L^2(\Omega)} \\ &\le C\big(\|u_\omega\|_{L^\infty(\Omega_{5R_0})} + \|\nabla u_\omega\|_{L^2(\Omega)} + |\psi[\omega]|\big) \\ &\le C\big(M(\alpha, \beta, F, w) + |b_\Omega[f]| + \|f\|_{L^2(\Omega_{6R_0})}\big)\,. \end{aligned} \tag{2.127}$$

On the other hand, from (2.101) and $F_{12} - F_{21} \in L^1(\Omega)$ we have for any $\gamma' \in [0, \gamma]$,

$$\begin{aligned} \|R[\omega]\|_{L^\infty_{1+\gamma'}(B^c_{4R_0})} \le \frac{C}{1-\gamma'}\bigg(&|\alpha|^{-\frac{\gamma'}{2}}\big|\log|\alpha|\big|\,\|G_\alpha(\beta, w) + H_\alpha(F)\|_{L^\infty_{2+\gamma'}(\Omega)} \\ &+ |\alpha|^{-\frac{1+\gamma'}{2}}\|G_\alpha(\beta, w) + H_\alpha(F)\|_{L^2(\Omega)} + d_{\gamma'}[F]\bigg)\,, \end{aligned} \tag{2.128}$$

$$d_{\gamma'}[F] \;=\; \sup_{|x|\ge 4R_0} |x|^{\gamma'}\big|\int_{2|y|\ge|x|} (F_{12} - F_{21})\,\mathrm{d}y\big|\,,$$

where C is independent of γ', γ, and α. Here we have used that $G_\alpha(\beta, w)$ is symmetric and that $U = 0$ for $|x| \ge 2R_0$ by its definition. Note that $d_0[F] \le \|F_{12} - F_{21}\|_{L^1(\Omega)}$ holds, which will be used later. Combining (2.127) with (2.128), (2.114), (2.115), (2.117), and (2.118), we obtain for $\gamma' \in [0, \gamma]$,

$$\begin{aligned} \|R[\omega]\|_{L^\infty_{1+\gamma'}(\Omega)} \le \frac{C}{1-\gamma'}\bigg\{&|b_\Omega[f]| + \|f\|_{L^2(\Omega_{6R_0})} + |\alpha|^{-\frac{1+\gamma'}{2}}M(\alpha, \beta, F, w) + d_{\gamma'}[F] \\ &+ |\alpha|^{-\frac{\gamma'}{2}}\big|\log|\alpha|\big|\,\big(|\alpha| + |\beta| + \|w\|_{L^\infty_1(\Omega)}\big)\|w\|_{L^\infty_{1+\gamma'}(\Omega)} \\ &+ |\alpha|^{-\frac{\gamma'}{2}}\big|\log|\alpha|\big|\,\big(|\alpha| + \|F\|_{L^\infty_{2+\gamma'}(\Omega)}\big)\bigg\}\,. \end{aligned} \tag{2.129}$$

Now we observe that for sufficiently small δ_1 and δ_2 (depending only on Ω so far) the function $M(\alpha, \beta, F, w)$ is bounded from above as

$$M(\alpha, \beta, F, w) \le \big(|\alpha| + \delta_1 + \delta_2|\log\delta_1|\big)\delta_1 + |\alpha| + \|F\|_{L^2(\Omega)}\,. \tag{2.130}$$

Here we have used the fact that $\rho(r) = r|\log r|$ is monotone increasing on $(0, e^{-1}]$, which implies $\|\nabla w\|_{L^2(\Omega)}|\log \|\nabla w\|_{L^2(\Omega)}| \leq \delta_1|\log \delta_1|$. Taking (2.126), (2.127), and (2.130) into account, we assume that $|\alpha|$, $\|F\|_{L^2(\Omega)}$, $|b_\Omega[f]|$, and $\|f\|_{L^2(\Omega_{6R_0})}$ are small enough that

$$\delta_1 = 16(C_0+1)\big(|\alpha| + \|F\|_{L^2(\Omega)} + |b_\Omega[f]| + \|f\|_{L^2(\Omega_{6R_0})}\big) < \frac{1}{16(C_0+1)}\,. \tag{2.131}$$

Here C_0 is the largest constant of C appearing in (2.126), (2.127), and (2.129) (larger than 1 without loss of generality), and then, C_0 is independent of γ and α. Then for $\delta_2 \in (0, \frac{1}{16(C_0+1)|\log \delta_1|}]$ we see from (2.130),

$$M(\alpha, \beta, F, w) \leq \frac{1}{4(C_0+1)}\delta_1\,. \tag{2.132}$$

Thus, (2.126) and (2.127) imply that for $\delta_2 \in (0, \frac{1}{16(C_0+1)|\log \delta_1|}]$,

$$|\psi[\omega]| + \|\nabla R[\omega]\|_{L^2(\Omega)} + \|R[\omega]\|_{L^\infty(\Omega_{5R_0})} \leq \frac{\delta_1}{2} \qquad \text{for all } \omega \in \mathcal{B}_{\vec{\delta},\gamma}\,.$$

Next we focus on $\|R[\omega]\|_{L^\infty_1(\Omega)}$. Taking (2.129) with $\gamma' = 0$ and (2.131) (with $|\alpha| < e^{-1}$) into account, we set δ_2 as

$$\delta_2 = \frac{16(C_0+1)}{|\log \delta_1|}\left(|\alpha|^{-\frac{1}{2}}\delta_1 + \big|\log|\alpha|\big|\left(|\alpha| + \|F\|_{L^\infty_2(\Omega)}\right) + \|F_{12} - F_{21}\|_{L^1(\Omega)}\right), \tag{2.133}$$

which is smaller than $\frac{1}{16(C_0+1)|\log \delta_1|}$ if $|\alpha|$ and the data related to F in (2.131) and (2.133) are small enough, while δ_2 is larger than δ_1 since $\delta_1 \geq |\alpha|$ and $|\alpha|^{\frac{1}{2}}\big|\log|\alpha|\big| \leq 1$ for $|\alpha| < e^{-1}$. Note that $d_0[F] \leq \|F_{12} - F_{21}\|_{L^1(\Omega)}$ is also taken into account in the choice of (2.133). The key observation here is that, when $f = F = 0$, the numbers δ_1 and δ_2 are of the order $O(|\alpha|)$ and $O(|\alpha|^{\frac{1}{2}})$ for $|\alpha| \ll 1$, respectively. Then the term $C\big|\log|\alpha|\big|\big(|\alpha| + |\beta| + \|w\|_{L^\infty_1(\Omega)}\big)$ in the right-hand side of (2.129) with $\gamma' = 0$ is bounded from above by

$$C_0\big|\log|\alpha|\big|\left(|\alpha| + \delta_1 + \delta_2\right) \leq \frac{1}{32}\,, \tag{2.134}$$

if $\gamma \in [0,1)$ and if $|\alpha|$ and the data related to F (and $f = \operatorname{div} F$) appearing in (2.131) and (2.133) are sufficiently small. Note that, since δ_2 is at best of the order $O(|\alpha|^{\frac{1}{2}})$, the condition $\gamma \in [0,1)$ is crucial to ensure (2.134). Precisely, we need the smallness such as

$$|\alpha|^{\frac{1}{2}}\,\big|\log|\alpha|\big| + \kappa_\alpha(F) < \epsilon(\Omega) \ll 1\,, \tag{2.135}$$

where

$$\begin{aligned}\kappa_\alpha(F) = {} & |\alpha|^{-\frac{1}{2}}\big|\log|\alpha|\big|\left(|b_\Omega[f]| + \|F\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega_{6R_0})}\right) \\ & + \big|\log|\alpha|\big|\,\|F_{12} - F_{21}\|_{L^1(\Omega)} + (\log|\alpha|)^2\,\|F\|_{L^\infty_2(\Omega)}\,.\end{aligned} \tag{2.136}$$

Here the number $\epsilon(\Omega)$ depends only on Ω and is independent of α and γ, and we also note that $\kappa_\alpha[F]$ does not contain the number γ in its definition. Under the above smallness condition we have from (2.129) with $\gamma' = 0$ and the choice of δ_2,

$$\|R[\omega]\|_{L^\infty_1(\Omega)} \leq \frac{\delta_2}{2} \qquad \text{for all } \omega \in \mathcal{B}_{\vec{\delta},\gamma}\,,$$

as desired. In the above argument the number δ_3 can be arbitrary.

Next we estimate the norm $\|R[\omega]\|_{L^\infty_{1+\gamma}(\Omega)}$ (in the case γ is positive). To bound the term

$$\frac{C}{1-\gamma}|\alpha|^{-\frac{\gamma}{2}}\big|\log|\alpha|\big|\big(|\alpha|+|\beta|+\|w\|_{L^\infty_1(\Omega)}\big)$$

in the right-hand side of (2.129) with $\gamma' = \gamma$, we need the additional smallness for δ_1 and δ_2 depending on γ:

$$\frac{C_0}{1-\gamma}|\alpha|^{-\frac{\gamma}{2}}\big|\log|\alpha|\big|\,\big(|\alpha|+\delta_1+\delta_2\big) \le \frac{1}{32}\,. \tag{2.137}$$

Precisely, in the case γ is positive, δ_1 and δ_2 are required to have the smallness as

$$|\alpha|^{\frac{1-\gamma}{2}}\,\big|\log|\alpha|\big| + |\alpha|^{-\frac{\gamma}{2}}\kappa_\alpha(F) < \epsilon_\gamma(\Omega) \ll 1\,, \tag{2.138}$$

where the number $\epsilon_\gamma(\Omega)$ depends Ω on γ, contrary to the case of $\epsilon(\Omega)$ in (2.135). We note that $\epsilon_0(\Omega) = \epsilon(\Omega)$ and $\epsilon_\gamma(\Omega)$ is taken so that it is monotone decreasing and continuous on $\gamma \in [0,1)$ in virtue of (2.129). Then we set δ_3 as

$$\delta_3 \;=\; 2\bigg(|\alpha|^{-\frac{1+\gamma}{2}}\delta_1 + |\alpha|^{-\frac{\gamma}{2}}\big|\log|\alpha|\big|\,\|F\|_{L^\infty_{2+\gamma}(\Omega)} + d_\gamma[F]\bigg)\,, \tag{2.139}$$

Then we can conclude from (2.129) with $\gamma' = \gamma$ and (2.134) that

$$\|R[\omega]\|_{L^\infty_{1+\gamma}(\Omega)} \le \frac{\delta_3}{2} \qquad \text{for all } \ \omega \in \mathcal{B}_{\vec{\delta},\gamma}\,.$$

It should be emphasized here that the argument works even if δ_3 itself is large. We have now shown that Φ is a mapping from $\mathcal{B}_{\vec{\delta},\gamma}$ into $\mathcal{B}_{\vec{\delta},\gamma}$ with the choice of δ_j in (2.131), (2.133), and (2.139) for $j = 1,2,3$, respectively.

Next let us show that Φ is a contraction mapping on $\mathcal{B}_{(\delta_1,\delta_2,\delta_2),0}$. For convenience we set $\vec{\beta} = (\beta_1,\beta_2)$, and $\mathbf{w} = (w_1,w_2)$ for $\omega_j = (\beta_j,w_j) \in \mathcal{B}_{(\delta_1,\delta_2,\delta_2),0}$, $j = 1,2$. We also set

$$h \;=\; \big(\psi[\omega_1]-\psi[\omega_2]\big)V + R[\omega_1] - R[\omega_2]\,, \tag{2.140}$$

which is equal to $u_{\omega_1} - u_{\omega_2}$, and hence, the velocity h satisfies

$$\begin{cases} -\Delta h - \alpha(x^\perp\cdot\nabla h - h^\perp) + \nabla q \;=\; \operatorname{div} G'_\alpha(\vec{\beta},\mathbf{w})\,, \quad \operatorname{div} h \;=\; 0\,, \qquad x\in\Omega\,, \\ \qquad\qquad h \;=\; 0\,, \qquad x\in\partial\Omega\,, \\ \qquad\qquad h \;\to\; 0\,, \qquad |x|\to\infty\,, \end{cases}$$

where $q = q_{\omega_1} - q_{\omega_2} \in W^{1,2}_{\mathrm{loc}}(\overline{\Omega})$. Here $G'_\alpha(\vec{\beta},\mathbf{w})$ is given by

$$\begin{aligned} G'_\alpha(\vec{\beta},\mathbf{w}) \;=\; &-\alpha(U\otimes(w_1-w_2)+(w_1-w_2)\otimes U) - (\beta_1-\beta_2)(V\otimes w_1 + w_1\otimes V) \\ &- \beta_2(V\otimes(w_1-w_2)+(w_1-w_2)\otimes V) - w_1\otimes(w_1-w_2) - (w_1-w_2)\otimes w_2\,. \end{aligned}$$

Below we give the estimates of $G'_\alpha(\vec{\beta},\mathbf{w})$, where the estimate for the L^2 norm of the term $V\otimes w_1 + w_1\otimes V$ has to be carefully computed: in principle, we need to estimate it by δ_1 rather than δ_2, for their dependence on $|\alpha|$ is essentially different. Due to the negative power on $|\alpha|$ in the linear estimate (2.101) this is crucial to show that Φ is a contraction mapping. Because of this reasoning we apply (2.161) in Lemma A.1 by recalling the bound $|V(x)| \leq C(1+|x|)^{-1}$, which yields

$$\|V \otimes w_1 + w_1 \otimes V\|_{L^2(\Omega)} \leq C\|\nabla w_1\|_{L^2(\Omega)} |\log \|\nabla w_1\|_{L^2(\Omega)}| \,. \tag{2.141}$$

Here we have used the smallness of $\|\nabla w_1\|_{L^2(\Omega)} + \|w_1\|_{L^\infty_1(\Omega)}$. Similarly, also for the nonlinear term in $G'_\alpha(\vec{\beta}, \mathbf{w})$ we will apply (2.161). Then it follows that

$$\begin{aligned}
&\|G'_\alpha(\vec{\beta}, \mathbf{w})\|_{L^2(\Omega)} \\
&\leq C\big(|\alpha| \, \|\nabla(w_1 - w_2)\|_{L^2(\Omega)} + |\beta_1 - \beta_2| \, \|\nabla w_1\|_{L^2(\Omega)} |\log \|\nabla w_1\|_{L^2(\Omega)}| \\
&\quad + |\beta_2| \, \|w_1 - w_2\|_{L^\infty_1(\Omega)} + \|w_1 - w_2\|_{L^\infty_1(\Omega)} \|\nabla \mathbf{w}\|_{L^2(\Omega)} \big|\log \|\nabla \mathbf{w}\|_{L^2(\Omega)}\big|\big) \\
&\leq C\big(|\alpha| \, \|\nabla(w_1 - w_2)\|_{L^2(\Omega)} + \delta_1 |\log \delta_1| \, |\beta_1 - \beta_2| + 3\delta_1 |\log \delta_1| \|w_1 - w_2\|_{L^\infty_1(\Omega)}\big) \\
&\leq C(|\alpha| + \delta_1 |\log \delta_1|) \|\omega_1 - \omega_2\|_{X_0} \,,
\end{aligned} \tag{2.142}$$

and on the other hand, it is not difficult to see

$$\begin{aligned}
\|G'_\alpha(\vec{\beta}, \mathbf{w})\|_{L^\infty_2(\Omega)} &\leq C\big(|\alpha| \, \|w_1 - w_2\|_{L^\infty_1(\Omega)} + |\beta_1 - \beta_2| \, \|w_1\|_{L^\infty_1(\Omega)} \\
&\qquad + |\beta_2| \, \|w_1 - w_2\|_{L^\infty_1(\Omega)} + \|\mathbf{w}\|_{L^\infty_1(\Omega)} \|w_1 - w_2\|_{L^\infty_1(\Omega)}\big) \\
&\leq C\big(\delta_2 |\beta_1 - \beta_2| + (|\alpha| + \delta_1 + 2\delta_2) \|w_1 - w_2\|_{L^\infty_1(\Omega)}\big) \\
&\leq C(|\alpha| + \delta_1 + \delta_2) \|\omega_1 - \omega_2\|_{X_0} \,.
\end{aligned} \tag{2.143}$$

Similarly, we observe that

$$\begin{aligned}
&\|\mathrm{div}\, G'_\alpha(\vec{\beta}, \mathbf{w})\|_{L^2(\Omega_{5R_0})} \\
&\leq C\big(|\alpha| \|\nabla(w_1 - w_2)\|_{L^2(\Omega)} + |\beta_1 - \beta_2| \|\nabla w_1\|_{L^2(\Omega_{5R_0})} + |\beta_2| \|\nabla(w_1 - w_2)\|_{L^2(\Omega_{5R_0})} \\
&\quad + \|w_1\|_{L^\infty(\Omega_{5R_0})} \|\nabla(w_1 - w_2)\|_{L^2(\Omega)} + \|\nabla w_2\|_{L^2(\Omega)} \|w_1 - w_2\|_{L^\infty(\Omega)}\big) \\
&\leq C\big(|\alpha| \|\nabla(w_1 - w_2)\|_{L^2(\Omega)} + \delta_1 |\beta_1 - \beta_2| + \delta_1 \|\nabla(w_1 - w_2)\|_{L^2(\Omega)} \\
&\quad + \delta_1 \|\nabla(w_1 - w_2)\|_{L^2(\Omega)} + \delta_1 \|w_1 - w_2\|_{L^\infty_1(\Omega)}\big) \\
&\leq C(|\alpha| + \delta_1) \|\omega_1 - \omega_2\|_{X_0} \,.
\end{aligned} \tag{2.144}$$

By applying Theorem 2.3.8, we have the representation of the velocity h as

$$h \;=\; \bigg(\int_{\partial\Omega} y^\perp \cdot T(h,q)\nu \; \mathrm{d}\sigma_y\bigg) V + \mathcal{R}_\Omega[\mathrm{div}\, G'_\alpha(\vec{\beta}, \mathbf{w})] \,. \tag{2.145}$$

Here we have used $b_\Omega[\mathrm{div}\, G'_\alpha(\vec{\beta}, \mathbf{w})] \;=\; 0$ again, which follows from the symmetry of $G'_\alpha(\vec{\beta}, \mathbf{w})$ and from the fact that the trace of $G'_\alpha(\vec{\beta}, \mathbf{w})$ on $\partial\Omega$ is zero. Since $h = u_{\omega_1} - u_{\omega_2}$ and $q = q_{\omega_1} - q_{\omega_2}$, we see from the definitions of $T(h,q)$ and $\psi[\omega_j]$ in (2.124),

$$\int_{\partial\Omega} y^\perp \cdot T(h,q)\nu \; \mathrm{d}\sigma_y \;=\; \psi[\omega_1] - \psi[\omega_2] \,,$$

and thus, we also have from (2.140) and (2.145),

$$\mathcal{R}_\Omega[\mathrm{div}\, G'_\alpha(\vec{\beta}, \mathbf{w})] \;=\; R[\omega_1] - R[\omega_2] \,.$$

In virtue of (2.86)–(2.88) we see

$$
\begin{aligned}
\Big|\int_{\partial\Omega} y^{\perp}\cdot T(h,q)\nu \,\mathrm{d}\sigma_y\Big| &\le C\big(\|\nabla h\|_{W^{1,2}(\Omega_{4R_0})}+\|q\|_{W^{1,2}(\Omega_{4R_0})}\big)\\
&\le C\big(\|G'_\alpha(\vec{\beta},\mathbf{w})\|_{L^2(\Omega)}+\|\mathrm{div}\, G'_\alpha(\vec{\beta},\mathbf{w})\|_{L^2(\Omega_{5R_0})}\big)\,. \qquad (2.146)
\end{aligned}
$$

A similar argument as in the derivation of (2.127) yields

$$
\begin{aligned}
&\|\mathcal{R}_\Omega[\mathrm{div}\, G'_\alpha(\vec{\beta},\mathbf{w})]\|_{L^\infty(\Omega_{4R_0})}+\|\nabla\mathcal{R}_\Omega[\mathrm{div}\, G'_\alpha(\vec{\beta},\mathbf{w})]\|_{L^2(\Omega)}\\
&\le C\big(\|G'_\alpha(\vec{\beta},\mathbf{w})\|_{L^2(\Omega)}+\|\mathrm{div}\, G'_\alpha(\vec{\beta},\mathbf{w})\|_{L^2(\Omega_{5R_0})}\big)\,.
\end{aligned} \qquad (2.147)
$$

Moreover, by applying (2.101) we see that the term $\mathcal{R}_\Omega[\mathrm{div}\, G'_\alpha(\vec{\beta},\mathbf{w})]$ satisfies

$$
\begin{aligned}
&\|\mathcal{R}_\Omega[\mathrm{div}\, G'_\alpha(\vec{\beta},\mathbf{w})]\|_{L^\infty_1(B^{\mathrm{c}}_{4R_0})}\\
&\le C\bigg(|\alpha|^{-\frac12}\|G'_\alpha(\vec{\beta},\mathbf{w})\|_{L^2(\Omega)}+\big|\log|\alpha|\big|\|G'_\alpha(\vec{\beta},\mathbf{w})\|_{L^\infty_2(\Omega)}\bigg)\,.
\end{aligned} \qquad (2.148)
$$

Here we have used again the symmetry of $G'_\alpha(\vec{\beta},\mathbf{w})$. Combining (2.146)–(2.148) with (2.142)–(2.144), we obtain for sufficiently small $|\alpha|\neq 0$ and $\kappa_\alpha[F]$ in (2.136),

$$
\begin{aligned}
&\|\Phi[\omega_1]-\Phi[\omega_2]\|_{X_0}\\
&= |\psi[\omega_1]-\psi[\omega_2]|+\|\nabla\big(R[\omega_1]-R[\omega_2]\big)\|_{L^2(\Omega)}+\|R[\omega_1]-R[\omega_2]\|_{L^\infty_1(\Omega)}\\
&\le C\bigg(|\alpha|^{-\frac12}\big(|\alpha|+\delta_1|\log\delta_1|\big)+\big|\log|\alpha|\big|\,\big(|\alpha|+\delta_1+\delta_2\big)\bigg)\|\omega_1-\omega_2\|_{X_0}\\
&\le \frac34\|\omega_1-\omega_2\|_{X_0}\,,
\end{aligned} \qquad (2.149)
$$

that is, the map Φ is a contraction on $\mathcal{B}_{(\delta_1,\delta_2,\delta_2),0}$. Here we have used the estimates $|\log\delta_1|\le \big|\log|\alpha|\big|$ and $\delta_1\le 2^{-1}|\alpha|^{\frac12}\big|\log|\alpha|\big|^{-1}$ if $\delta_1\ge|\alpha|$ and the data related to F in (2.131) are small enough. Therefore, there exists a fixed point $\omega=(\beta,w)$ of Φ in $\mathcal{B}_{\vec{\delta},\gamma}$, which is unique in $\mathcal{B}_{(\delta_1,\delta_2,\delta_2),0}$. By the definition of Φ in (2.125), the fixed point $\omega=(\beta,w)$ satisfies

$$
u_\omega = u_{(\beta,w)} = \psi[\omega]V+R[\omega] = \beta V+w\,,
$$

which is the solution to (NS'_α), as desired. Let us set $v=\beta V+w$ for the fixed point $(\beta,w)\in\mathcal{B}_{\vec{\delta},\gamma}$. The local regularity of $v\in W^{2,2}_{\mathrm{loc}}(\overline{\Omega})^2$ as well as $\nabla q\in L^2_{\mathrm{loc}}(\overline{\Omega})^2$ follows from the standard elliptic regularity of the Stokes operator by regarding the nonlinear term, which belongs to $L^2(\Omega)^2$ by the above construction, as a given external force. This leads to the regularity $u\in W^{2,2}_{\mathrm{loc}}(\overline{\Omega})^2$ and $\nabla p\in L^2_{\mathrm{loc}}(\overline{\Omega})^2$ for the solution $(u,\nabla p)$ to (NS_α) by (2.111). Next we observe that $v=\beta V+w$ solves

$$
\left\{
\begin{aligned}
-\Delta v-\alpha(x^\perp\cdot\nabla w-w^\perp)+\nabla\tilde{q} &= -\mathrm{div}\,(\alpha U\otimes v+v\otimes\alpha U+v\otimes v)\\
&\qquad +\mathrm{div}\, H_\alpha(F)\,,\qquad x\in\Omega\,,\\
\mathrm{div}\, v &= 0\,,\qquad x\in\Omega\,,\\
v &= 0\,,\qquad x\in\partial\Omega\,,\\
v &\to 0\,,\qquad |x|\to\infty\,.
\end{aligned}
\right. \qquad (\mathrm{NS}''_\alpha)
$$

Here we have used the identity $x^\perp \cdot \nabla V - V^\perp = 0$ by the definition of V. Let us take the approximation of v of the form

$$v^{(N)} = \chi_N \beta V + w^{(N)}\,, \quad w^{(N)} = \chi_N w - \mathbb{B}_N[\nabla \chi_N \cdot w]\,, \quad N \gg 1\,, \tag{2.150}$$

where $\chi_N(|x|)$ is the radial cut-off function satisfying $\chi_N = 1$ for $|x| \le N$, $\chi_N = 0$ for $|x| \ge 2N$, and $|\nabla \chi_N| \le CN^{-1}$, while $\mathbb{B}_N$ is the Bogovskii operator in the closed annulus $A_N = \{N \le |x| \le 2N\}$ which satisfies

$$\operatorname{supp} \mathbb{B}_N[\nabla \chi_N \cdot w] \subset A_N\,, \qquad \operatorname{div} \mathbb{B}_N[\nabla \chi_N \cdot w] = \nabla \chi_N \cdot w$$

and

$$\begin{aligned} N^{-1}\|\mathbb{B}_N[\nabla \chi_N \cdot w]\|_{L^2(\Omega)} + \|\nabla \mathbb{B}_N[\nabla \chi_N \cdot w]\|_{L^2(\Omega)} &\le C\|\nabla \mathbb{B}_N[\nabla \chi_N \cdot w]\|_{L^2(\Omega)} \\ &\le C\|\nabla \chi_N \cdot w\|_{L^2(\Omega)}\,. \end{aligned} \tag{2.151}$$

Here C is independent of N; see, e.g. Borchers and Sohr [6, Theorem 2.10]. Then, by multiplying $v^{(N)}$ both sides of the first equation in (NS''_α) and integrating over Ω, we obtain

$$\begin{aligned} &\langle \nabla v, \nabla v^{(N)} \rangle_{L^2(\Omega)} + \alpha \langle w, x^\perp \cdot \nabla w^{(N)} - (w^{(N)})^\perp \rangle_{L^2(\Omega)} \\ &= \langle v \otimes v + \alpha U \otimes \bar{v} + v \otimes \alpha U, \nabla v^{(N)} \rangle_{L^2(\Omega)} - \langle H_\alpha(F), \nabla v^{(N)} \rangle_{L^2(\Omega)} \end{aligned} \tag{2.152}$$

from the integration by parts. Here we have used again the identity for the radial circular flow: $x^\perp \cdot \nabla(\chi_N V) - \chi_N V^\perp = 0$. It is easy to see from (2.151) and $w \in \dot{W}^{1,2}_{0,\sigma}(\Omega) \cap L^\infty_{1+\gamma}(\Omega)^2$ that

$$\begin{aligned} \langle \nabla v, \nabla v^{(N)} \rangle_{L^2(\Omega)} &\to \langle \nabla v, \nabla v \rangle_{L^2(\Omega)}\,, \\ \langle v \otimes v, \nabla v^{(N)} \rangle_{L^2(\Omega)} &\to \langle v \otimes v, \nabla v \rangle_{L^2(\Omega)} = 0\,, \\ \langle \alpha U \otimes v + v \otimes \alpha U, \nabla v^{(N)} \rangle_{L^2(\Omega)} &\to \langle \alpha U \otimes v + v \otimes \alpha U, \nabla v \rangle_{L^2(\Omega)} = \alpha \langle U \otimes v, \nabla v \rangle_{L^2(\Omega)}\,, \\ \langle H_\alpha(F), \nabla v^{(N)} \rangle_{L^2(\Omega)} &\to \langle H_\alpha(F), \nabla v \rangle_{L^2(\Omega)}\,, \end{aligned}$$

as $N \to \infty$. As for the term $\langle w, (w^{(N)})^\perp \rangle_{L^2(\Omega)}$ we see

$$\begin{aligned} |\langle w, (w^{(N)})^\perp \rangle_{L^2(\Omega)}| &= |\langle w, \mathbb{B}_N[\nabla \chi_N \cdot w]^\perp \rangle_{L^2(\Omega)}| \\ &\le \|w\|_{L^2(\{N \le |x| \le 2N\})} \|\mathbb{B}_N[\nabla \chi_N \cdot w]\|_{L^2(\Omega)} \\ &\le CN\|w\|_{L^2(\{N \le |x| \le 2N\})} \|\nabla \chi_N \cdot w\|_{L^2(\Omega)} \\ &\le CN^{-2\gamma}\|w\|^2_{L^\infty_{1+\gamma}(\Omega)} \\ &\begin{cases} \to 0 \quad (N \to \infty) & \text{if} \quad \gamma > 0\,, \\ \le C\|w\|^2_{L^\infty_1(\Omega)} & \text{if} \quad \gamma = 0\,. \end{cases} \end{aligned}$$

It remains to consider the term $\langle w, x^\perp \cdot \nabla w^{(N)} \rangle_{L^2(\Omega)}$. From the integration by parts and from $x^\perp \cdot \nabla \chi_N = 0$, $\operatorname{div}(x^\perp \chi_N) = 0$, and $\operatorname{supp} \mathbb{B}_N[\nabla \chi_N \cdot w] \subset A_N$ we have

$$\begin{aligned} |\langle w, x^\perp \cdot \nabla w^{(N)} \rangle_{L^2(\Omega)}| &= |\langle w, x^\perp \cdot \nabla \mathbb{B}_N[\nabla \chi_N \cdot w] \rangle_{L^2(\Omega)}| \\ &\le N\|w\|_{L^2(\{N \le |x| \le 2N\})} \|\nabla \mathbb{B}_N[\nabla \chi_N \cdot w]\|_{L^2(\Omega)} \\ &\le CN^{-2\gamma}\|w\|^2_{L^\infty_{1+\gamma}(\Omega)} \\ &\begin{cases} \to 0 \quad (N \to \infty) & \text{if} \quad \gamma > 0\,, \\ \le C\|w\|^2_{L^\infty_1(\Omega)} & \text{if} \quad \gamma = 0\,. \end{cases} \end{aligned}$$

Here we have also used (2.151). Collecting these above, we have arrived at the identity

$$\langle \nabla v, \nabla v\rangle_{L^2(\Omega)} \;=\; \alpha\langle U\otimes v, \nabla v\rangle_{L^2(\Omega)} - \langle H_\alpha(F), \nabla v\rangle_{L^2(\Omega)} \qquad \text{when} \quad \gamma>0\,. \tag{2.153}$$

In particular, from the Poincaré inequality $|\langle U\otimes v, \nabla v\rangle_{L^2(\Omega)}| \le C\|\nabla v\|^2_{L^2(\Omega)}$ we obtain the estimate

$$(1-C|\alpha|)\|\nabla v\|^2_{L^2(\Omega)} \le \|F+\alpha\nabla U\|^2_{L^2(\Omega)} \qquad \text{when} \quad \gamma>0\,, \tag{2.154}$$

which shows (2.106) for the case $\gamma>0$ by the relation $u=\alpha U+v$. Note that the constant C in (2.154) depends only on R_0 and is independent of α and γ. To obtain the energy inequality for the case $\gamma=0$ we first consider the approximation of F and f such that

$$F_n(x) \;=\; e^{-\frac{1}{n}|x|^2}F(x)\,, \qquad f_n \;=\; \operatorname{div} F_n\,. \tag{2.155}$$

Then $F_n \in L^\infty_{2+\gamma}(\Omega)^{2\times 2}$ for $\gamma>0$ and

$$\begin{aligned} &\lim_{n\to\infty} b_\Omega[f_n-f] \;=\; \lim_{n\to\infty}\|F-F_n\|_{L^2(\Omega)} \;=\; \lim_{n\to\infty}\|f_n-f\|_{L^2(\Omega_{6R_0})} \;=\; 0\,,\\ &\lim_{n\to\infty}\|(F-F_n)_{12}-(F-F_n)_{21}\|_{L^1(\Omega)} \;=\; 0\,, \qquad \|F_n\|_{L^\infty_2(\Omega)} \le \|F\|_{L^\infty_2(\Omega)}\,. \end{aligned} \tag{2.156}$$

Here we have used $F_{12}-F_{21}\in L^1(\Omega)$ for the convergence of $b_\Omega[f_n]$. Assume that

$$|\alpha|^{\frac12}\big|\log|\alpha|\big| + \kappa_\alpha[F] < \epsilon(\Omega)\,,$$

and we fix α. Then there is a unique fixed point (β, w) of Φ in $\mathcal{B}_{(\delta_1,\delta_2,\delta_2),0}$. On the other hand, since α is fixed, there is $\gamma_0>0$ such that

$$\sup_{0\le\gamma\le\gamma_0}\big(|\alpha|^{\frac{1-\gamma}{2}}\big|\log|\alpha|\big| + |\alpha|^{-\frac{\gamma}{2}}\kappa_\alpha[F]\big) < \epsilon_{\gamma_0}(\Omega)\,.$$

Here we have used the fact that $\epsilon_0(\Omega)=\epsilon(\Omega)$ and $\epsilon_\gamma(\Omega)$ is continuous on $\gamma\in[0,1)$. Hence, in view of (2.156) and (2.136), there is $N\gg 1$ such that

$$\sup_{n\ge N}\sup_{0\le\gamma\le\gamma_0}\big(|\alpha|^{\frac{1-\gamma}{2}}\big|\log|\alpha|\big| + |\alpha|^{-\frac{\gamma}{2}}\kappa_\alpha[F_n]\big) < \epsilon_{\gamma_0}(\Omega)\,.$$

Let $(v_n, \nabla\tilde q_n)$ with $v_n=\beta_n V+w_n$, $n\ge N$, be the unique solution to (NS''_α) with F replaced by F_n such that $(\beta_n, w_n)\in\mathcal{B}_{(\delta_1,\delta_2,\delta_3^{(n)}),\gamma}\subset\mathcal{B}_{(\delta_1,\delta_2,\delta_2),0}$ with some $\gamma\in(0,\gamma_0]$. Note that for sufficiently large n, we can take the same δ_1 and δ_2. Then (2.153) implies

$$\|\nabla v_n\|^2_{L^2(\Omega)} \;=\; \alpha\langle U\otimes v_n, \nabla v_n\rangle_{L^2(\Omega)} - \langle H_\alpha(F), \nabla v_n\rangle_{L^2(\Omega)}\,. \tag{2.157}$$

Since $(\beta_n, w_n)\in\mathcal{B}_{(\delta_1,\delta_2,\delta_2),0}$ we have uniform estimates of $(v_n, \nabla\tilde q_n)$, and thus, we find a subsequence, denoted again by $(v_n, \nabla\tilde q_n)$, such that $\beta_n\to\beta_\infty$,

$$\begin{aligned} &w_n \rightharpoonup w_\infty \quad \text{in} \quad W^{2,2}_{\mathrm{loc}}(\overline{\Omega})^2\,, \qquad \tilde q_n \rightharpoonup \tilde q_\infty \quad \text{in} \quad W^{1,2}_{\mathrm{loc}}(\overline{\Omega})\,,\\ &\nabla w_n \rightharpoonup \nabla w_\infty \quad \text{in} \quad L^2(\Omega)^{2\times2}\,, \qquad w_n \rightharpoonup^* w_\infty \quad \text{in} \quad L^\infty_1(\Omega)^2\,, \end{aligned}$$

and $w_n\to w_\infty$ strongly in $W^{1,2}_{\mathrm{loc}}(\overline{\Omega})^2$. Moreover, we observe from (2.153) that $v_\infty = \beta_\infty V + w_\infty$ satisfies the energy inequality

$$\|\nabla v_\infty\|^2_{L^2(\Omega)} \le \alpha\,\langle U\otimes v_\infty, \nabla v_\infty\rangle_{L^2(\Omega)} - \langle H_\alpha(F), \nabla v_\infty\rangle_{L^2(\Omega)}\,. \tag{2.158}$$

It is also easy to see that $(v_\infty, \nabla\tilde{q}_\infty)$ is a solution to (NS$''_\alpha$) and $(\beta_\infty, w_\infty) \in \mathcal{B}_{(\delta_1,\delta_2,\delta_2),0}$. By the uniqueness of the fixed point of Φ in $\mathcal{B}_{(\delta_1,\delta_2,\delta_2),0}$, we have $(\beta_\infty, w_\infty) = (\beta, w)$. Therefore, (2.158) holds with v_∞ replaced by $v = \beta V + w$, as desired. Thus we have (2.106) also when $F \in L^\infty_2(\Omega)^{2\times 2}$ and $F_{12} - F_{21} \in L^1(\Omega)$.

The estimates (2.108) and (2.109) follow from the fact $\|w\|_{L^\infty_1(\Omega)} \le \delta_2$ and $\|w\|_{L^\infty_{1+\gamma}(\Omega)} \le \delta_3$ together with the definitions of δ_j in (2.133), (2.139), and $d_\gamma[F] \le C\gamma^{-1}\|F\|_{L^\infty_{2+\gamma}(\Omega)}$ when $\gamma > 0$. As for the identity (2.107) on the coefficient β, we observe from (2.124),

$$\beta = \int_{\partial\Omega} y^\perp \cdot \big(T(v,q)\nu\big)\, \mathrm{d}\sigma_y + b_\Omega[f]\,.$$

Since $v = u - \alpha x^\perp$ and $q = p + P$ near $\partial\Omega$, where $P = P(|x|)$ is a radial function and taken so that $\nabla P = \operatorname{div}\,[(\alpha U + \beta V) \otimes (\alpha U + \beta V)]$, the straightforward calculations yield

$$\int_{\partial\Omega} y^\perp \cdot \big(T(v,q)\nu\big)\, \mathrm{d}\sigma_y = \int_{\partial\Omega} y^\perp \cdot \big(T(u,p)\nu\big)\, \mathrm{d}\sigma_y\,.$$

Thus (2.107) holds. The proof of Theorem 2.4.1 is complete. □

Finally we consider the case $F \in L^\infty_{2,0}(\Omega)^{2\times 2}$. Combining Theorem 2.4.1 with Theorem 2.4.3 below, we obtain Theorem 2.1.1.

Theorem 2.4.3 *Assume that $f = \operatorname{div} F$ satisfies the conditions in Theorem 2.4.1 for $\gamma = 0$. Assume in addition that $F \in L^\infty_{2,0}(\Omega)^{2\times 2}$. Then the remainder w in Theorem 2.4.1 belongs to $L^\infty_{1,0}(\Omega)^2$.*

Proof: The proof is very similar to the derivation of the energy inequality for the case $\gamma = 0$ in the proof of Theorem 2.4.1. We set F_n and f_n as in (2.155). Then F_n and f_n satisfy (2.156), and moreover, the additional condition $F \in L^\infty_{2,0}(\Omega)^{2\times 2}$ implies

$$\|F_n - F\|_{L^\infty_2(\Omega)} \to 0\,, \qquad n \to \infty\,. \tag{2.159}$$

The proof of (2.159) is as follows: for any small number $\epsilon > 0$, there exists $R > 0$ such that $\|F_n - F\|_{L^\infty_2(B^c_R)} \le 2\epsilon\, \|F\|_{L^\infty_2(\Omega)}$ by the condition $F \in L^\infty_{2,0}(\Omega)^{2\times 2}$. Then we have

$$\begin{aligned}\limsup_{n\to\infty} \|F_n - F\|_{L^\infty_2(\Omega)} &\le \limsup_{n\to\infty} \big(\|F_n - F\|_{L^\infty_2(B_R)} + \|F_n - F\|_{L^\infty_2(B^c_R)}\big) \\ &\le \limsup_{n\to\infty} \big((1 - e^{-\frac{R^2}{n}}) + 2\epsilon\big)\|F\|_{L^\infty_2(\Omega)} = 2\epsilon\, \|F\|_{L^\infty_2(\Omega)}\,,\end{aligned}$$

which implies (2.159). As in the proof of Theorem 2.4.1, let $(v_n, \nabla q_n)$, $v_n = \beta_n V + w_n$, $n \gg 1$, be the solution to (NS$''_\alpha$) with F replaced by F_n such that $(\beta_n, w_n) \in \mathcal{B}_{(\delta_1,\delta_2,\delta^{(n)}_{3,}),\gamma} \subset \mathcal{B}_{(\delta_1,\delta_2,\delta_2),0}$ with some $\gamma \in (0,1)$. Since $w_n \in L^\infty_{1+\gamma}(\Omega)^2$ and $\gamma > 0$, it suffices to show that (β_n, w_n) converges to (β, w) in $\mathbb{R} \times L^\infty_1(\Omega)^2$, where $v = \beta V + w$ is the solution to (NS$''_\alpha$). To prove this we observe that the difference $h = v - v_n$ solves

$$\begin{cases} -\Delta h - \alpha(x^\perp \cdot \nabla h - h^\perp) + \nabla q = \operatorname{div} G'_\alpha(\vec{\beta}, \mathbf{w}) + \operatorname{div}(F - F_n)\,, & x \in \Omega\,, \\ \operatorname{div} h = 0\,, & x \in \Omega\,, \\ h = 0\,, & x \in \partial\Omega\,. \\ h \to 0\,, & |x| \to \infty\,. \end{cases}$$

Here we have set $\vec{\beta} = (\beta, \beta_n)$, $\mathbf{w} = (w, w_n)$, and

$$\begin{aligned} G'_\alpha(\vec{\beta}, \mathbf{w}) \; &= \; -\alpha(U \otimes (w - w_n) + (w - w_n) \otimes U) - (\beta - \beta_n)(V \otimes w + w \otimes V) \\ &\quad - \beta_n(V \otimes (w - w_n) + (w - w_n) \otimes V) - w \otimes (w - w_n) - (w - w_n) \otimes w_n \,. \end{aligned}$$

Then the same argument as in the derivation of (2.149) shows

$$\begin{aligned} \|(\beta, w) - (\beta_n, w_n)\|_{X_0} &\leq \frac{3}{4}\|(\beta, w) - (\beta_n, w_n)\|_{X_0} \\ &\quad + C\Big(|b_\Omega[f - f_n]| + \|F - F_n\|_{L^2(\Omega)} + \|f - f_n\|_{L^2(\Omega_{6R_0})} \\ &\quad + \|(F - F_n)_{12} - (F - F_n)_{21}\|_{L^1(\Omega)} + \|F - F_n\|_{L^\infty_2(\Omega)}\Big) \,, \end{aligned}$$

where C is independent of n. Thus, (β_n, w_n) converges to (β, w) in $\mathbb{R} \times L^\infty_1(\Omega)^2$, which shows $w \in L^\infty_{1,0}(\Omega)^2$. The proof is complete. □

2.5 Appendix

We will prove the Hardy type inequality in two-dimensional exterior domains, which has been used in the proof of Theorem 2.4.1.

Lemma A.1 *Let Ω be an exterior domain in $\mathbb{R}^2$. Then it follows that*

$$\|\frac{f}{1 + |x|}\|_{L^2(\Omega)} \leq C\|\nabla f\|_{L^2(\Omega)} \log\left(e + \frac{\|f\|_{L^\infty_1(\Omega)}}{\|\nabla f\|_{L^2(\Omega)}}\right) \tag{2.160}$$

for any $f \in \dot{W}^{1,2}_0(\Omega) \cap L^\infty_1(\Omega)$. Here C depends only on Ω. In particular, if

$$e\|\nabla f\|_{L^2(\Omega)} + \|f\|_{L^\infty_1(\Omega)} \leq 1 \,,$$

then

$$\|\frac{f}{1 + |x|}\|_{L^2(\Omega)} \leq C\|\nabla f\|_{L^2(\Omega)} \big| \log \|\nabla f\|_{L^2(\Omega)}\big| \,. \tag{2.161}$$

Proof: Take $x_0 \in \mathbb{R}^2 \setminus \overline{\Omega}$ and $0 < r_0 < e^{-1}$ so that $B_{r_0}(x_0) \subset \mathbb{R}^2 \setminus \overline{\Omega}$. By considering the zero extension of f to $\mathbb{R}^2$, it suffices to show (2.160) for $\Omega = \mathbb{R}^2$ and $f \in \dot{W}^{1,2}(\mathbb{R}^2) \cap L^\infty_1(\mathbb{R}^2)$ such that $f = 0$ in $B_{r_0}(x_0)$. Fix $R > 2|x_0|$. By the condition $f(x_0) = 0$ and the mean value theorem in the integral form we have

$$\begin{aligned} \frac{|f(x)|}{1 + |x|} &\leq \frac{|x - x_0|}{1 + |x|} \int_0^1 |(\nabla f)(\tau(x - x_0) + x_0)| \, \mathrm{d}\tau \\ &\leq (1 + |x_0|) \int_{\frac{r_0}{|x - x_0|}}^1 |(\nabla f)(\tau(x - x_0) + x_0)| \, \mathrm{d}\tau \,, \quad x \in \mathbb{R}^2 \setminus B_{r_0}(x_0) \,, \end{aligned}$$

which gives

$$\begin{aligned} \|\frac{f}{1 + |x|}\|_{L^2(\{|x - x_0| \leq R\})} &\leq (1 + |x_0|) \int_{\frac{r_0}{R}}^1 \tau^{-1} \|\nabla f\|_{L^2(\mathbb{R}^2)} \, \mathrm{d}\tau \\ &\leq (1 + |x_0|)\big(|\log R| + |\log r_0|\big)\|\nabla f\|_{L^2(\mathbb{R}^2)} \,. \end{aligned} \tag{2.162}$$

On the other hand, we have

$$\|\frac{f}{1+|x|}\|_{L^2(\{|x-x_0|\geq R\})} \leq \|\frac{1}{(1+|x|)^2}\|_{L^2(\{|x|\geq \frac{R}{2}\})}\|f\|_{L_1^\infty(\mathbb{R}^2)}$$
$$\leq \frac{C}{R}\|f\|_{L_1^\infty(\mathbb{R}^2)}\,. \tag{2.163}$$

If $\|f\|_{L_1^\infty(\mathbb{R}^2)} \leq 2|x_0|\|\nabla f\|_{L^2(\mathbb{R}^2)}$ then we obtain (2.160) from (2.162) and (2.163) with $R = 2|x_0|+1$. If $\|f\|_{L_1^\infty(\mathbb{R}^2)} \geq 2|x_0|\|\nabla f\|_{L^2(\mathbb{R}^2)}$ then we take $R = e + \frac{\|f\|_{L_1^\infty(\mathbb{R}^2)}}{\|\nabla f\|_{L^2(\mathbb{R}^2)}}$, which yields again from (2.162) and (2.163) that

$$\|\frac{f}{1+|x|}\|_{L^2(\mathbb{R}^2)} \leq C|\log r_0|(1+|x_0|)\|\nabla f\|_{L^2(\mathbb{R}^2)}\log\left(e+\frac{\|f\|_{L_1^\infty(\mathbb{R}^2)}}{\|\nabla f\|_{L^2(\mathbb{R}^2)}}\right). \tag{2.164}$$

Here we have used $|\log r_0| \geq 1$ and $|\log R| \geq 1$, and C is a numerical constant. Thus (2.160) holds. The proof is complete. □

Chapter 3

On stationary two-dimensional flows around a fast rotating disk

Abstract In the previous chapter, we proved the unique existence of solutions to the two-dimensional stationary Navier-Stokes equations describing the flows around a rotating obstacle, when the rotation speed is sufficiently small. We will study the fast rotation case in this chapter under the assumption that the obstacle is a unit disk.

Thanks to the symmetry of the fluid domain, we can establish the unique existence of solutions for any rotating speed contrary to the previous chapter, and moreover, we can relax the summability conditions in Theorem 2.1.1 on the external force and on the class of solutions. Finally, the qualitative effects of a large rotation are described precisely by exhibiting a boundary layer structure and an axisymmetrization of the flow.

3.1 Introduction

As we have seen in Chapter 2, the rotation of a two-dimensional rigid body (obstacle) leads to a drastic change in the decay structure of its surrounding fluid as in the translation case explained in Chapter 1. Moreover, the obstacle's rotation yields a significant localizing effect that enables us to construct corresponding steady state solutions to the Navier-Stokes equations when the rotation is slow enough, and in particular, the Reynolds number is sufficiently small; see Galdi [19] for the translation case results. Although on the one hand a faster motion of the obstacle might give a stronger localizing and stabilizing effect, on the other hand it produces a rapid flow and creates a strong shear near the boundary that can be a source of instability. As a result, rigorous analysis becomes quite difficult for the nonlinear problem in general. Hence it is useful to study the problem under a simple geometrical setting and to understand a typical fluid structure that describes these two competitive mechanisms; localizing and stabilizing effects on the one hand, and the presence of a rapid flow and the boundary layer created by the fast motion of the obstacle on the other hand.

In this chapter we study two-dimensional flows around a rotating obstacle assuming that the obstacle is a unit disk centered at the origin, especially in the case when the rotation speed is sufficiently fast and the Reynolds number is high. Note that in a three-dimensional setting, these flows are considered as a model for two-dimensional flows around a rotating infinite cylinder with a uniform cross section which is a unit disk.

After taking the same change of variables procedure as in Chapter 2, we consider the following stationary Navier-Stokes equations in the domain $\Omega = \{x \in \mathbb{R}^2 \mid |x| > 1\}$:

$$\begin{cases} -\Delta u - \alpha(x^\perp \cdot \nabla u - u^\perp) + \nabla p \,=\, -u \cdot \nabla u + f\,, \qquad x \in \Omega\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad \operatorname{div} u \,=\, 0\,, \qquad x \in \Omega\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad u \,=\, \alpha x^\perp\,, \qquad x \in \partial\Omega\,. \end{cases} \tag{NS$_\alpha$}$$

Here the unknown functions $u = u(x) = (u_1(x), u_2(x))^\top$ and $p = p(x)$ are respectively the velocity field and the pressure field of the fluid in the coordinates attached to the rotating disk, and $f = f(x) = (f_1(x), f_2(x))^\top$ is an external force given in this reference coordinates. We use the same notations as in Chapter 2 for differential operators with respect to $x = (x_1, x_2)^\top$. We note again that in the original coordinates the stationary solution to (NS$_\alpha$) gives a specific time periodic flow with a periodicity $\frac{2\pi}{|\alpha|}$. Due to the symmetry of the domain there is an explicit stationary solution to (NS$_\alpha$) when $f = 0$:

$$\big(\alpha U, \alpha^2 \nabla P\big) \quad \text{with} \quad U(x) \,=\, \frac{x^\perp}{|x|^2}\,, \qquad P(x) \,=\, -\frac{1}{2|x|^2}\,\cdot \tag{3.1}$$

Thus it is natural to consider an expansion around this explicit solution. By using the identity $u \cdot \nabla u = \frac{1}{2}\nabla|u|^2 + u^\perp \operatorname{rot} u$ with $\operatorname{rot} u = \partial_1 u_2 - \partial_2 u_1$ and the condition $\operatorname{rot} U = 0$ for $x \neq 0$, the equations for $v = u - \alpha U$ can be written as

$$\begin{cases} -\Delta v - \alpha(x^\perp \cdot \nabla v - v^\perp) + \nabla q + \alpha U^\perp \operatorname{rot} v \,=\, -v^\perp \operatorname{rot} v + f\,, \quad x \in \Omega\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \operatorname{div} v \,=\, 0\,, \qquad\quad x \in \Omega\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad v \,=\, 0\,, \qquad\quad x \in \partial\Omega\,. \end{cases} \tag{$\widetilde{\text{NS}}_\alpha$}$$

The goal of this chapter is to show the existence and uniqueness of solutions to ($\widetilde{\text{NS}}_\alpha$) for arbitrary $\alpha \in \mathbb{R}\backslash\{0\}$ under a suitable condition on the external force f in terms of regularity and summability. Moreover, we shall give a detailed qualitative analysis for the fast rotation case $|\alpha| \gg 1$ that exhibits a boundary layer structure and an axisymmetrization of the flow.

For the known results related to the problem in this chapter, we mainly refer the reader to the papers in Chapter 2 in order to avoid overlapping, but let us compare our results with Hillairet and Wittwer [32] in which they consider the stationary Navier-Stokes equations

$$\begin{cases} -\Delta w + \nabla r \,=\, -w \cdot \nabla w\,, \qquad y \in \Omega\,, \\ \qquad\quad \operatorname{div} w \,=\, 0\,, \qquad y \in \Omega\,, \\ \qquad\qquad\;\; w \,=\, \alpha x^\perp + b\,, \qquad y \in \partial\Omega\,, \end{cases}$$

in the exterior unit disk as in this chapter and establish the existence of solutions around (3.1) when $|\alpha|$ is sufficiently large and the time-independent given data $b = b(x)$ is small enough. Our problem is in fact essentially different from the one discussed in [32]. Indeed, the stationary solution to (NS$_\alpha$) is a time periodic solution in the original frame, and therefore, the result in [32] is not applicable to our problem and vice versa.

We summarize the novelty of the results in this chapter as follows:

(1) Existence and uniqueness of solutions to ($\widetilde{\text{NS}}_\alpha$) for arbitrary $\alpha \in \mathbb{R} \setminus \{0\}$.

(2) Relaxed summability condition on f and on the class of solutions, which allows slow spatial decay with respect to scaling.

(3) Qualitative analysis of solutions in the fast rotation case $|\alpha| \gg 1$.

As for (1), the result is new compared with the ones in Chapter 2 in which the stationary solutions are obtained only for nonzero but small $|\alpha|$, though there is no restriction on the shape of the obstacle in Chapter 2. The reason why we can construct solutions for all nonzero α in the exterior unit disk is a remarkable coercive estimate for the term $-\alpha(x^\perp\cdot\nabla v - v^\perp) + \alpha U^\perp \mathrm{rot}\, v$ in polar coordinates; see (3.16) below. As for (2), we note that the given data f and the class of solutions in Chapter 2 are in a scale critical space. A typical condition for f assumed in Chapter 2 is that $f = \mathrm{div}\, F$ with $F(x) = O(|x|^{-2})$, and then the solution v satisfies the estimate $|v(x)| \le C|x|^{-1}$ for $|x| \gg 1$. In this chapter the summability condition on f is much weaker than this scaling; see (3.4) below. Moreover, the radial part of the solution constructed in this chapter only behaves like $o(1)$ as $|x|\to\infty$ in general, which is considerably slow, while the nonradial part of the solution belongs to $L^2(\Omega)$ which is just in the scale critical regime. The point (3) is important both physically and mathematically. Understanding the fluid structure around the fast rotating obstacle up to the boundary is one of the main subjects of this chapter, and we show the appearance of a boundary layer as well as an axisymmetrization mechanism due to the fast rotation.

Let us state our functional setting. Due to the symmetry of the domain it is natural to introduce the relevant function spaces in terms of polar coordinates. As usual, we set

$$\begin{aligned} x_1 &= r\cos\theta\,, \quad x_2 = r\sin\theta\,, \qquad r = |x| \ge 1\,, \quad \theta\in[0,2\pi)\,,\\ \mathbf{e}_r &= \frac{x}{|x|}\,, \qquad \mathbf{e}_\theta = \frac{x^\perp}{|x|} = \partial_\theta \mathbf{e}_r\,, \end{aligned}$$

and

$$v = v_r\,\mathbf{e}_r + v_\theta\,\mathbf{e}_\theta\,, \qquad v_r = v\cdot\mathbf{e}_r\,, \qquad v_\theta = v\cdot\mathbf{e}_\theta\,.$$

Next, for each $n\in\mathbb{Z}$, we denote by $\mathcal{P}_n$ the projection on the Fourier mode n with respect to the angular variable θ:

$$\mathcal{P}_n v = v_{r,n}e^{in\theta}\mathbf{e}_r + v_{\theta,n}e^{in\theta}\mathbf{e}_\theta\,, \tag{3.2}$$

where

$$\begin{aligned} v_{r,n}(r) &= \frac{1}{2\pi}\int_0^{2\pi} v_r(r\cos\theta, r\sin\theta)e^{-in\theta}\,\mathrm{d}\theta\,,\\ v_{\theta,n}(r) &= \frac{1}{2\pi}\int_0^{2\pi} v_\theta(r\cos\theta, r\sin\theta)e^{-in\theta}\,\mathrm{d}\theta\,. \end{aligned}$$

We also set for $m\in\mathbb{N}\cup\{0\}$,

$$\mathcal{Q}_m v = \sum_{|n|=m+1}^{\infty} \mathcal{P}_n v\,. \tag{3.3}$$

For notational convenience we will often write v_n for $\mathcal{P}_n v$. Each $\mathcal{P}_n$ is an orthogonal projection in $L^2(\Omega)^2$, and the space $L^2_\sigma(\Omega) := \overline{\{f\in C_0^\infty(\Omega)^2 \mid \mathrm{div}\, f = 0\}}^{L^2(\Omega)^2}$ is invariant under the action of $\mathcal{P}_n$. Note that $v_0 := \mathcal{P}_0 v$ is the radial part of v, and thus, $\mathcal{Q}_0 v$ is the nonradial part of v. We will set $\mathcal{P}_n L^2(\Omega)^2 := \{f\in L^2(\Omega)^2 \mid f = \mathcal{P}_n f\}$, and similar notation will be used for $L^2_\sigma(\Omega)$ and $\mathcal{Q}_0$. A vector field f in Ω is formally identified with the pair $(\mathcal{P}_0 f, \mathcal{Q}_0 f)$. Then, for the class of external forces we introduce the product space

$$Y := \mathcal{P}_0 L^1(\Omega)^2 \times \mathcal{Q}_0 L^2(\Omega)^2\,. \tag{3.4}$$

For the class of solutions we set

$$X := \mathcal{P}_0 W_0^{1,\infty}(\Omega)^2 \times \mathcal{Q}_0 W_0^{1,2}(\Omega)^2 \,. \tag{3.5}$$

Here $W_0^{1,r}(\Omega) := \{f \in W^{1,r}(\Omega) \mid f = 0 \text{ on } \partial\Omega\}$ for $1 < r \leq \infty$. In this chapter we say that a couple $(v, \nabla q)$ is a solution to $(\widetilde{\mathrm{NS}}_\alpha)$ with $f = (\mathcal{P}_0 f, \mathcal{Q}_0 f) \in Y$ if

(i) $v \in X \cap L^\infty(\Omega)^2 \cap W^{2,1}_{\mathrm{loc}}(\overline{\Omega})^2$ and $q \in W^{1,1}_{\mathrm{loc}}(\overline{\Omega})$,
(ii) $(v, \nabla q)$ satisfies the first and the second equations in $(\widetilde{\mathrm{NS}}_\alpha)$ for $a.e.\ x \in \Omega$.

We note that the Dirichlet boundary condition on v is implemented in the function space X. Our first result is stated as follows.

Theorem 3.1.1 *There exists $\gamma > 0$ such that the following statements hold.*
(i) *Let $0 < |\alpha| < 1$. Then for any external force $f = (\mathcal{P}_0 f, \mathcal{Q}_0 f) \in Y$ satisfying*

$$\|(\mathcal{P}_0 f)_\theta\|_{L^1(\Omega)} \leq \gamma |\alpha| \,, \qquad \|\mathcal{Q}_0 f\|_{L^2(\Omega)} \leq \gamma |\alpha|^2 \,, \tag{3.6}$$

there exists a unique solution $(v, \nabla q) \in X \cap L^\infty(\Omega)^2 \cap W^{2,1}_{\mathrm{loc}}(\overline{\Omega})^2 \times L^1_{\mathrm{loc}}(\overline{\Omega})^2$ to $(\widetilde{\mathrm{NS}}_\alpha)$ satisfying

$$\|\mathcal{P}_0 v\|_{L^\infty(\Omega)} + \big\|\nabla \mathcal{P}_0 v\big\|_{L^\infty(\Omega)} \leq C \|(\mathcal{P}_0 f)_\theta\|_{L^1(\Omega)} + \frac{C}{|\alpha|^{\frac{3}{2}}} \|\mathcal{Q}_0 f\|^2_{L^2(\Omega)} \,, \tag{3.7}$$

$$\|\mathcal{Q}_0 v\|_{L^2(\Omega)} \leq \frac{C}{|\alpha|} \|\mathcal{Q}_0 f\|_{L^2(\Omega)} \,, \tag{3.8}$$

$$\sum_{|n| \geq 1} \|\mathcal{P}_n v\|_{L^\infty(\Omega)} \leq \frac{C}{|\alpha|^{\frac{3}{4}}} \|\mathcal{Q}_0 f\|_{L^2(\Omega)} \,, \tag{3.9}$$

$$\|\nabla \mathcal{Q}_0 v\|_{L^2(\Omega)} \leq \frac{C}{|\alpha|^{\frac{1}{2}}} \|\mathcal{Q}_0 f\|_{L^2(\Omega)} \,. \tag{3.10}$$

(ii) *Let $|\alpha| \geq 1$. Then for any external force $f = (\mathcal{P}_0 f, \mathcal{Q}_0 f) \in Y$ satisfying*

$$\|(\mathcal{P}_0 f)_\theta\|_{L^1(\Omega)} \leq \gamma \,, \qquad \|\mathcal{Q}_0 f\|_{L^2(\Omega)} \leq \gamma \,, \tag{3.11}$$

there exists a unique solution $(v, \nabla q) \in X \cap L^\infty(\Omega)^2 \cap W^{2,1}_{\mathrm{loc}}(\overline{\Omega})^2 \times L^1_{\mathrm{loc}}(\overline{\Omega})^2$ to $(\widetilde{\mathrm{NS}}_\alpha)$ satisfying

$$\|\mathcal{P}_0 v\|_{L^\infty(\Omega)} + \big\|\nabla \mathcal{P}_0 v\big\|_{L^\infty(\Omega)} \leq C \|(\mathcal{P}_0 f)_\theta\|_{L^1(\Omega)} + \frac{C}{|\alpha|^{\frac{1}{2}}} \|\mathcal{Q}_0 f\|^2_{L^2(\Omega)} \,, \tag{3.12}$$

$$\|\mathcal{Q}_0 v\|_{L^2(\Omega)} \leq \frac{C}{|\alpha|^{\frac{1}{2}}} \|\mathcal{Q}_0 f\|_{L^2(\Omega)} \,, \tag{3.13}$$

$$\sum_{|n| \geq 1} \|\mathcal{P}_n v\|_{L^\infty(\Omega)} \leq \frac{C}{|\alpha|^{\frac{1}{4}}} \|\mathcal{Q}_0 f\|_{L^2(\Omega)} \,, \tag{3.14}$$

$$\|\nabla \mathcal{Q}_0 v\|_{L^2(\Omega)} \leq C \|\mathcal{Q}_0 f\|_{L^2(\Omega)} \,. \tag{3.15}$$

Note that the summability of f assumed in Theorem 3.1.1 is much weaker than the scale-critical one. For the radial part $\mathcal{P}_0 v$ we can show $\lim_{|x|\to\infty} |\mathcal{P}_0 v(x)| = 0$ but there is no rate in general under the assumptions of Theorem 3.1.1. If the external force f decays fast enough as $|x| \to \infty$ then it is expected that the solution in Theorem 3.1.1 behaves like a constant multiple of the circular flow U for any size of α, as in the case $0 < |\alpha| \ll 1$ which is proved in Chapter 2 and Hishida and Kyed [36]. Although such an asymptotic behavior at spatial infinity is also an important problem, we will not go into the details about this topic in this chapter. Theorem 3.1.1 already exhibits the axisymmetrizing effect of the fast rotating obstacle in L^2 and L^∞, which will be further extended in Theorems 3.1.2 and 3.1.3 below. The proof of Theorem 3.1.1 consists in two ingredients: the analysis of the linearized problem (S_α) (defined and studied in Section 3.3), and the estimate of the interaction between the radial part and the nonradial part in the nonlinear problem (see Section 3.4). The linear result used in Theorem 3.1.1 is stated in Proposition 3.3.1, and the proof is based on an energy method. Although the proof of the linear result is not so difficult, there is a key observation for the term $-\alpha(x^\perp \cdot \nabla v - v^\perp) + \alpha U^\perp \mathrm{rot}\, v$. Indeed, for the linearized problem (S_α) the energy computation for $v_n = \mathcal{P}_n v$ with $n \neq 0$ gives

$$\begin{aligned}
&\alpha n \bigg(\|v_{r,n}\|^2_{L^2(\Omega)} - (1 - \frac{2}{n^2}) \|\frac{v_{r,n}}{r}\|^2_{L^2(\Omega)} + \|v_{\theta,n}\|^2_{L^2(\Omega)} - \|\frac{v_{\theta,n}}{r}\|^2_{L^2(\Omega)} \bigg) \\
&\qquad = -\mathrm{Im}\langle f_n \,, v_n \rangle_{L^2(\Omega)} \,.
\end{aligned} \tag{3.16}$$

Here f_n denotes $\mathcal{P}_n f$ and the norm $\|g\|_{L^2(\Omega)}$ for the function $g : [1,\infty) \to \mathbb{C}$ is defined as $(2\pi)^{\frac{1}{2}} \|g\|_{L^2((1,\infty); r\,\mathrm{d}r)}$. The key point here is that the bracket in (3.16) is nonnegative and provides a bound for $\|\frac{\sqrt{|x|^2-1}}{|x|} v_n\|^2_{L^2(\Omega)}$ since $\Omega = \{|x| > 1\}$. Then by combining with an interpolation inequality of the form

$$\|g\|_{L^2(\Omega)} \le C \|\partial_r g\|^{\frac{1}{3}}_{L^2(\Omega)} \|\frac{\sqrt{r^2-1}}{r} g\|^{\frac{2}{3}}_{L^2(\Omega)} + C \|\frac{\sqrt{r^2-1}}{r} g\|_{L^2(\Omega)} \tag{3.17}$$

for any scalar function $g \in W^{1,2}((1,\infty); r\,\mathrm{d}r)$ and the dissipation from the Laplacian in the energy computation, we can close the energy estimate for all $\alpha \neq 0$. The proof of (3.17) is given in Appendix 3.5.2. In solving the nonlinear problem the key observation is that the product of the radial parts in the nonlinear term can always be written in a gradient form and thus regarded as a pressure term, which yields the identity

$$v^\perp \mathrm{rot}\, v = v_0^\perp \mathrm{rot}\, \mathcal{Q}_0 v + (\mathcal{Q}_0 v)^\perp \mathrm{rot}\, v_0 + (\mathcal{Q}_0 v)^\perp \mathrm{rot}\, \mathcal{Q}_0 v + \nabla \tilde{q} \tag{3.18}$$

for a suitable $\tilde{q}$. Indeed, this identity is valid since v is solenoidal and its normal trace vanishes at the boundary $|x| = 1$. Since $\mathcal{P}_0\big(v_0^\perp \mathrm{rot}\, \mathcal{Q}_0 v + (\mathcal{Q}_0 v)^\perp \mathrm{rot}\, v_0\big) = 0$ as long as $\mathcal{Q}_0 v \in W^{1,2}_0(\Omega)^2$ the radial part of the velocity in the right-hand side of (3.18) (neglecting $\nabla\tilde{q}$) belongs to $L^1(\Omega)^2$, which is the same summability as the space Y. This is a brief explanation for the reason why we can close the nonlinear estimate and solve ($\widetilde{\mathrm{NS}}_\alpha$) in X for a source $f \in Y$.

Our second result is focused on the fast rotation case $|\alpha| \gg 1$. In this regime there are three fundamental mechanisms in our system:

(I) an axisymmetrization due to the fast rotation of the obstacle,

(II) the presence of a boundary layer for the nonradial part of the flow due to the no-slip boundary condition,

(III) the diffusion in high angular frequencies due to the viscosity.

(I) and (II) are potentially in a competitive relation, for the no-slip boundary condition and the boundary layer can suppress the effect of the fast rotation to some extent.

(II) and (III) are also competitive. Indeed, it is natural that if the viscosity is strong enough then the boundary layer is diffused and is no longer observable. The important task here is to determine the regime of angular frequencies in which the boundary layer appears, and to estimate the thickness of the boundary layer. We show that the boundary layer appears in the regime $1 \le |n| \ll O(|\alpha|^{\frac{1}{2}})$, and the thickness of the boundary layer is $(2|\alpha n|)^{-\frac{1}{3}}$ for each n in this regime. In constructing the boundary layer the term $\alpha U^{\perp}\mathrm{rot}\, v$ plays a crucial role as well as the term $-\alpha(x^{\perp}\cdot\nabla v - v^{\perp})$. In fact, if we drop the term $\alpha U^{\perp}\mathrm{rot}\, v$ as a model problem then the thickness of the boundary layer arising from the rotation term $-\alpha(x^{\perp}\cdot\nabla v - v^{\perp})$ is $|\alpha n|^{-\frac{1}{2}}$, and the leading boundary layer profile is simply described by exponential functions. The term $\alpha U^{\perp}\mathrm{rot}\, v$ leads to a significant change both in the thickness and in the profile of the boundary layer, and we need to introduce the Airy function to describe the profile of the boundary layer associated with the term $-\alpha(x^{\perp}\cdot\nabla v - v^{\perp}) + \alpha U^{\perp}\mathrm{rot}\, v$.

By performing the boundary layer analysis we can improve the result stated in (ii) of Theorem 3.1.1 in the regime $|\alpha| \gg 1$, which is briefly described as follows.

Theorem 3.1.2 *There exists $\gamma > 0$ such that the following statement holds. For all sufficiently large $|\alpha| \ge 1$ and for any external force $f = (\mathcal{P}_0 f, \mathcal{Q}_0 f) \in Y$ satisfying*

$$\|(\mathcal{P}_0 f)_\theta\|_{L^1(\Omega)} \le \gamma|\alpha|^{\frac{1}{3}}\,, \qquad \|\mathcal{Q}_0 f\|_{L^2(\Omega)} \le \gamma|\alpha|^{\frac{1}{3}}\,, \tag{3.19}$$

there exists a unique solution $(v,\nabla q) \in X\cap L^\infty(\Omega)^2 \cap W^{2,1}_{\mathrm{loc}}(\overline{\Omega})^2 \times L^1_{\mathrm{loc}}(\overline{\Omega})^2$ to $(\widetilde{\mathrm{NS}}_\alpha)$ satisfying

$$\|\mathcal{P}_0 v\|_{L^\infty(\Omega)} + \|\nabla\mathcal{P}_0 v\|_{L^\infty(\Omega)} \le C\|(\mathcal{P}_0 f)_\theta\|_{L^1(\Omega)} + \frac{C}{|\alpha|}\|\mathcal{Q}_0 f\|^2_{L^2(\Omega)}\,, \tag{3.20}$$

$$\|\mathcal{Q}_0 v\|_{L^2(\Omega)} \le \frac{C}{|\alpha|^{\frac{2}{3}}}\|\mathcal{Q}_0 f\|_{L^2(\Omega)}\,, \tag{3.21}$$

$$\sum_{|n|\ge 1}\|\mathcal{P}_n v\|_{L^\infty(\Omega)} \le \frac{C\big(\log|\alpha|\big)^{\frac{1}{2}}}{|\alpha|^{\frac{1}{2}}}\|\mathcal{Q}_0 f\|_{L^2(\Omega)}\,, \tag{3.22}$$

$$\|\nabla\mathcal{Q}_0 v\|_{L^2(\Omega)} \le \frac{C}{|\alpha|^{\frac{1}{3}}}\|\mathcal{Q}_0 f\|_{L^2(\Omega)}\,. \tag{3.23}$$

This solution is unique in a suitable subset of X.

By fixing the external force f we can state Theorem 3.1.2 in a different but more convenient way to understand the qualitative behavior of solutions in the fast rotation limit.

Theorem 3.1.3 *For any $f = (\mathcal{P}_0 f, \mathcal{Q}_0 f) \in Y$ there is $\alpha_0 = \alpha_0(\|f\|_Y) \ge 1$ such that the following statements hold. If $|\alpha| \ge \alpha_0$ then there exists a solution $(v^{(\alpha)}, \nabla q^{(\alpha)}) \in X\cap L^\infty(\Omega)^2\cap W^{2,1}_{\mathrm{loc}}(\overline{\Omega})^2 \times L^1_{\mathrm{loc}}(\overline{\Omega})^2$ to $(\widetilde{\mathrm{NS}}_\alpha)$ satisfying*

$$\|v^{(\alpha)} - v_0^{\mathrm{linear}}\|_{L^\infty(\Omega)} \le \frac{C(\log|\alpha|)^{\frac{1}{2}}}{|\alpha|^{\frac{1}{2}}}\,. \tag{3.24}$$

Here $v_0^{\rm linear}$ is the solution to the linearized problem (S_α) *defined in page 72 with f replaced by $\mathcal{P}_0 f$ which is in fact independent of α, and C depends only on $\|f\|_Y$. Moreover, there exists $\kappa > 0$ independent of α and f such that if $1 \le |n| \le \kappa|\alpha|^{\frac12}$ then $v_n^{(\alpha)} = \mathcal{P}_n v^{(\alpha)}$ is written in the form*

$$v_n^{(\alpha)} = v_n^{(\alpha),{\rm slip}} + v_n^{(\alpha),{\rm slow}} + v_{n,{\rm BL}}^{(\alpha)} + \widetilde{v}_n^{(\alpha)}. \tag{3.25}$$

Here $v_n^{(\alpha),{\rm slip}}$ satisfies $v_{n,r}^{(\alpha),{\rm slip}} = {\rm rot}\, v_n^{(\alpha),{\rm slip}} = 0$ on $\partial\Omega$, $v_n^{(\alpha),{\rm slow}}$ is irrotational in Ω, and $v_{n,{\rm BL}}^{(\alpha)}$ possesses a boundary layer structure with the boundary layer thickness $|2\alpha n|^{-\frac13}$. Finally the following estimates hold:

$$\|v_n^{(\alpha),{\rm slip}}\|_{L^2(\Omega)} + \|v_n^{(\alpha),{\rm slow}}\|_{L^2(\Omega)} + \|v_{n,{\rm BL}}^{(\alpha)}\|_{L^2(\Omega)} \le \frac{C}{|\alpha n|^{\frac23}},$$

while $\widetilde{v}_n^{(\alpha)}$ is a remainder which satisfies

$$\|\widetilde{v}_n^{(\alpha)}\|_{L^2(\Omega)} \le \frac{C}{|\alpha n|}.$$

Here the constant C depends only on $\|f\|_Y$.

By going back to (NS_α), Theorems 3.1.2 and 3.1.3 show that there exists a unique solution $u = u^{(\alpha)}$ which satisfies

$$\|u^{(\alpha)} - \alpha U - v_0^{\rm linear}\|_{L^\infty(\Omega)} \le \frac{C(\log|\alpha|)^{\frac12}}{|\alpha|^{\frac12}}, \qquad |\alpha| \gg 1. \tag{3.26}$$

The expansion (3.26) verifies the axisymmetrizing effect (measured in L^∞) due to the fast rotation. The logarithmic factor $(\log|\alpha|)^{\frac12}$ is simply due to the regularity of f, and if f has more regularity such as $\sum_{n\ne0}\|\mathcal{P}_n f\|_{L^2(\Omega)}^s < \infty$ for some $s < 2$ then the factor $(\log|\alpha|)^{\frac12}$ in (3.24) and (3.26) can be dropped. Moreover, the power $|\alpha|^{-\frac12}$ can be also improved by assuming enough regularity of f. For example, if $\mathcal{Q}_0 f \in W_0^{1,2}(\Omega)^2$ in addition, then $|\alpha|^{-\frac12}$ is replaced by $|\alpha|^{-\frac34}$, though we do not go into the detail on this point. The new ingredient of the proof of Theorems 3.1.2 and 3.1.3 is stated in Proposition 3.3.2 and consists in refined estimates for the linearized problem (S_α). The nonlinear problem is handled exactly in the same manner as in the proof of Theorem 3.1.1. For (S_α) we observe that in polar coordinates the angular mode n of the streamfunction satisfies the ODE in $r \in (1,\infty)$

$$\Big(\frac{{\rm d}^2}{{\rm d}r^2} + \frac1r\frac{\rm d}{{\rm d}r} - \frac{n^2}{r^2} + i\alpha n\big(1-\frac{1}{r^2}\big)\Big)\big(\frac{{\rm d}^2}{{\rm d}r^2} + \frac1r\frac{\rm d}{{\rm d}r} - \frac{n^2}{r^2}\big)\psi_n = 0, \tag{3.27}$$

with the boundary condition $\psi_n(1) = \frac{{\rm d}\psi_n}{{\rm d}r}(1) = 0$ when $|n| \ge 1$. The thickness of the boundary layer originating from the fast rotation is determined by the balance between $\frac{{\rm d}^2}{{\rm d}r^2}$ and $i\alpha n(1-\frac{1}{r^2}) \approx 2i\alpha n(r-1)$ near $r=1$ as long as the dissipation $-\frac{n^2}{r^2} \approx -n^2$ is moderate. This implies that the thickness is $|2\alpha n|^{-\frac13}$. Then the regime of n where the dissipation is relatively moderate is estimated from the condition $n^2 \ll \frac{{\rm d}^2}{{\rm d}r^2} \approx O(|\alpha n|^{\frac23})$, which leads to $|n| \ll O(|\alpha|^{\frac12})$. From this observation we employ the boundary layer analysis when the angular frequency n satisfies $1 \le |n| \ll O(|\alpha|^{\frac12})$, while we just apply Proposition 3.3.1

in the regime $|n| \geq O(|\alpha|^{\frac{1}{2}})$ where the boundary layer due to the fast rotation is no longer present. In the regime $1 \leq |n| \ll O(|\alpha|^{\frac{1}{2}})$ we first consider (S_α) with $f = \mathcal{P}_n f$ but under the slip boundary condition $v_{r,n} = \mathrm{rot}\, v_n = 0$ on $\partial\Omega$. The estimate of this *slip* solution is obtained by the energy method for the vorticity equations thanks to the boundary condition $\mathrm{rot}\, v_n = 0$ on $\partial\Omega$. The key point is that the term $U^\perp \mathrm{rot}$ in the velocity equation becomes $U \cdot \nabla$ in the vorticity equation which is antisymmetric because of $\mathrm{div}\, U = 0$, and thus, it is easy to apply the energy method for the vorticity under the slip boundary condition. The *no-slip* solution is then obtained by correcting the boundary condition. To this end we construct the boundary layer solution called the fast mode. The leading profile of the boundary layer is given by a suitable integral of the Airy function. In order to recover the no-slip boundary condition, the fact that $\displaystyle\int_0^\infty \mathrm{Ai}(s)\, \mathrm{d}s \neq 0$ is crucial and it plays the role of a nondegeneracy condition in our construction of the solution. This construction gives a formula as in (3.25) for the solution to (S_α). Compared with Proposition 3.3.1, which is based only on an energy computation for the velocity field, the estimate of the n mode v_n is drastically improved for $1 \leq |n| \ll O(|\alpha|^{\frac{1}{2}})$ thanks to the boundary layer analysis. On the other hand, in the regime $|n| \geq O(|\alpha|^{\frac{1}{2}})$, Proposition 3.3.1 for the *no-slip* solution already gives the same decay estimates as in Proposition 3.3.4 for the *slip* solution, as expected.

This chapter is organized as follows. In Section 3.2 we recall basic facts on operators and vector fields in polar coordinates. In Section 3.3 the linearized problem (S_α) is studied. This section is the core of the chapter. In Subsection 3.3.1 we prove the linear estimates which are valid for all $\alpha \in \mathbb{R} \setminus \{0\}$. These are summarized in Proposition 3.3.1. Subsection 3.3.2 is devoted to the linear analysis for the case $|\alpha| \gg 1$, and the main result of this section is Proposition 3.3.2. The nonlinear problem is discussed in Section 3.4. Some basics on the Airy function and the proof of the interpolation inequality (3.17) are given in the appendix.

3.2 Preliminaries

In this preliminary section we state basic results on some differential operators and vector fields in polar coordinates.

3.2.1 Operators in polar coordinates

The following formulas will be used frequently:

$$\mathrm{div}\, v = \partial_1 v_1 + \partial_2 v_2 = \frac{1}{r}\partial_r(r v_r) + \frac{1}{r}\partial_\theta v_\theta\,, \tag{3.28}$$

$$\mathrm{rot}\, v = \partial_1 v_2 - \partial_2 v_1 = \frac{1}{r}\partial_r(r v_\theta) - \frac{1}{r}\partial_\theta v_r\,, \tag{3.29}$$

$$|\nabla v|^2 = |\partial_r v_r|^2 + |\partial_r v_\theta|^2 + \frac{1}{r^2}\big(|\partial_\theta v_r - v_\theta|^2 + |v_r + \partial_\theta v_\theta|^2\big)\,, \tag{3.30}$$

and

$$\begin{aligned} -\Delta v = {} & \Big(-\partial_r\big(\frac{1}{r}\partial_r(r v_r)\big) - \frac{1}{r^2}\partial_\theta^2 v_r + \frac{2}{r^2}\partial_\theta v_\theta \Big)\mathbf{e}_r \\ & + \Big(-\partial_r\big(\frac{1}{r}\partial_r(r v_\theta)\big) - \frac{1}{r^2}\partial_\theta^2 v_\theta - \frac{2}{r^2}\partial_\theta v_r \Big)\mathbf{e}_\theta\,, \end{aligned} \tag{3.31}$$

$$\mathbf{e}_r \cdot \nabla v \;=\; (\partial_r v_r)\,\mathbf{e}_r \;+\; (\partial_r v_\theta)\,\mathbf{e}_\theta\,,$$
$$\mathbf{e}_\theta \cdot \nabla v \;=\; \frac{\partial_\theta v_r - v_\theta}{r}\,\mathbf{e}_r \;+\; \frac{\partial_\theta v_\theta + v_r}{r}\,\mathbf{e}_\theta\,.$$

In particular, we have

$$\begin{aligned} x^\perp \cdot \nabla v - v^\perp &= |x|\big(\mathbf{e}_\theta \cdot \nabla v\big) - \big(v_r \mathbf{e}_r^\perp + v_\theta \mathbf{e}_\theta^\perp\big) \\ &= (\partial_\theta v_r - v_\theta)\,\mathbf{e}_r \;+\; (\partial_\theta v_\theta + v_r)\,\mathbf{e}_\theta - \big(v_r \mathbf{e}_r^\perp + v_\theta \mathbf{e}_\theta^\perp\big) \\ &= \partial_\theta v_r\,\mathbf{e}_r + \partial_\theta v_\theta\,\mathbf{e}_\theta\,. \end{aligned} \tag{3.32}$$

From (3.30) and the definition of $\mathcal{P}_n$ in (3.2) it follows that for $n \in \mathbb{N} \cup \{0\}$ and for v in $W^{1,2}(\Omega)^2$,

$$\begin{aligned} \|\nabla v\|_{L^2(\Omega)}^2 &= \sum_{n\in\mathbb{Z}} \|\nabla \mathcal{P}_n v\|_{L^2(\Omega)}^2\,, \\ |\nabla \mathcal{P}_n v|^2 &= |\partial_r v_{r,n}|^2 + \frac{1+n^2}{r^2}|v_{r,n}|^2 \\ &\quad + |\partial_r v_{\theta,n}|^2 + \frac{1+n^2}{r^2}|v_{\theta,n}|^2 - \frac{4n}{r^2}\mathrm{Im}(v_{\theta,n}\overline{v_{r,n}})\,. \end{aligned}$$

In particular, we have

$$|\nabla \mathcal{P}_n v|^2 \geq |\partial_r v_{r,n}|^2 + \frac{(|n|-1)^2}{r^2}|v_{r,n}|^2 + |\partial_r v_{\theta,n}|^2 + \frac{(|n|-1)^2}{r^2}|v_{\theta,n}|^2\,, \tag{3.33}$$

and thus, from the definition of $\mathcal{Q}_m$ in (3.3),

$$\|\nabla \mathcal{Q}_m v\|_{L^2(\Omega)}^2 \geq \|\partial_r (\mathcal{Q}_m v)_r\|_{L^2(\Omega)}^2 + \|\partial_r (\mathcal{Q}_m v)_\theta\|_{L^2(\Omega)}^2 + m^2 \|\frac{v}{|x|}\|_{L^2(\Omega)}^2\,.$$

3.2.2 The Biot-Savart law in polar coordinates

For a given scalar field ω in Ω, the streamfunction ψ is formally defined as the solution to the Poisson equation: $-\Delta\psi = \omega$ in Ω, $\psi = 0$ on $\partial\Omega$. For $n \in \mathbb{Z}$ and $\omega \in L^2(\Omega)$ we set

$$\begin{aligned} \mathcal{P}_n \omega &:= \frac{1}{2\pi}\int_0^{2\pi} \omega(r\cos s, r\sin s) e^{-ins}\,\mathrm{d}s\, e^{in\theta}\,, \\ \omega_n &:= \big(\mathcal{P}_n \omega\big) e^{-in\theta}\,. \end{aligned} \tag{3.34}$$

By using the Laplace operator in polar coordinates, the Poisson equation for the Fourier mode n is given by

$$H_n \psi_n \;:=\; -\psi_n^{''} - \frac{1}{r}\psi_n^{'} + \frac{n^2}{r^2}\psi_n = \omega_n\,, \quad r > 1\,, \qquad \psi_n(1) = 0\,. \tag{3.35}$$

Let $|n| \geq 1$. Then the solution $\psi_n = \psi_n[\omega_n]$ to the ordinary differential equation (3.35) decaying at spatial infinity is formally given as

$$\begin{aligned} \psi_n[\omega_n](r) &= \frac{1}{2|n|}\bigg(- \frac{d_n[\omega_n]}{r^{|n|}} + \frac{1}{r^{|n|}}\int_1^r s^{1+|n|}\omega_n(s)\,\mathrm{d}s + r^{|n|}\int_r^\infty s^{1-|n|}\omega_n(s)\,\mathrm{d}s \bigg)\,, \\ d_n[\omega_n] &:= \int_1^\infty s^{1-|n|}\omega_n(s)\,\mathrm{d}s\,. \end{aligned}$$

The Biot-Savart law $V_n[\omega_n]$ is then written as

$$\begin{aligned} V_n[\omega_n] \;&:=\; V_{r,n}[\omega_n]\, e^{in\theta}\, \mathbf{e}_r \;+\; V_{\theta,n}[\omega_n]\, e^{in\theta}\, \mathbf{e}_\theta\,, \\ V_{r,n}[\omega_n] \;&:=\; \frac{in}{r}\psi_n[\omega_n]\,, \qquad V_{\theta,n}[\omega_n] \;:=\; -\frac{\mathrm{d}}{\mathrm{d}r}\psi_n[\omega_n]\,. \end{aligned} \tag{3.36}$$

The velocity $V_n[\omega_n]$ is well defined at least when $r^{1-|n|}\omega_n \in L^1((1,\infty))$, and it is straightforward to see that

$$\begin{aligned} &\operatorname{div} V_n[\omega_n] \;=\; 0\,, \quad \operatorname{rot} V_n[\omega_n] \;=\; \omega_n\, e^{in\theta} \qquad \text{in } \Omega\,, \\ &\mathbf{e}_r \cdot V_n[\omega_n] \;=\; 0 \quad \text{on } \partial\Omega\,. \end{aligned} \tag{3.37}$$

The condition $r^{1-|n|}\omega_n \in L^1((1,\infty))$ is automatically satisfied when $\omega \in L^2(\Omega)$ and $|n| \geq 2$. When $|n| = 1$ the integral in the definition of $\psi_n[\omega_n]$ does not always converge absolutely for general $\omega \in L^2(\Omega)$. However, it is well-defined if $\omega = \operatorname{rot} u$ for some $u \in W^{1,2}(\Omega)^2$, for one can apply the integration by parts that ensures the convergence of $\lim_{N\to\infty} \int_r^N \omega_n \,\mathrm{d}r$ even when $|n| = 1$. As a result, we can check that for any solenoidal vector field v in $L^2_\sigma(\Omega) \cap W^{1,2}(\Omega)^2$, the n mode $v_n = \mathcal{P}_n v$ is expressed in terms of its vorticity ω_n by the formula (3.36) when $|n| \geq 1$.

3.3 Analysis of the linearized system

The linearized system around αU for $(\widetilde{\mathrm{NS}}_\alpha)$ is

$$\begin{cases} -\Delta v - \alpha(x^\perp \cdot \nabla v - v^\perp) + \nabla q + \alpha U^\perp \operatorname{rot} v \;=\; f\,, & x \in \Omega\,, \\ \operatorname{div} v \;=\; 0\,, & x \in \Omega\,, \\ v \;=\; 0\,, & x \in \partial\Omega\,. \end{cases} \tag{S_α}$$

In this section we study (S_α) for $\alpha \in \mathbb{R} \setminus \{0\}$.

3.3.1 General estimate

In this subsection we establish estimates on solutions to (S_α) that are valid for all $\alpha \neq 0$. For convenience we set for scalar functions $g, h : [1,\infty) \to \mathbb{C}$,

$$\langle g, h\rangle_{L^2(\Omega)} := 2\pi \int_1^\infty g(r)\overline{h(r)} r \,\mathrm{d}r\,, \qquad \|g\|^2_{L^2(\Omega)} := 2\pi \int_1^\infty |g(r)|^2 r \,\mathrm{d}r\,.$$

Before going into details let us give a remark on the verification of the energy argument. Let us assume that $f \in L^2(\Omega)^2$ and $v \in W^{1,2}_0(\Omega)^2 \cap W^{2,2}_{\mathrm{loc}}(\overline{\Omega})^2$ is a solution to (S_α) for some $q \in W^{1,2}_{\mathrm{loc}}(\overline{\Omega})$. We have to be careful when applying the energy argument due to the presence of the term $x^\perp \cdot \nabla v$ in the first equation of (S_α), for this term involves a linearly growing coefficient, and therefore it is not clear whether the inner product $\langle x^\perp \cdot \nabla v, v\rangle_{L^2(\Omega)}$ makes sense or not. A similar difficulty appears in taking the inner product $\langle \nabla q, v\rangle_{L^2(\Omega)}$, since we are assuming only $q \in W^{1,2}_{\mathrm{loc}}(\overline{\Omega})$. The most convenient way to overcome this difficulty is to consider the equation for $v_n := \mathcal{P}_n v$, which is identified with $(v_{r,n}, v_{\theta,n})$. Note that $v_{r,n}, v_{\theta,n} \in W^{1,2}_0((1,\infty); r\,\mathrm{d}r) \cap W^{2,2}_{\mathrm{loc}}([1,\infty))$ if $v \in W^{1,2}_0(\Omega)^2 \cap W^{2,2}_{\mathrm{loc}}(\overline{\Omega})^2$.

Let us denote by $\omega_n = \omega_n(r)$ the n mode of the vorticity of v in the polar coordinates, i.e., $\omega_n(r) = (\operatorname{rot}\mathcal{P}_n v)e^{-in\theta}$. Similarly, we set $q_n = q_n(r) = (\mathcal{P}_n q)e^{-in\theta}$, where the projection $\mathcal{P}_n$ for the scalar q is defined as

$$\mathcal{P}_n q := \frac{1}{2\pi}\int_0^{2\pi} q(r\cos s, r\sin s)e^{-ins}\,\mathrm{d}s\, e^{in\theta}\,.$$

Then, from (3.31) and (3.32), $v_{r,n}$ and $v_{\theta,n}$ obey the following equations:

$$-\partial_r\big(\frac{1}{r}\partial_r(rv_{r,n})\big) + \frac{n^2}{r^2}v_{r,n} + \frac{2in}{r^2}v_{\theta,n} - i\alpha n v_{r,n} - \alpha\frac{\omega_n}{r} + \partial_r q_n = f_{r,n}\,, \qquad (3.38)$$

$$-\partial_r\big(\frac{1}{r}\partial_r(rv_{\theta,n})\big) + \frac{n^2}{r^2}v_{\theta,n} - \frac{2in}{r^2}v_{r,n} - i\alpha n v_{\theta,n} + inq_n = f_{\theta,n}\,, \qquad (3.39)$$

together with the divergence free condition $\partial_r(rv_{r,n}) + inv_{\theta,n} = 0$ and the boundary condition $v_{r,n}(1) = v_{\theta,n}(1) = 0$. Then the key observation is that the factor $-i\alpha n$ is regarded as a resolvent parameter, and by setting $\lambda = -i\alpha n$, the above system is equivalent to

$$(\lambda - \Delta)v_n + \nabla\mathcal{P}_n q + \alpha U^{\perp}\operatorname{rot} v_n = f_n\,, \qquad x \in \Omega\,, \qquad (3.40)$$

with $\operatorname{div} v_n = 0$ and $v_n|_{\partial\Omega} = 0$, where $f_n = \mathcal{P}_n f \in \mathcal{P}_n L^2(\Omega)^2$. Indeed, system (3.40) in polar coordinates is exactly (3.38) and (3.39). The key point is that there is no term involving a linearly growing coefficient in (3.40), and therefore we can apply the standard regularity theory of the Stokes resolvent system with a resolvent parameter λ. Let us assume that $n \neq 0$. Then $\lambda \neq 0$ since we are assuming that $\alpha \neq 0$. If $v_n \in L^2_\sigma(\Omega) \cap W^{1,2}_0(\Omega)^2 \cap W^{2,2}_{\mathrm{loc}}(\overline{\Omega})^2$ and $\mathcal{P}_n q \in W^{1,2}_{\mathrm{loc}}(\overline{\Omega})$ is a solution to (3.40), then $(v_n, \mathcal{P}_n q)$ is a weak solution to the Stokes system with source term $f_n - \alpha U^{\perp}\operatorname{rot} v_n$ which clearly belongs to $L^2(\Omega)^2$, and thus, the regularity theory of the Stokes system implies that $v_n \in W^{2,2}(\Omega)^2$ and $\nabla\mathcal{P}_n q \in L^2(\Omega)^2$. In this way we can recover the summability of $\nabla^2 v_n, \nabla\mathcal{P}_n q \in L^2(\Omega)^2$. Then, by going back to the system (3.38) and (3.39), we also find that $q_n \in L^2((1,\infty); r\,\mathrm{d}r)$ from (3.39), for all the other terms in (3.39) belong to $L^2([1,\infty); r\,\mathrm{d}r)$. As a summary, for any solution $v \in L^2_\sigma(\Omega) \cap W^{1,2}_0(\Omega)^2 \cap W^{2,2}_{\mathrm{loc}}(\overline{\Omega})^2$ and $q \in W^{1,2}_{\mathrm{loc}}(\overline{\Omega})$ to (S_α), we can rigorously verify the energy computation for the system (3.38)-(3.39) in each n mode $(v_{r,n}, v_{\theta,n})$ with $n \neq 0$. The estimate for the 0 mode is handled in a different way from the energy method, and is discussed in Subsection 3.3.1 below.

Our main result in this subsection is stated as follows. Let us recall that the projection $\mathcal{Q}_0$ is defined as $\mathcal{Q}_0 := \sum_{n\neq 0}\mathcal{P}_n$.

Proposition 3.3.1 *Let $\alpha \in \mathbb{R}\setminus\{0\}$.*
(i) *For any external force $f_0 \in \mathcal{P}_0 L^1(\Omega)^2$ the system* (S_α) *admits a unique solution $(v_0, \nabla q)$ with $v_0 \in \mathcal{P}_0 L^\infty(\Omega)^2 \cap W^{1,\infty}_0(\Omega)^2$, $\nabla^2 v_0 \in L^1_{\mathrm{loc}}(\overline{\Omega})^{2\times 2}$, $q \in W^{1,1}_{\mathrm{loc}}(\overline{\Omega})$. Moreover, $v_0 = v_{\theta,0}\mathbf{e}_\theta$ and*

$$\|v_0\|_{L^\infty(\Omega)} + \big\||x|\nabla v_0\big\|_{L^\infty(\Omega)} \le C\|f_{\theta,0}\|_{L^1(\Omega)}\,. \qquad (3.41)$$

Here $f_{\theta,0} := f_0\cdot\mathbf{e}_\theta$ and C is independent of α.

(ii) *For any external force* $f \in \mathcal{Q}_0 L^2(\Omega)^2$ *the system* (S_α) *admits a unique solution* $(v, \nabla q)$ *with* $v \in \mathcal{Q}_0 L^2_\sigma(\Omega) \cap W^{1,2}_0(\Omega)^2 \cap W^{2,2}_{\mathrm{loc}}(\overline{\Omega})^2$ *and* $q \in W^{1,2}_{\mathrm{loc}}(\overline{\Omega})$. *Moreover,* $v_n = \mathcal{P}_n v$ *satisfies the following estimates: if* $1 \le |n| < 1 + \sqrt{2|\alpha|}$ *then*

$$\|v_n\|_{L^2(\Omega)} \le \frac{C}{|\alpha|^{\frac12}} \big(\frac{1}{|n|} + \frac{1}{|\alpha|^{\frac12}}\big) \|f_n\|_{L^2(\Omega)}\,, \tag{3.42}$$

$$\big\|\frac{\sqrt{|x|^2-1}}{|x|} v_n\big\|_{L^2(\Omega)} \le \frac{C}{|\alpha n|^{\frac12} |\alpha|^{\frac14}} \big(\frac{1}{|n|} + \frac{1}{|\alpha|^{\frac12}}\big)^{\frac12} \|f_n\|_{L^2(\Omega)}\,, \tag{3.43}$$

$$\|v_n\|_{L^\infty(\Omega)} \le \frac{C}{|\alpha|^{\frac14}} \big(\frac{1}{|n|} + \frac{1}{|\alpha|^{\frac12}}\big) \|f_n\|_{L^2(\Omega)}\,, \tag{3.44}$$

$$\|\nabla v_n\|_{L^2(\Omega)} \le C \big(\frac{1}{|n|} + \frac{1}{|\alpha|^{\frac12}}\big) \|f_n\|_{L^2(\Omega)}\,, \tag{3.45}$$

while if $|n| \ge 1 + \sqrt{2|\alpha|}$ *then*

$$\|v_n\|_{L^2(\Omega)} \le \frac{C}{|n|} \big(\frac{1}{|n|} + \frac{1}{|\alpha|}\big) \|f_n\|_{L^2(\Omega)}\,, \tag{3.46}$$

$$\big\|\frac{\sqrt{|x|^2-1}}{|x|} v_n\big\|_{L^2(\Omega)} \le \frac{C}{|\alpha n|^{\frac12} |n|^{\frac12}} \big(\frac{1}{|n|} + \frac{1}{|\alpha|}\big)^{\frac12} \|f_n\|_{L^2(\Omega)}\,, \tag{3.47}$$

$$\|v_n\|_{L^\infty(\Omega)} \le \frac{C}{|n|^{\frac34}} \big(\frac{1}{|n|} + \frac{1}{|\alpha|}\big)^{\frac34} \|f_n\|_{L^2(\Omega)}\,, \tag{3.48}$$

$$\|\nabla v_n\|_{L^2(\Omega)} \le \frac{C}{|n|^{\frac12}} \big(\frac{1}{|n|} + \frac{1}{|\alpha|}\big)^{\frac12} \|f_n\|_{L^2(\Omega)}\,. \tag{3.49}$$

Finally if $|\alpha| \gg 1$ *and* $|n| = O(|\alpha|^{\frac12})$ *then*

$$\|v_n\|_{L^2(\Omega)} \le \frac{C}{|\alpha|} \|f_n\|_{L^2(\Omega)}\,, \tag{3.50}$$

$$\|v_n\|_{L^\infty(\Omega)} \le \frac{C}{|\alpha|^{\frac34}} \|f_n\|_{L^2(\Omega)}\,, \tag{3.51}$$

$$\|\nabla v_n\|_{L^2(\Omega)} \le \frac{C}{|\alpha|^{\frac12}} \|f_n\|_{L^2(\Omega)}\,. \tag{3.52}$$

Here $f_n := \mathcal{P}_n f$ *and* C *is independent of* n *and* α*.*

Structure and estimate of the 0 mode

Firstly we observe that if v_0 satisfies $v_0 \in \mathcal{P}_0 L^\infty(\Omega)^2 \cap W^{1,\infty}_0(\Omega)^2$ then the divergence-free condition in polar coordinates (3.28) implies that

$$\frac{\mathrm{d}(r v_{r,0})}{\mathrm{d}r} = 0\,,$$

and thus, $v_{r,0} = \dfrac{C}{r}$ with some constant C. Then the no-slip boundary condition leads to $C = 0$, and therefore, $v_{r,0} = 0$. So it suffices to consider the angular part $v_{\theta,0}$. From (3.32) we have

$$x^\perp \cdot \nabla v_0^\perp - v_0^\perp = 0\,,$$

and we also note that the term $U^\perp \mathrm{rot}\, v_0$ in (S$_\alpha$) with $U^\perp = -\dfrac{x}{|x|^2}$ is always written in a gradient form, so can be absorbed in a pressure term.

Collecting these remarks, we see that any solution $(v_0, \nabla q)$ to (S$_\alpha$) with $f_0 \in \mathcal{P}_0 L^1(\Omega)^2$ satisfying $v_0 \in \mathcal{P}_0 L^\infty(\Omega)^2 \cap W_0^{1,\infty}(\Omega)^2$, $\nabla^2 v_0 \in L^1_{\mathrm{loc}}(\overline{\Omega})^{2\times 2}$, $q \in W^{1,1}_{\mathrm{loc}}(\overline{\Omega})$ must be written as $v_0 = v_{0,\theta} \mathbf{e}_\theta$, where $v_{\theta,0} = v_{\theta,0}(r)$ obeys from (3.39) the ODE

$$-\frac{\mathrm{d}^2 v_{\theta,0}}{\mathrm{d}r^2} - \frac{1}{r}\frac{\mathrm{d}v_{\theta,0}}{\mathrm{d}r} + \frac{v_{\theta,0}}{r^2} = f_{\theta,0}\,, \quad r > 1\,, \qquad v_{\theta,0}(1) = 0\,. \tag{3.53}$$

The bounded solution to (3.53) is written as

$$v_{\theta,0}(r) = \frac{1}{2}\Big(-\frac{1}{r}\int_1^\infty f_{\theta,0}\,\mathrm{d}s + \frac{1}{r}\int_1^r s^2 f_{\theta,0}\,\mathrm{d}s + r\int_r^\infty f_{\theta,0}\,\mathrm{d}s\Big)\,. \tag{3.54}$$

We note that

$$\|f_{\theta,0}\|_{L^1(\Omega)} = 2\pi \int_1^\infty |f_{\theta,0}|\, s\,\mathrm{d}s\,.$$

Thus we see from (3.54) that

$$\|v_{\theta,0}\|_{L^\infty(\Omega)} + \|r\frac{\mathrm{d}v_{\theta,0}}{\mathrm{d}r}\|_{L^\infty(\Omega)} \le C\|f_{\theta,0}\|_{L^1(\Omega)}\,,$$

which implies (3.41).

A priori estimate of the n mode with $|n| \ge 1$

Let v denote a solution to (S$_\alpha$) satisfying $v \in \mathcal{Q}L^2_\sigma(\Omega) \cap W_0^{1,2}(\Omega)^2 \cap W^{2,2}_{\mathrm{loc}}(\overline{\Omega})^2$ for $f \in \mathcal{Q}_0 L^2(\Omega)^2$ with some $q \in W^{1,2}_{\mathrm{loc}}(\overline{\Omega})$. Then, as we have already seen in the beginning of Section 3.3.1, for each $n \neq 0$, the n mode $v_n = \mathcal{P}_n v$ belongs in addition to $W^{2,2}(\Omega)^2$ and we also have that $\mathcal{P}_n q$ belongs to $W^{1,2}(\Omega)$. Hence the energy computation below for $(v_{r,n}, v_{\theta,n})$ to the system (3.38)–(3.39) is rigorously verified. With this important remark in mind we multiply both sides of (3.38) by $r\bar{v}_{r,n}$ and of (3.39) by $r\bar{v}_{\theta,n}$ and integrate over $[1,\infty)$, which results in the following identities:

$$\|\nabla v_n\|^2_{L^2(\Omega)} = -\alpha \mathrm{Re}\langle U^\perp \mathrm{rot}\, v_n\,, v_n\rangle_{L^2(\Omega)} + \mathrm{Re}\langle f_n\,, v_n\rangle_{L^2(\Omega)}\,, \tag{3.55}$$

$$-\alpha n\big(\|v_{r,n}\|^2_{L^2(\Omega)} + \|v_{\theta,n}\|^2_{L^2(\Omega)}\big) + \alpha \mathrm{Im}\langle U^\perp \mathrm{rot}\, v_n\,, v_n\rangle_{L^2(\Omega)} = \mathrm{Im}\langle f_n\,, v_n\rangle_{L^2(\Omega)}\,. \tag{3.56}$$

Note that $U^\perp(x) = -\dfrac{x}{|x|^2}$ thus we see from $\mathrm{rot}\, v_n = -\Delta\psi_n$ by definition of the stream-function ψ_n for the n mode with $n \neq 0$,

$$\begin{aligned}
\mathrm{Re}\langle U^\perp \mathrm{rot}\, v_n\,, v_n\rangle_{L^2(\Omega)} &= \mathrm{Re}\langle \frac{1}{|x|}\Delta\psi_n\,, v_{r,n}\rangle_{L^2(\Omega)} \\
&= 2\pi n\,\mathrm{Im}\int_1^\infty (\partial_r^2\psi_n + \frac{1}{r}\partial_r\psi_n - \frac{n^2}{r^2}\psi_n)\overline{\psi_n}\frac{\mathrm{d}r}{r} \\
&= 4\pi n\,\mathrm{Im}\int_1^\infty \frac{1}{r}\partial_r\psi_n\,\overline{\psi_n}\frac{\mathrm{d}r}{r}\,.
\end{aligned}$$

This gives the bound

$$\Big|\mathrm{Re}\langle U^\perp \mathrm{rot}\, v_n\,, v_n\rangle_{L^2(\Omega)}\Big| \le 2\big\|\frac{v_{r,n}}{r}\big\|_{L^2(\Omega)}\big\|\frac{v_{\theta,n}}{r}\big\|_{L^2(\Omega)}\,. \tag{3.57}$$

Therefore, (3.55) and (3.57) with the lower bound (3.33) imply

$$\begin{aligned}\|\partial_r v_{r,n}\|^2_{L^2(\Omega)} + \|\partial_r v_{\theta,n}\|^2_{L^2(\Omega)} + \big((|n|-1)^2 - |\alpha|\big)\big(\|\frac{v_{r,n}}{r}\|^2_{L^2(\Omega)} + \|\frac{v_{\theta,n}}{r}\|^2_{L^2(\Omega)}\big)\\ \le |\mathrm{Re}\langle f_n\,, v_n\rangle_{L^2(\Omega)}|\,.\end{aligned} \tag{3.58}$$

Next we study identity (3.56). We see that

$$\begin{aligned}\mathrm{Im}\langle U^\perp \mathrm{rot}\, v_n\,, v_n\rangle_{L^2(\Omega)} &= \mathrm{Im}\langle \frac{1}{|x|}\Delta\psi_n\,, v_{r,n}\rangle_{L^2(\Omega)}\\ &= -2\pi n\,\mathrm{Re}\int_1^\infty (\partial_r^2\psi_n + \frac{1}{r}\partial_r\psi_n - \frac{n^2}{r^2}\psi_n)\overline{\psi_n}\frac{\mathrm{d}r}{r}\,.\end{aligned}$$

Integrations by parts yield

$$\begin{aligned}&\mathrm{Re}\int_1^\infty (\partial_r^2\psi_n + \frac{1}{r}\partial_r\psi_n - \frac{n^2}{r^2}\psi_n)\overline{\psi_n}\frac{\mathrm{d}r}{r}\\ &= -\int_1^\infty |\partial_r\psi_n|^2\frac{\mathrm{d}r}{r} + 2\mathrm{Re}\int_1^\infty \frac{1}{r}\partial_r\psi_n\overline{\psi_n}\frac{\mathrm{d}r}{r} - \int_1^\infty \frac{n^2|\psi_n|^2}{r^2}\frac{\mathrm{d}r}{r}\\ &= -\int_1^\infty |\partial_r\psi_n|^2\frac{\mathrm{d}r}{r} - (n^2-2)\int_1^\infty \frac{|\psi_n|^2}{r^2}\frac{\mathrm{d}r}{r}\,.\end{aligned}$$

Hence,

$$\mathrm{Im}\langle U^\perp \mathrm{rot}\, v_n\,, v_n\rangle_{L^2(\Omega)} = n\big(\|\frac{v_{\theta,n}}{r}\|^2_{L^2(\Omega)} + (1-\frac{2}{n^2})\|\frac{v_{r,n}}{r}\|^2_{L^2(\Omega)}\big)\,. \tag{3.59}$$

Hence, (3.56) and (3.59) give

$$\begin{aligned}\alpha n\bigg(\|v_{r,n}\|^2_{L^2(\Omega)} - (1-\frac{2}{n^2})\|\frac{v_{r,n}}{r}\|^2_{L^2(\Omega)} + \|v_{\theta,n}\|^2_{L^2(\Omega)} - \|\frac{v_{\theta,n}}{r}\|^2_{L^2(\Omega)}\bigg)\\ = -\mathrm{Im}\langle f_n\,, v_n\rangle_{L^2(\Omega)}\,,\end{aligned} \tag{3.60}$$

which in particular leads to

$$\big\|\frac{\sqrt{r^2-1}}{r}v_{r,n}\big\|^2_{L^2(\Omega)} + \big\|\frac{\sqrt{r^2-1}}{r}v_{\theta,n}\big\|^2_{L^2(\Omega)} \le \frac{1}{|\alpha n|}\big|\mathrm{Im}\langle f_n\,, v_n\rangle_{L^2(\Omega)}\big|\,, \tag{3.61}$$

$$\big\|\frac{v_{r,n}}{r}\big\|^2_{L^2(\Omega)} \le \big|\frac{n}{2\alpha}\big|\,\big|\mathrm{Im}\langle f_n\,, v_n\rangle_{L^2(\Omega)}\big|\,. \tag{3.62}$$

Then, from (3.60) and (3.62) we gather

$$\begin{aligned}\|v_{r,n}\|^2_{L^2(\Omega)} &\le \|\frac{v_{r,n}}{r}\|^2_{L^2(\Omega)} + \frac{1}{|\alpha n|}\big|\mathrm{Im}\langle f_n\,, v_n\rangle_{L^2(\Omega)}\big|\\ &\le \big(\big|\frac{n}{2\alpha}\big| + \frac{1}{|\alpha n|}\big)\big|\mathrm{Im}\langle f_n\,, v_n\rangle_{L^2(\Omega)}\big|\\ &\le \big|\frac{3n}{2\alpha}\big|\,\big|\mathrm{Im}\langle f_n\,, v_n\rangle_{L^2(\Omega)}\big|\,, \qquad |n|\ge 3\,,\end{aligned} \tag{3.63}$$

while for $|n| = 1, 2$, a slightly finer estimate is available from (3.60):

$$\|v_{r,n}\|_{L^2(\Omega)}^2 \le \frac{1}{|\alpha n|}\big|\mathrm{Im}\langle f_n\,, v_n\rangle_{L^2(\Omega)}\big|\,, \qquad |n| = 1, 2\,. \tag{3.64}$$

To obtain the estimate of $\|v_n\|_{L^2(\Omega)}$ we first observe that the following interpolation inequality holds for any scalar function $g \in W^{1,2}((1,\infty); r\,\mathrm{d}r)$:

$$\|g\|_{L^2(\Omega)} \le C\|\partial_r g\|_{L^2(\Omega)}^{\frac{1}{3}}\|\frac{\sqrt{r^2-1}}{r}g\|_{L^2(\Omega)}^{\frac{2}{3}} + C\|\frac{\sqrt{r^2-1}}{r}g\|_{L^2(\Omega)}\,. \tag{3.65}$$

See Appendix 3.5.2 for the proof of (3.65). From (3.65) and (3.61) we have

$$\begin{aligned}
|\alpha|\big(\|v_{r,n}\|_{L^2(\Omega)}^2 + \|v_{\theta,n}\|_{L^2(\Omega)}^2\big) &\le C|\alpha|^{\frac{3}{2}}\big(\|\frac{\sqrt{r^2-1}}{r}v_{r,n}\|_{L^2(\Omega)}^2 + \|\frac{\sqrt{r^2-1}}{r}v_{\theta,n}\|_{L^2(\Omega)}^2\big) \\
&\quad + \frac{1}{4}\big(\|\partial_r v_{r,n}\|_{L^2(\Omega)}^2 + \|\partial_r v_{\theta,n}\|_{L^2(\Omega)}^2\big) \\
&\quad + C|\alpha|\big(\|\frac{\sqrt{r^2-1}}{r}v_{r,n}\|_{L^2(\Omega)}^2 + \|\frac{\sqrt{r^2-1}}{r}v_{\theta,n}\|_{L^2(\Omega)}^2\big) \\
&\le C(\frac{|\alpha|^{\frac{1}{2}}}{|n|} + \frac{1}{|n|})\|f_n\|_{L^2(\Omega)}\|v_n\|_{L^2(\Omega)} \\
&\quad + \frac{1}{4}\big(\|\partial_r v_{r,n}\|_{L^2(\Omega)}^2 + \|\partial_r v_{\theta,n}\|_{L^2(\Omega)}^2\big)\,,
\end{aligned}$$

and thus,

$$\begin{aligned}
&|\alpha|\big(\|v_{r,n}\|_{L^2(\Omega)}^2 + \|v_{\theta,n}\|_{L^2(\Omega)}^2\big) \\
&\le \frac{C}{n^2}(1 + \frac{1}{|\alpha|})\|f_n\|_{L^2(\Omega)}^2 + \frac{1}{2}\big(\|\partial_r v_{r,n}\|_{L^2(\Omega)}^2 + \|\partial_r v_{\theta,n}\|_{L^2(\Omega)}^2\big)\,.
\end{aligned} \tag{3.66}$$

Hence (3.58) and (3.66) imply

$$\|\partial_r v_{r,n}\|_{L^2(\Omega)}^2 + \|\partial_r v_{\theta,n}\|_{L^2(\Omega)}^2 \le \frac{C}{n^2}(1 + \frac{1}{|\alpha|})\|f_n\|_{L^2(\Omega)}^2 + 2|\mathrm{Re}\langle f_n\,, v_n\rangle_{L^2(\Omega)}|\,. \tag{3.67}$$

Then (3.66) and (3.67) yield

$$\|v_{r,n}\|_{L^2(\Omega)}^2 + \|v_{\theta,n}\|_{L^2(\Omega)}^2 \le \frac{C}{|\alpha| n^2}(1 + \frac{1}{|\alpha|})\|f_n\|_{L^2(\Omega)}^2 + \frac{1}{|\alpha|}\|f_n\|_{L^2(\Omega)}\|v_n\|_{L^2(\Omega)}\,,$$

that is,

$$\|v_n\|_{L^2(\Omega)}^2 \le C\big(\frac{1}{|\alpha| n^2} + \frac{1}{|\alpha|^2}\big)\|f_n\|_{L^2(\Omega)}^2\,. \tag{3.68}$$

This proves (3.42) and (3.50). Note that the factor $\frac{1}{|\alpha| n^2}$ dominates $\frac{1}{\alpha^2}$ in the regime $|n| \le O(|\alpha|^{\frac{1}{2}})$ and $|\alpha| \ge 1$. Once we have proved (3.68) the following estimates are immediately obtained from (3.61) and (3.67):

$$\|\frac{\sqrt{r^2-1}}{r}v_{r,n}\|_{L^2(\Omega)}^2 + \|\frac{\sqrt{r^2-1}}{r}v_{\theta,n}\|_{L^2(\Omega)}^2 \le \frac{C}{|\alpha n|}\big(\frac{1}{|\alpha|^{\frac{1}{2}}|n|} + \frac{1}{|\alpha|}\big)\|f_n\|_{L^2(\Omega)}^2\,, \tag{3.69}$$

$$\|\partial_r v_{r,n}\|_{L^2(\Omega)}^2 + \|\partial_r v_{\theta,n}\|_{L^2(\Omega)}^2 \le C\big(\frac{1}{n^2} + \frac{1}{|\alpha|}\big)\|f_n\|_{L^2(\Omega)}^2\,. \tag{3.70}$$

The constant C in (3.68), (3.69), and (3.70) is independent of $|n| \geq 1$ and α. Inequality in (3.69) proves (3.43). Moreover, (3.68) and (3.58) prove (3.45) and (3.52) since

$$|\alpha| \| \frac{v_n}{|x|} \|_{L^2(\Omega)}^2 \leq |\alpha| \|v_n\|_{L^2(\Omega)}^2 \,.$$

Finally (3.44) and (3.51) are obtained from the interpolation inequality for functions in the space $W_0^{1,2}((1,\infty);\, \mathrm{d}r)$.

A priori estimate of the n mode with $|n| \gg O(\sqrt{1+|\alpha|})$

If $|n| \geq 1 + \sqrt{2|\alpha|}$ then (3.58) yields

$$\begin{aligned} \|\partial_r v_{r,n}\|_{L^2(\Omega)}^2 + \|\partial_r v_{\theta,n}\|_{L^2(\Omega)}^2 + \frac{n^2}{8} \big(\| \frac{v_{r,n}}{r} \|_{L^2(\Omega)}^2 + \| \frac{v_{\theta,n}}{r} \|_{L^2(\Omega)}^2 \big) \\ \leq \big| \mathrm{Re} \langle f_n \,, v_n \rangle_{L^2(\Omega)} \big| \,. \end{aligned} \tag{3.71}$$

Then (3.60) and (3.71) give

$$\begin{aligned} \|v_n\|_{L^2(\Omega)}^2 &\leq \| \frac{v_n}{|x|} \|_{L^2(\Omega)}^2 + \frac{1}{|\alpha n|} \|f_n\|_{L^2(\Omega)} \|v_n\|_{L^2(\Omega)} \\ &\leq \big(\frac{8}{n^2} + \frac{1}{|\alpha n|} \big) \|f_n\|_{L^2(\Omega)} \|v_n\|_{L^2(\Omega)} \,, \end{aligned}$$

which shows

$$\|v_n\|_{L^2(\Omega)}^2 \leq \big(\frac{8}{n^2} + \frac{1}{|\alpha n|} \big)^2 \|f_n\|_{L^2(\Omega)}^2 \,. \tag{3.72}$$

Note that (3.72) is better than (3.68) in the regime $|n| \gg 1 + \sqrt{2|\alpha|}$, while both are of the same order in the regime $|n| = O(\sqrt{1+|\alpha|})$. The estimates (3.71) and (3.72) yield

$$\|\nabla v_n\|_{L^2(\Omega)}^2 \leq C \big(\frac{1}{n^2} + \frac{1}{|\alpha n|} \big) \|f_n\|_{L^2(\Omega)}^2 \,, \tag{3.73}$$

while (3.61) and (3.72) lead to

$$\| \frac{\sqrt{r^2-1}}{r} v_{r,n} \|_{L^2(\Omega)}^2 + \| \frac{\sqrt{r^2-1}}{r} v_{\theta,n} \|_{L^2(\Omega)}^2 \leq \frac{1}{|\alpha n|} \big(\frac{8}{n^2} + \frac{1}{|\alpha n|} \big) \|f_n\|_{L^2(\Omega)}^2 \,. \tag{3.74}$$

Again, (3.73) and (3.74) are better than (3.70) and (3.69) in the regime $|n| \gg 1 + \sqrt{2|\alpha|}$. The estimates (3.72), (3.73), and (3.74) show (3.46), (3.49), and (3.47). Then (3.48) follows by interpolation using (3.46) and (3.49)

Proof of Proposition 3.3.1

The statement (i) of Proposition 3.3.1 is proved in Subsection 3.3.1. In particular, we have that $v_0 = v_{\theta,0} \mathrm{e}_\theta$ and $v_{\theta,0}$ is given by (3.54). It remains to prove (ii) of Proposition 3.3.1.

(Estimates and Uniqueness) We have already proved the a priori estimates of solutions in Subsections 3.3.1 and 3.3.1, which give (3.42)–(3.49). The uniqueness of solutions directly follows from these a priori estimates.

(Existence) By considering the Helmholtz-Leray projection which is bounded in L^2 and invariant under the action of $\mathcal{P}_n$, we may assume that f_n belongs to $\mathcal{Q}_0 L^2_\sigma(\Omega)$, rather than $\mathcal{Q}_0 L^2(\Omega)^2$. To show the existence of solutions we consider the operator

$$\mathbb{A}_\alpha := \mathbb{P}\Delta - \alpha \mathbb{P} U^\perp \mathrm{rot}$$

in the L^2 framework, where $\mathbb{P} : L^2(\Omega)^2 \to L^2_\sigma(\Omega)$ is the Helmholtz-Leray projection and Δ is the Dirichlet Laplacian in $L^2(\Omega)$. The operator $\mathbb{P}\Delta$ is thus the standard Stokes operator in $L^2_\sigma(\Omega)$. Let us consider the operator $\mathbb{A}_\alpha$ in the invariant space $\mathcal{P}_n L^2_\sigma(\Omega)$, $n \neq 0$. Note that the spectrum of the Stokes operator $\mathbb{P}\Delta$ in $\mathcal{P}_n L^2_\sigma(\Omega)$ is included in the half real line $\overline{\mathbb{R}_-} = \{\lambda \leq 0\}$, while the operator $\mathbb{P}U^\perp \mathrm{rot}$ is relatively compact with respect to $\mathbb{P}\Delta$ in $\mathcal{P}_n L^2_\sigma(\Omega)$, for $U^\perp$ is smooth and decays at infinity. Thus, the difference between the spectrum of $\mathbb{A}_\alpha$ and the one of $\mathbb{P}\Delta$ consists only of discrete eigenvalues with finite multiplicities. Then the scalar $\lambda := -i\alpha n$ must belong to the resolvent set of $\mathbb{A}_\alpha$ in $\mathcal{P}_n L^2_\sigma(\Omega)$, otherwise $-i\alpha n$ is an eigenvalue of $\mathbb{A}_\alpha$ in $\mathcal{P}_n L^2_\sigma(\Omega)$ but this cannot be true due to the a priori estimates on solutions of (3.40) with $\lambda = -i\alpha n$ which we have shown above. Hence, when $\lambda = -i\alpha n$, for any $f_n = \mathcal{P}_n f \in \mathcal{P}_n L^2_\sigma(\Omega)$ there exists a unique solution v_n to (3.40) belonging to the space $\mathcal{P}_n L^2_\sigma(\Omega) \cap W^{1,2}_0(\Omega)^2 \cap W^{2,2}(\Omega)^2$ with a suitable pressure field belonging to $W^{1,2}_{\mathrm{loc}}(\overline{\Omega})$. Then $v = \sum_{n\neq 0} v_n$ belongs to $\mathcal{Q}_0 L^2_\sigma(\Omega) \cap W^{1,2}_0(\Omega)^2 \cap W^{2,2}(\Omega)^2$ and solves (S_α) by construction, for (3.40) with $\lambda = -i\alpha n$ is equivalent to (3.38) and (3.39) for each $n \neq 0$. The proof of (ii) of Proposition 3.3.1 is complete. $\square$

3.3.2 Analysis in the fast rotation case $|\alpha| \gg 1$

In this subsection we focus on the behavior of solutions to (S_α) in the case $|\alpha| \gg 1$. Let us define the parameter

$$\beta := (-2i\alpha n)^{\frac{1}{3}} = \begin{cases} (2|\alpha n|)^{\frac{1}{3}} c_- & \text{if} \quad \alpha n > 0\,, \\ (2|\alpha n|)^{\frac{1}{3}} c_+ & \text{if} \quad \alpha n < 0\,, \end{cases} \tag{3.75}$$

where

$$c_\pm := \frac{\sqrt{3} \pm i}{2}\,. \tag{3.76}$$

Our goal is to prove the following structure result on solutions to (S_α).

Proposition 3.3.2 *There is a constant $\kappa \in (0,1)$ such that as long as $1 \leq |n| \leq \kappa |\alpha|^{\frac{1}{2}}$ and $|\beta|$ large enough (implying $|\alpha|$ large enough), the n mode of the velocity field v_n solving (S_α) with $f = f_n \in \mathcal{P}_n L^2(\Omega)^2$ may be decomposed into four parts*

$$v_n = v_n^{\mathrm{slip}} + v_n^{\mathrm{slow}} + v_{n,\mathrm{BL}} + \widetilde{v}_n\,,$$

where

- *the term v_n^{slip} satisfies the system (mS_α) and the estimates* (3.83)-(3.85) *of Proposition* 3.3.4.
- *the stream function of v_n^{slow} is given by*

$$\psi_n^{\mathrm{slow}}(r) = a_n r^{-|n|}\,,$$

where

$$|a_n| \leq C|\alpha n|^{-\frac{5}{6}} \|f_n\|_{L^2}\,.$$

- *the boundary layer term $v_{n,\mathrm{BL}}$ is given by*

$$v_{n,\mathrm{BL},r}(r) = \frac{inb_n}{r} G_{n,\alpha}\big(|\beta|(r-1)\big)\,, \quad v_{n,\mathrm{BL},\theta} = -|\beta| b_n G'_{n,\alpha}\big(|\beta|(r-1)\big)$$

with $G_{n,\alpha}$ a smooth function, decaying exponentially at infinity, uniformly in n and α, and where

$$|b_n| \le C|\alpha n|^{-\frac{5}{6}} \|f_n\|_{L^2}\,.$$

- *the term $\widetilde{v}_n$ is a remainder term in the sense that*

$$\begin{aligned} \|\widetilde{v}_n\|_{L^2(\Omega)} \le C|\alpha n|^{-1}\|f_n\|_{L^2}\,, \quad \|\widetilde{v}_n\|_{L^\infty(\Omega)} \le C|\alpha n|^{-\frac{5}{6}}\|f_n\|_{L^2}\,, \\ \text{and} \quad \|\nabla \widetilde{v}_n\|_{L^2(\Omega)} \le C|\alpha n|^{-\frac{2}{3}}\|f_n\|_{L^2}\,. \end{aligned} \tag{3.77}$$

In particular, the following estimates hold for v_n:

$$\begin{aligned} \|v_n\|_{L^2(\Omega)} \le C|\alpha n|^{-\frac{2}{3}}\|f_n\|_{L^2}\,, \quad \|v_n\|_{L^\infty(\Omega)} \le C|\alpha n|^{-\frac{1}{2}}\|f_n\|_{L^2}\,, \\ \text{and} \quad \|\nabla v_n\|_{L^2(\Omega)} \le C|\alpha n|^{-\frac{1}{3}}\|f_n\|_{L^2}\,. \end{aligned} \tag{3.78}$$

The constant C is independent of n, α, and f_n.

Remark 3.3.3 (1) The velocity fields v_n^{slip}, v_n^{slow} and $v_{n,\mathrm{BL}}$ have the same decay order in the spaces L^2, L^∞ and H^1, although their structure is different. Contrary to the other velocity fields, the term $v_{n,\mathrm{BL}}$ is negligible away from the boundary due to its rapid decay. The key point is that the boundary layer analysis enables us to improve the decay order in the low and middle frequencies $|n| \ll O(|\alpha|^{\frac{1}{2}})$, compared with the results of Proposition 3.3.1 which are based only on energy computations. Indeed, it is not clear whether (3.78) can be shown only from energy computations without using the boundary layer analysis.
(2) The vorticity of v_n^{slow} vanishes in Ω, that is, v_n^{slow} is irrotational in Ω. The function $G_{n,\alpha}$ and its derivatives of finite order are uniformly bounded in n and α which satisfy $|\alpha| \ge 1$ and $1 \le |n| \le |\alpha|^{\frac{1}{2}}$. The uniform decay estimate of $G_{n,\alpha}(\rho)$ for $\rho \gg 1$ is stated in (3.117) below. The smallness of κ in Proposition 3.3.2 is required only in obtaining some lower bound for the quantity $\Big||\beta| G'_{n,\alpha}(0) + |n| G_{n,\alpha}(0)\Big|$ which is essential to construct the solution satisfying the no-slip boundary condition and possessing the structure stated in Proposition 3.3.2.

Proof of Proposition 3.3.2: We first consider a linearized problem similar to (S_α) but with a different boundary condition, such that the vorticity vanishes on the boundary:

$$\begin{cases} -\Delta v - \alpha(x^\perp \cdot \nabla v - v^\perp) + \nabla q + \alpha U^\perp \mathrm{rot}\, v = f\,, & x \in \Omega\,, \\ \mathrm{div}\, v = 0\,, & x \in \Omega\,, \\ v_r = \mathrm{rot}\, v = 0\,, & x \in \partial\Omega\,. \end{cases} \tag{mS_α}$$

Note that the vorticity field ω then satisfies the following equations

$$\begin{cases} -\Delta\omega - \alpha x^\perp \cdot \nabla\omega + \alpha U \cdot \nabla\omega = \mathrm{rot}\, f\,, \quad x \in \Omega\,, \\ \omega|_{\partial\Omega} = 0\,. \end{cases} \tag{3.79}$$

Let us prove the following proposition.

Proposition 3.3.4 *For any $\alpha \in \mathbb{R}$ with $|\alpha| \geq 1$ and external force $f \in \mathcal{Q}_0 L^2(\Omega)^2$ the system* (mS_α) *admits a unique solution $(v, \nabla q)$ with $v \in \mathcal{Q}_0 L^2_\sigma(\Omega) \cap W^{1,2}(\Omega)^2 \cap W^{2,2}_{\mathrm{loc}}(\overline{\Omega})$ and $q \in W^{1,2}_{\mathrm{loc}}(\overline{\Omega})$. Moreover, $\omega = \mathrm{rot}\, v \in \mathcal{Q}_0 W^{1,2}_0(\Omega)$ satisfies for $n \neq 0$,*

$$\|\mathcal{P}_n \omega\|_{L^2(\Omega)} \leq \frac{C}{|\alpha n|^{\frac{1}{3}}} \|f_n\|_{L^2(\Omega)}\,, \tag{3.80}$$

$$\|\frac{\sqrt{|x|^2-1}}{|x|} \mathcal{P}_n \omega\|_{L^2(\Omega)} \leq \frac{1}{|\alpha n|^{\frac{1}{2}}} \|f_n\|_{L^2(\Omega)}\,, \tag{3.81}$$

$$\|\nabla \mathcal{P}_n \omega\|_{L^2(\Omega)} \leq \|f_n\|_{L^2(\Omega)}\,, \tag{3.82}$$

while $v_n = \mathcal{P}_n v$ satisfies for $1 \leq |n| \leq 2|\alpha|^{\frac{1}{2}}$,

$$\|v_n\|_{L^2(\Omega)} \leq \frac{C}{|\alpha n|^{\frac{2}{3}}} \|f_n\|_{L^2(\Omega)}\,, \tag{3.83}$$

$$\|v_n\|_{L^\infty(\Omega)} \leq \frac{C}{|\alpha n|^{\frac{1}{2}}} \|f_n\|_{L^2(\Omega)}\,, \tag{3.84}$$

$$\|\frac{\sqrt{|x|^2-1}}{|x|} v_n\|_{L^2(\Omega)} \leq \frac{C}{|\alpha n|^{\frac{5}{6}}} \|f_n\|_{L^2(\Omega)}\,. \tag{3.85}$$

Here $f_n := \mathcal{P}_n f$ and C is independent of n and α.

Remark 3.3.5 Note that

$$\|\nabla v_n\|_{L^2(\Omega)} \leq C\big(\|\mathcal{P}_n \omega\|_{L^2(\Omega)} + \|\frac{v_n}{|x|}\|_{L^2(\Omega)}\big) \leq \frac{C}{|\alpha n|^{\frac{1}{3}}} \|f_n\|_{L^2(\Omega)}\,. \tag{3.86}$$

The decay order in each estimate of (3.86) and (3.83)–(3.85) is better than the order in (3.42)–(3.45) for the solution subject to the no-slip boundary condition. This faster decay is shown by an energy estimate thanks to the slip condition in (mS_α), which reduces the magnitude of the boundary layer arising from the fast rotation in low and middle frequencies.

Proof of Proposition 3.3.4: For simplicity we set $\omega_n = \omega_n(r) := (\mathcal{P}_n \omega) e^{-in\theta}$.
(A priori estimates) We first show the a priori estimates stated in (3.80)–(3.85). As in the proof of Proposition 3.3.1, for each n mode the energy computation for v_n based on the integration by parts is verified for any solution $(v, \nabla q)$ to (mS_α) such that $v \in \mathcal{Q}_0 L^2_\sigma(\Omega) \cap W^{1,2}(\Omega)^2 \cap W^{2,2}_{\mathrm{loc}}(\overline{\Omega})$ and $q \in W^{1,2}_{\mathrm{loc}}(\overline{\Omega})$ (see the argument at the beginning of Subsection 3.3.1). We also note that the n mode of the vorticity $\omega_n \in W^{1,2}_0((1,\infty); r\,\mathrm{d}r)$ satisfies the following ordinary differential equations on $(1,\infty)$ in the weak sense:

$$-\Big(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{n^2}{r^2} + i\alpha n\big(1 - \frac{1}{r^2}\big)\Big)\omega_n = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r f_{\theta,n}) - \frac{in}{r} f_{r,n}\,, \tag{3.87}$$

with the Dirichlet boundary condition $\omega_n(1) = 0$. We can multiply both sides by $r\bar{\omega}_n$ and integrate over $[1,\infty)$, which gives the following identities:

$$\int_1^\infty \big(|\partial_r \omega_n|^2 + n^2 \frac{|\omega_n|^2}{r^2}\big) r\,\mathrm{d}r = -\mathrm{Re}\Big(\int_1^\infty f_{\theta,n} \frac{\mathrm{d}\bar{\omega}_n}{\mathrm{d}r} r\,\mathrm{d}r + in\int_1^\infty f_{r,n}\bar{\omega}_n\,\mathrm{d}r\Big)\,, \tag{3.88}$$

$$\alpha n \int_1^\infty (1 - \frac{1}{r^2})|\omega_n|^2 r\,\mathrm{d}r = \mathrm{Im}\Big(\int_1^\infty f_{\theta,n} \frac{\mathrm{d}\bar{\omega}_n}{\mathrm{d}r} r\,\mathrm{d}r + in\int_1^\infty f_{r,n}\bar{\omega}_n\,\mathrm{d}r\Big)\,. \tag{3.89}$$

The first identity (3.88) gives the bound

$$\|\nabla \mathcal{P}_n\omega\|_{L^2(\Omega)} \le \|f_n\|_{L^2(\Omega)}\,,$$

and then the second identity (3.89) yields

$$\Big\|\frac{\sqrt{|x|^2-1}}{|x|}\mathcal{P}_n\omega\Big\|^2_{L^2(\Omega)} \le \frac{1}{|\alpha n|}\|f_n\|_{L^2(\Omega)}\|\nabla\mathcal{P}_n\omega\|_{L^2(\Omega)} \le \frac{1}{|\alpha n|}\|f_n\|^2_{L^2(\Omega)}\,.$$

Thus (3.81)–(3.82) hold. The estimate (3.80) follows from (3.81)–(3.82) by the interpolation inequality (3.65) and $|\alpha| \ge 1$. The estimates on the velocity v_n are obtained from the estimates for the streamfunction ψ_n which is a solution to the equation (3.35). Let us first estimate $\dfrac{\mathrm{d}^2\psi_n}{\mathrm{d}r^2}$ and $\dfrac{\psi_n}{r^2}\cdot$ We write $\psi_n' = \dfrac{\mathrm{d}\psi_n}{\mathrm{d}r}$ and $\psi_n'' = \dfrac{\mathrm{d}^2\psi_n}{\mathrm{d}r^2}$, and then observe that

$$\int_1^\infty \frac{1}{r}\psi_n'\bar{\psi}_n'' r\,\mathrm{d}r = \int_1^\infty \Big(\frac{\psi_n}{r}\Big)'\bar{\psi}_n'' r\,\mathrm{d}r + \int_1^\infty \frac{\psi_n}{r^2}\bar{\psi}_n'' r\,\mathrm{d}r\,.$$

A similar computation yields

$$-\mathrm{Re}\int_1^\infty \frac{\psi_n}{r^2}\bar{\psi}_n'' r\,\mathrm{d}r = \mathrm{Re}\int_1^\infty \Big(\frac{\psi_n}{r}\Big)'\bar{\psi}_n'\,\mathrm{d}r = \int_1^\infty \Big|\Big(\frac{\psi_n}{r}\Big)'\Big|^2 r\,\mathrm{d}r$$

and

$$\mathrm{Re}\int_1^\infty \frac{1}{r}\psi_n'\frac{\bar{\psi}_n}{r^2} r\,\mathrm{d}r = \int_1^\infty \Big|\frac{\psi_n}{r^2}\Big|^2 r\,\mathrm{d}r\,.$$

Hence we have from (3.35), by multiplying both sides by $r\bar{\psi}_n''$ and by integrating over $[1,\infty)$,

$$\int_1^\infty |\psi_n''|^2 r\,\mathrm{d}r + \mathrm{Re}\int_1^\infty \Big(\frac{\psi_n}{r}\Big)'\bar{\psi}_n'' r\,\mathrm{d}r + (n^2-1)\int_1^\infty \Big|\Big(\frac{\psi_n}{r}\Big)'\Big|^2 r\,\mathrm{d}r = -\mathrm{Re}\int_1^\infty \omega_n\bar{\psi}_n'' r\,\mathrm{d}r\,,$$

and similarly, by multiplying by $\frac{\bar{\psi}_n}{r}$ in (3.35) and integrating over $[1,\infty)$,

$$\int_1^\infty \Big|\Big(\frac{\psi_n}{r}\Big)'\Big|^2 r\,\mathrm{d}r + (n^2-1)\int_1^\infty \Big|\frac{\psi_n}{r^2}\Big|^2 r\,\mathrm{d}r = \mathrm{Re}\int_1^\infty \omega_n\frac{\bar{\psi}_n}{r^2} r\,\mathrm{d}r\,.$$

Combining these two identities gives the following bounds: when $|n| \ge 2$,

$$\|\psi_n''\|_{L^2(\Omega)} + \big\|(\frac{n\psi_n}{r})'\big\|_{L^2(\Omega)} + n^2\big\|\frac{\psi_n}{r^2}\big\|_{L^2(\Omega)} \le C\|\omega_n\|_{L^2(\Omega)}\,, \tag{3.90}$$

while when $|n| = 1$,

$$\begin{aligned} \|(\frac{\psi_n}{r})'\|^2_{L^2(\Omega)} &\le \|\omega_n\|_{L^2(\Omega)}\big\|\frac{\psi_n}{r^2}\big\|_{L^2(\Omega)}\,,\\ \|\psi_n''\|^2_{L^2(\Omega)} &\le C\|\omega_n\|_{L^2(\Omega)}\big(\|\omega_n\|_{L^2(\Omega)} + \big\|\frac{\psi_n}{r^2}\big\|_{L^2(\Omega)}\big)\,, \end{aligned} \tag{3.91}$$

where C is independent of n. Next we observe that the identity (3.60) holds even under the slip boundary condition, and thus, (3.61) and (3.62) are valid also for solutions to (mS_α).

Hence, by recalling the relation $v_{r,n} = \frac{in}{r}\psi_n$ and using the interpolation inequality (3.65), we obtain for $|n| \geq 2$, thanks to (3.90) and (3.61),

$$\begin{aligned}
\|v_{r,n}\|_{L^2(\Omega)} &\leq C\|\partial_r v_{r,n}\|_{L^2(\Omega)}^{\frac{1}{3}} \big\|\frac{\sqrt{r^2-1}}{r} v_{r,n}\big\|_{L^2(\Omega)}^{\frac{2}{3}} + C\big\|\frac{\sqrt{r^2-1}}{r} v_{r,n}\big\|_{L^2(\Omega)} \\
&\leq C\|\omega_n\|_{L^2(\Omega)}^{\frac{1}{3}} \big(\frac{1}{|\alpha n|}\|f_n\|_{L^2(\Omega)}\|v_n\|_{L^2(\Omega)}\big)^{\frac{1}{3}} + C\big(\frac{1}{|\alpha n|}\|f_n\|_{L^2(\Omega)}\|v_n\|_{L^2(\Omega)}\big)^{\frac{1}{2}},
\end{aligned}$$

while for $|n| = 1$ we use (3.91) and also (3.62), which give

$$\begin{aligned}
\|v_{r,n}\|_{L^2(\Omega)} &\leq C\|\partial_r v_{r,n}\|_{L^2(\Omega)}^{\frac{1}{3}} \big\|\frac{\sqrt{r^2-1}}{r} v_{r,n}\big\|_{L^2(\Omega)}^{\frac{2}{3}} + C\big\|\frac{\sqrt{r^2-1}}{r} v_{r,n}\big\|_{L^2(\Omega)} \\
&\leq C\|\omega_n\|_{L^2(\Omega)}^{\frac{1}{6}} \big(\frac{1}{|\alpha|}\|f_n\|_{L^2(\Omega)}\|v_n\|_{L^2(\Omega)}\big)^{\frac{5}{12}} + C\big(\frac{1}{|\alpha|}\|f_n\|_{L^2(\Omega)}\|v_n\|_{L^2(\Omega)}\big)^{\frac{1}{2}}.
\end{aligned}$$

Similarly, we have from $v_{\theta,n} = -\psi_n'$, (3.65), (3.90), and (3.61), for $|n| \geq 2$,

$$\|v_{\theta,n}\|_{L^2(\Omega)} \leq C\|\omega_n\|_{L^2(\Omega)}^{\frac{1}{3}} \big(\frac{1}{|\alpha n|}\|f_n\|_{L^2(\Omega)}\|v_n\|_{L^2(\Omega)}\big)^{\frac{1}{3}} + C\big(\frac{1}{|\alpha n|}\|f_n\|_{L^2(\Omega)}\|v_n\|_{L^2(\Omega)}\big)^{\frac{1}{2}},$$

on the other hand, for $|n| = 1$, by applying (3.65), (3.91), (3.61) and (3.62),

$$\begin{aligned}
\|v_{\theta,n}\|_{L^2(\Omega)} &\leq C\|\omega_n\|_{L^2(\Omega)}^{\frac{1}{3}} \big(\frac{1}{|\alpha|}\|f_n\|_{L^2(\Omega)}\|v_n\|_{L^2(\Omega)}\big)^{\frac{1}{3}} \\
&\quad + C\|\omega_n\|_{L^2(\Omega)}^{\frac{1}{6}} \big(\frac{1}{|\alpha|}\|f_n\|_{L^2(\Omega)}\|v_n\|_{L^2(\Omega)}\big)^{\frac{5}{12}} + C\big(\frac{1}{|\alpha|}\|f_n\|_{L^2(\Omega)}\|v_n\|_{L^2(\Omega)}\big)^{\frac{1}{2}}.
\end{aligned}$$

Hence we have from (3.80) for the estimate of $\|\omega_n\|_{L^2(\Omega)}$ and $|\alpha| \geq 1$,

$$\|v_n\|_{L^2(\Omega)} \leq \frac{C}{|\alpha n|^{\frac{2}{3}}}\|f_n\|_{L^2(\Omega)}, \qquad |n| \geq 1. \tag{3.92}$$

The estimate (3.85) follows from (3.61) and (3.92). As for the L^∞ estimate in the case $|n| \geq 2$, we have from the interpolation inequality and (3.90),

$$\begin{aligned}
\|v_{r,n}\|_{L^\infty(\Omega)}^2 \leq C\|\partial_r v_{r,n}\|_{L^2(\Omega)}\|v_{r,n}\|_{L^2(\Omega)} &\leq C\|\omega_n\|_{L^2(\Omega)}\|v_{r,n}\|_{L^2(\Omega)} \\
&\leq \frac{C}{|\alpha n|}\|f_n\|_{L^2(\Omega)}^2,
\end{aligned}$$

and similarly,

$$\|v_{\theta,n}\|_{L^\infty(\Omega)}^2 \leq \frac{C}{|\alpha n|}\|f_n\|_{L^2(\Omega)}^2.$$

The case $|n| = 1$ is handled in the same manner, and in this case we use (3.91) instead of (3.90) to estimate $\|\partial_r v_{r,n}\|_{L^2(\Omega)}$ and $\|\partial_r v_{\theta,n}\|_{L^2(\Omega)}$. This modification does not produce any change in the final estimate

$$\|v_n\|_{L^\infty(\Omega)} \leq \frac{C}{|\alpha|^{\frac{1}{2}}}\|f_n\|_{L^2(\Omega)}.$$

The details are omitted here. Thus (3.80)–(3.85) hold.

(Existence and uniqueness) The uniqueness follows from the a priori estimates. The existence is shown by the same way as in the proof of Proposition 3.3.1 in Subsection 3.3.1 (the proof for the statement (ii)), so we omit the details here. The proof is complete. □

Let us return to the proof of Proposition 3.3.2. We denote by $v^{\rm slip}$ the solution to $({\rm mS}_\alpha)$ with the external force f in Proposition 3.3.4. Note that $v^{\rm slip}$ satisfies the desired estimate (3.78), while $v^{\rm slip}$ is not necessarily subject to the no-slip boundary condition. Hence, starting from the perfect-slip solution $v^{\rm slip}$, our next task is to recover the no-slip boundary condition by the boundary layer analysis. To this end we consider the following equations related to the stream function:

$$\begin{cases} \Big(\dfrac{{\rm d}^2}{{\rm d}r^2}+\dfrac{1}{r}\dfrac{\rm d}{{\rm d}r}-\dfrac{n^2}{r^2}+i\alpha n\big(1-\dfrac{1}{r^2}\big)\Big)\big(\dfrac{{\rm d}^2}{{\rm d}r^2}+\dfrac{1}{r}\dfrac{\rm d}{{\rm d}r}-\dfrac{n^2}{r^2}\big)\varphi_n=0\,, \quad r>1\,, \\ |\varphi_n(1)|\leq 1\,, \qquad \lim_{r\to\infty}\varphi_n=0\,. \end{cases} \tag{3.93}$$

Indeed, the first equation in (3.93) is nothing but the equation for the n mode of the stream function in our problem (in polar coordinates), while here we do not need to impose the exact boundary value on the boundary $r=1$, and the key point is that there exists a solution to (3.93) which possess a boundary layer structure due to $|\alpha n|\gg 1$. We shall decompose it into a boundary layer part and a remainder:

$$\varphi_n=\varphi_{n,{\rm BL}}+\widetilde{\varphi}_n\,, \tag{3.94}$$

where $\varphi_{n,{\rm BL}}$ is a function on the boundary layer variable $|\beta|(r-1)$ and $\widetilde{\varphi}_n$ vanishes at $r=1$ and is smaller than $\varphi_{n,{\rm BL}}$ up to its derivatives. The precise construction of $\varphi_{n,{\rm BL}}$ and $\widetilde{\varphi}_n$ will be stated later.

Once φ_n is constructed as in (3.94), the n mode of the no-slip solution v_n to (3.93), or equivalently its stream function ψ_n, is obtained by the following argument. Noticing that (3.93) has an explicit solution given by $r\mapsto r^{-|n|}$ (corresponding to a vorticity-free solution), we look for the stream function ψ_n under the following form:

$$\psi_n(r)=a_n r^{-|n|}+b_n\varphi_n(r)+\psi_n^{\rm slip}(r) \tag{3.95}$$

where $\psi_n^{\rm slip}$ is the n mode of the streamfunction for $v_n^{\rm slip}$, namely,

$$\psi_n^{\rm slip}=\frac{r}{in}v_{r,n}^{\rm slip}\,, \qquad n\neq 0\,, \tag{3.96}$$

and the coefficients a_n and b_n are determined from the prescription that ψ_n and ψ_n' vanish at the boundary $r=1$:

$$\begin{aligned} a_n+b_n\varphi_n(1)&=0 \\ -|n|a_n+b_n\frac{{\rm d}\varphi_n}{{\rm d}r}(1)&=-\frac{{\rm d}\psi_n^{\rm slip}}{{\rm d}r}(1)\,\big(=v_{\theta,n}^{\rm slip}(1)\big)\,. \end{aligned}$$

In other words there holds

$$b_n\Big(\frac{{\rm d}\varphi_n}{{\rm d}r}(1)+|n|\varphi_n(1)\Big)=v_{\theta,n}^{\rm slip}(1)\,, \qquad a_n=-b_n\varphi_n(1)\,. \tag{3.97}$$

The key point is that for $|n|\leq\kappa|\alpha|^{\frac12}$ for some small enough κ, the term $\dfrac{{\rm d}\varphi_n}{{\rm d}r}(1)+|n|\varphi_n(1)$ will be shown to be nonzero and actually large, due to a specific boundary layer structure of φ_n. Combining (3.94) with (3.95) gives the formula

$$\psi_n(r)=\psi_n^{\rm slip}(r)+a_n r^{-|n|}+b_n\varphi_{n,{\rm BL}}(r)+b_n\widetilde{\varphi}_n(r)\,.$$

Thus, with the notation of Proposition 3.3.2, the remainder velocity $\widetilde{v}_n$ is given in terms of the stream function $\widetilde{\psi}_n$ defined as

$$\widetilde{\psi}_n(r) := b_n \widetilde{\varphi}_n(r)\,.$$

We now focus on the construction of φ_n and its associate velocity field. To estimate a possible boundary layer thickness we observe from the first equation of (3.93) that there is a natural scale balance between $-\frac{\mathrm{d}^2}{\mathrm{d}r^2}$ and $-i\alpha n(1-\frac{1}{r^2}) \approx -2i\alpha n(r-1)$ near the boundary $r=1$, which formally implies that the thickness of the boundary layer is $|2\alpha n|^{-\frac{1}{3}} = |\beta|^{-1}$ where β is defined in (3.75). Before stating the result leading to the construction of the boundary layer term, let us recall the notation introduced in (3.35):

$$H_n := -\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{n^2}{r^2}\,, \tag{3.98}$$

and let us denote

$$A_n := H_n - i\alpha n\Big(1-\frac{1}{r^2}\Big) \tag{3.99}$$

so that the first equation in (3.93) translates into $A_n H_n \varphi_n = 0$. The next proposition is the construction of $\varphi_{n,\mathrm{BL}}$ in (3.94), which describes the leading part of the boundary layer.

Proposition 3.3.6 *There exist $\kappa \in (0,1)$ and $C>0$ such that the following statement holds. If $|n| \le \kappa|\alpha|^{\frac{1}{2}}$ and if $|\beta| = (2|\alpha n|)^{\frac{1}{3}}$ is large enough, then there exist smooth functions $\varphi_{n,\mathrm{BL}}$ and $g_{n,\mathrm{BL}}$ on $[1,\infty)$ such that*

$$A_n H_n \varphi_{n,\mathrm{BL}} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r g_{n,\mathrm{BL}})\,, \quad \textit{with} \quad \|g_{n,\mathrm{BL}}\|_{L^2(\Omega)} \le C|\beta|^{\frac{3}{2}}\,, |\varphi_{n,\mathrm{BL}}(1)| \le 1\,, \tag{3.100}$$

and

- *there holds*

$$\Big|\frac{\mathrm{d}\varphi_{n,\mathrm{BL}}}{\mathrm{d}r}(1) + |n|\varphi_{n,\mathrm{BL}}(1)\Big| \ge \frac{\kappa}{2}|\beta|\,, \tag{3.101}$$

- *there is a smooth function $G_{n,\alpha}$ decaying exponentially at infinity uniformly in n and α such that*

$$\varphi_{n,\mathrm{BL}}(r) = G_{n,\alpha}\big(|\beta|(r-1)\big)\,, \qquad r \ge 1\,. \tag{3.102}$$

More precisely, $G_{n,\alpha}$ in (3.102) satisfies the estimate (3.117) stated below. Let us postpone the proof of Proposition 3.3.6 and conclude the proof of Proposition 3.3.2. We construct a couple $(\varphi_{n,\mathrm{BL}}, g_{n,\mathrm{BL}})$ as in Proposition 3.3.6, which produces a boundary layer vector field

$$v_{r,n,\mathrm{BL}}(r) := \frac{inb_n}{r} G_{n,\alpha}\big(|\beta|(r-1)\big)\,, \quad v_{\theta,n,\mathrm{BL}}(r) := -|\beta| b_n G'_{n,\alpha}\big(|\beta|(r-1)\big)\,. \tag{3.103}$$

Next we fix $\varphi_n(1) = \varphi_{n,\mathrm{BL}}(1)$, so that the remainder $\widetilde{\varphi}_n$ as in (3.94) vanishes at the boundary. Moreover, from the requirement $A_n H_n \varphi_n = 0$ and thanks to (3.100), $\widetilde{\varphi}_n$ is obtained as the solution to

$$A_n H_n \widetilde{\varphi}_n = -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r g_{n,\mathrm{BL}}) \qquad r > 1\,, \qquad (H_n\widetilde{\varphi}_n)(1) = \widetilde{\varphi}_n(1) = 0\,.$$

More precisely, the construction of $\widetilde{\varphi}_n$ is as follows: we first construct $\widetilde{\omega}_n(r)e^{in\theta}$ as the solution to (3.79) with right-hand side $-\mathrm{rot}\,\big(g_{n,\mathrm{BL}}e^{in\theta}\mathbf{e}_\theta\big)$. Then $\widetilde{\varphi}_n$ is obtained as the solution to $H_n\widetilde{\varphi}_n = \widetilde{\omega}_n$ under the boundary condition $\widetilde{\varphi}_n(1) = 0$. Let us denote by $\widetilde{w}_n$ the velocity field whose stream function is $\widetilde{\varphi}_n$. Then the estimates of $\widetilde{w}_n$ and $\widetilde{\omega}_n$ follow from Proposition 3.3.4 and the fact that $\|g_{n,\mathrm{BL}}\|_{L^2(\Omega)} \le C|\beta|^{\frac{3}{2}}$ as stated in (3.100). In particular, (3.84) in Proposition 3.3.4 produces from $|\beta| = |2\alpha n|^{\frac{1}{3}}$ that

$$\Big|\frac{\mathrm{d}\widetilde{\varphi}_n}{\mathrm{d}r}(1)\Big| \le \|\widetilde{w}_{\theta,n}\|_{L^\infty(\Omega)} \lesssim |\beta|^{-\frac{3}{2}}\|g_{n,\mathrm{BL}}\|_{L^2(\Omega)} \le C\,.$$

Together with (3.94) and (3.101), we find that

$$\begin{aligned}\Big|\,\frac{\mathrm{d}\varphi_n}{\mathrm{d}r}(1) + |n|\varphi_n(1)\Big| &\ge \Big|\,\frac{\mathrm{d}\varphi_{n,\mathrm{BL}}}{\mathrm{d}r}(1) + |n|\varphi_{n,\mathrm{BL}}(1)\Big| - \Big|\,\frac{\mathrm{d}\widetilde{\varphi}_n}{\mathrm{d}r}(1)\Big|\\ &\ge \frac{\kappa}{2}|\beta| - C\\ &\ge \frac{\kappa}{4}|\beta|\end{aligned}$$

for $|\beta| \gg \kappa^{-1}$. Let us estimate the coefficients a_n and b_n, which are defined in (3.97). Since

$$|v^{\mathrm{slip}}_{\theta,n}(1)| \le \|v^{\mathrm{slip}}_{\theta,n}\|_{L^\infty(\Omega)} \lesssim |\alpha n|^{-\frac{1}{2}}\|f_n\|_{L^2(\Omega)}$$

thanks to Proposition 3.3.4 we infer that the parameter b_n satisfies

$$|b_n| = \Big|\frac{v^{\mathrm{slip}}_{\theta,n}(1)}{\frac{\mathrm{d}\varphi_n}{\mathrm{d}r}(1) + |n|\varphi_n(1)}\Big| \lesssim |\alpha n|^{-\frac{5}{6}}\|f_n\|_{L^2(\Omega)}$$

while since $|\varphi_n(1)| \le 1$,

$$|a_n| = |b_n\varphi_n(1)| \lesssim |\alpha n|^{-\frac{5}{6}}\|f_n\|_{L^2(\Omega)}\,.$$

Thus, the estimates of the velocity $v_{n,\mathrm{BL}}$ easily follow from its definition in (3.103). The estimates of the remainder velocity $\widetilde{v}_n$ follow from Proposition 3.3.4 and the estimates of b_n since $\widetilde{\psi}_n = b_n\widetilde{\varphi}_n$ that implies $\widetilde{v}_n = b_n\widetilde{w}_n$. This concludes the proof of Proposition 3.3.2. □

Proof of Proposition 3.3.6: Without loss of generality we assume from now on that $\alpha > 0$. As already mentioned, we formally estimate the thickness of the boundary layer to be of the order $|\beta| = |2\alpha n|^{\frac{1}{3}}$. One important remark here is the size of $\frac{n^2}{r^2}$ in the operators H_n and A_n defined in (3.98) and (3.99). Recall that we are interested in the regime $|n| \le O(\alpha^{\frac{1}{2}})$. If $|n| = O(\alpha^{\frac{1}{2}})$ then we observe that $|\beta| = |2\alpha n|^{\frac{1}{3}} = O((\alpha^{\frac{3}{2}})^{\frac{1}{3}}) = O(|n|)$, and thus, the term $\frac{n^2}{r^2}$ has the same size near the boundary as ∂_r^2 and $\alpha n(r-1)$. Hence, in the construction of the boundary layer we also need to take into account the term $\frac{n^2}{r^2}$, for this term is no longer small in the regime $|n| = O(\alpha^{\frac{1}{2}})$. With this remark in mind let us rewrite H_n and A_n in more convenient forms: we define

$$\widetilde{H}_n = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + n^2 \quad \text{and} \quad \widetilde{A}_n = \widetilde{H}_n - 2i\alpha n(r-1) = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \beta^3\big(r - 1 + \frac{in}{2\alpha}\big)\,,$$

so that

$$H_n = \widetilde{H}_n - \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + n^2\big(\frac{1}{r^2} - 1\big)$$

and by using $1-\frac{1}{r^2}=2(r-1)+(r-1)^2\frac{1+2r}{r^2}$,

$$\begin{aligned}A_n &= H_n - 2i\alpha n(r-1) - i\alpha n(r-1)^2\frac{1+2r}{r^2}\\ &= \widetilde{H}_n - 2i\alpha n(r-1) - \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + n^2(\frac{1}{r^2}-1) - i\alpha n(r-1)^2\frac{1+2r}{r^2}\\ &= \widetilde{A}_n - \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + n^2(\frac{1}{r^2}-1) - 2i\alpha n(r-1)^2\frac{1+2r}{r^2}\,.\end{aligned}$$

That is, $\widetilde{H}_n$ and $\widetilde{A}_n$ have the leading size of H_n and A_n, respectively, for the boundary layer functions. Then we write A_nH_n as

$$A_nH_n = \widetilde{A}_n\widetilde{H}_n + \big(A_n-\widetilde{A}_n\big)H_n + \widetilde{A}_n\big(H_n-\widetilde{H}_n\big)\,, \tag{3.104}$$

and we claim that

$$\big(A_n-\widetilde{A}_n\big)H_n + \widetilde{A}_n\big(H_n-\widetilde{H}_n\big) = \mathrm{rot}_n\, R_n \tag{3.105}$$

with a suitable operator R_n, where rot_n is defined in polar coordinates with the n mode for the angular variable. Note that the leading operator $\widetilde{A}_n\widetilde{H}_n$, when applied to a boundary layer function of the form $h(|\beta|(r-1))$, formally has size $O(|\beta|^4)$, the term $\big(A_n-\widetilde{A}_n\big)H_n+\widetilde{A}_n\big(H_n-\widetilde{H}_n\big)$ has size $O(|\beta|^3)$, and then R_n is of size $O(|\beta|^2)$.

Let us look for the boundary layer $\varphi_{n,\mathrm{BL}}$ as a solution to

$$\widetilde{A}_n\widetilde{H}_n\varphi_{n,\mathrm{BL}} = 0\,. \tag{3.106}$$

Since $\widetilde{H}_n$ is easily inverted for $|n|\geq 1$, we start by considering the homogeneous problem $\widetilde{A}_n\phi=0$. By its very definition the operator $\widetilde{A}_n$ is nothing but the Airy operator with a complex coefficient. Hence we introduce the Airy function $\mathrm{Ai}(z)$ which is a solution to

$$\frac{\mathrm{d}^2\mathrm{Ai}}{\mathrm{d}z^2} - z\mathrm{Ai} = 0$$

in $\mathbb{C}$; for details, see Appendix 3.5.1. Then we define

$$\widetilde{G}_{n,\alpha}(\rho) := \mathrm{Ai}\big(c_-(\rho+\frac{in|\beta|}{2\alpha})\big)\,, \qquad \rho>0\,, \tag{3.107}$$

which satisfies from $c_-^3=-i$,

$$\Big(\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + i(\rho+\frac{in}{2\alpha}|\beta|)\Big)\widetilde{G}_{n,\alpha} = 0\,, \qquad \rho>0\,. \tag{3.108}$$

Next we set

$$G_{n,\alpha}(\rho) := -\int_\rho^\infty e^{-\frac{|n|}{|\beta|}(\rho-\tau)}\int_\tau^\infty e^{-\frac{|n|}{|\beta|}(\sigma-\tau)}\widetilde{G}_{n,\alpha}(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau\,, \tag{3.109}$$

which satisfies

$$-\frac{\mathrm{d}^2G_{n,\alpha}}{\mathrm{d}\rho^2} + \frac{n^2}{|\beta|^2}G_{n,\alpha} = \widetilde{G}_{n,\alpha}\,, \qquad \rho>0\,. \tag{3.110}$$

Finally we define

$$C_{0,n,\alpha} = \begin{cases} \dfrac{1}{G_{n,\alpha}(0)} & \text{if} \quad |G_{n,\alpha}(0)| \geq 1\,, \\ 1 & \text{otherwise} \end{cases} \tag{3.111}$$

and we set

$$\varphi_{n,\mathrm{BL}}(r) := C_{0,n,\alpha} G_{n,\alpha}\big(|\beta|(r-1)\big)\,, \tag{3.112}$$

which satisfies from $-i|\beta|^3 = \beta^3$,

$$\widetilde{A}_n \widetilde{H}_n \varphi_{n,\mathrm{BL}} = 0\,, \qquad r > 1\,,$$

as desired. Notice also that

$$|\varphi_{n,\mathrm{BL}}(1)| \leq 1\,.$$

The key quantity is

$$\begin{aligned} \frac{\mathrm{d}\varphi_{n,\mathrm{BL}}}{\mathrm{d}r}|_{r=1} &= C_{0,n,\alpha}|\beta| \, \frac{\mathrm{d}G_{n,\alpha}}{\mathrm{d}\rho}|_{\rho=0} \\ &= C_{0,n,\alpha}|\beta| \Big(\int_0^\infty e^{-\frac{|n|}{|\beta|}\sigma} \widetilde{G}_{n,\alpha}(\sigma)\,\mathrm{d}\sigma - \frac{|n|}{|\beta|} G_{n,\alpha}(0) \Big) \\ &= C_{0,n,\alpha}|\beta| \Big(\int_0^\infty e^{-\frac{|n|}{|\beta|}\sigma} \mathrm{Ai}\big(c_-(\sigma + \frac{in|\beta|}{2\alpha})\big)\,\mathrm{d}\sigma - \frac{|n|}{|\beta|} G_{n,\alpha}(0) \Big) \\ &= C_{0,n,\alpha}|\beta| \Big(\frac{1}{c_-} \int_0^\infty e^{-\lambda s} \mathrm{Ai}(s + \frac{in|\beta|c_-}{2\alpha})\,\mathrm{d}s - \frac{|n|}{|\beta|} G_{n,\alpha}(0) \Big)\,, \end{aligned}$$

with

$$\lambda = \lambda_{n,\beta} := \frac{|n|}{|\beta| c_-} = \frac{|n| c_+}{|\beta|}\,.$$

We find from $\beta^3 = |\beta|^3 c_-^3$ that

$$\frac{in|\beta|c_-}{2\alpha} = \frac{n^2|\beta|c_-}{-2i\alpha n} = \frac{n^2|\beta|c_-}{\beta^3} = \frac{n^2|\beta|}{|\beta|^3 c_-^2} = \frac{n^2 c_+^2}{|\beta|^2} = \lambda^2 \tag{3.113}$$

so

$$\frac{\mathrm{d}\varphi_{n,\mathrm{BL}}}{\mathrm{d}r}|_{r=1} = C_{0,n,\alpha}|\beta| \Big(\frac{1}{c_-} \int_0^\infty e^{-\lambda s} \mathrm{Ai}(s + \lambda^2)\,\mathrm{d}s - \frac{|n|}{|\beta|} G_{n,\alpha}(0) \Big)\,. \tag{3.114}$$

Note that $|\lambda|^2 = 4^{-1}|n|^{\frac{4}{3}}\alpha^{-\frac{2}{3}}$ is small when $|n| \ll \alpha^{\frac{1}{2}}$. The proof of the following lemma is postponed to Appendix 3.5.1.

Lemma 3.3.7 *There holds*

$$\widetilde{C}_0 := \inf \Big\{ \, |C_{0,n,\alpha}| \;\; | \;\; \alpha \geq 1\,,\; 1 \leq |n| \leq \alpha^{\frac{1}{2}} \Big\} > 0\,, \tag{3.115}$$

and there is a constant $\varepsilon \in (0,1)$ *such that defining*

$$\Sigma_\varepsilon := \Big\{ \mu \in \mathbb{C} \,|\, \arg\, \mu = \frac{\pi}{6}\,,\, 0 \leq |\mu| \leq \varepsilon \Big\}\,,$$

then

$$\widetilde{\kappa}_\varepsilon := \inf_{\mu\in\Sigma_\varepsilon}\Big|\int_0^\infty e^{-\mu s}\mathrm{Ai}(s+\mu^2)\,\mathrm{d}s\Big| > 0\,. \tag{3.116}$$

Moreover, the function $G_{n,\alpha}$ defined in (3.109) *satisfies for R large enough*

$$\sup_{\alpha\geq 1}\sup_{1\leq|n|\leq\alpha^{1/2}}\sup_{\rho\geq R} e^\rho\,|\frac{\mathrm{d}^k G_{n,\alpha}}{\mathrm{d}\rho^k}(\rho)| < \infty\,, \qquad k=0,1,2,3\,. \tag{3.117}$$

Now let us return to the proof of Proposition 3.3.6. We define $\varphi_{n,\mathrm{BL}}$ as in (3.112). Note that thanks to (3.114), (3.116) and (3.115) there holds as long as $\frac{|n|}{|\beta|} = |\lambda| \leq \varepsilon$,

$$\begin{aligned}|\frac{\mathrm{d}\varphi_{n,\mathrm{BL}}}{\mathrm{d}r}(1)| &\geq \widetilde{C}_0|\beta|\Big(\widetilde{\kappa}_\varepsilon - \frac{|n|}{|\beta|}|G_{n,\alpha}(0)|\Big)\\ &\geq \widetilde{C}_0|\beta|\big(\widetilde{\kappa}_\varepsilon - \frac{\varepsilon}{\widetilde{C}_0}\big)\,.\end{aligned}$$

Here we have also used (3.111). Hence

$$|\frac{\mathrm{d}\varphi_{n,\mathrm{BL}}}{\mathrm{d}r}(1)| \geq \frac{\widetilde{C}_0\widetilde{\kappa}_\varepsilon}{2}|\beta| \tag{3.118}$$

as long as $2\varepsilon \leq \widetilde{C}_0\widetilde{\kappa}_\varepsilon$, which is possible since $\widetilde{\kappa}_\varepsilon$ is nonincreasing in $\varepsilon > 0$. This proves (3.101) since

$$\begin{aligned}\Big|\frac{\mathrm{d}\varphi_{n,\mathrm{BL}}}{\mathrm{d}r}(1) + |n|\varphi_{n,\mathrm{BL}}(1)\Big| &\geq \Big|\frac{\mathrm{d}\varphi_{n,\mathrm{BL}}}{\mathrm{d}r}(1)\Big| - |n||\varphi_{n,\mathrm{BL}}(1)|\\ &\geq \frac{\widetilde{C}_0\widetilde{\kappa}_\varepsilon}{2}|\beta| - |n|\\ &\geq \frac{\breve{\kappa}}{2}|\beta|\end{aligned}$$

with $\breve{\kappa} := \widetilde{C}_0\widetilde{\kappa}_\varepsilon/2$, and the last inequality holds as long as $|n| \leq \min(\varepsilon, \breve{\kappa}/2)|\beta|$. It then suffices to choose $\kappa \leq \min(\sqrt{2}\varepsilon^{\frac{3}{2}}, \breve{\kappa}^{\frac{3}{2}}/2)$ which ensures that

$$|n| \leq \kappa\alpha^{\frac{1}{2}} \Rightarrow |n| \leq \min(\varepsilon, \frac{\breve{\kappa}}{2})\,|\beta|\,.$$

The result (3.102) is an obvious consequence of the previous construction so it remains to prove that (3.100) is satisfied for a suitable $g_{n,\mathrm{BL}}$. Let us recall (3.104). We define $g_{n,\mathrm{BL}}$ as

$$g_{n,\mathrm{BL}}(r) := -\frac{1}{r}\int_r^\infty s\Big(\widetilde{A}_n\big(H_n - \widetilde{H}_n\big) + \big(A_n - \widetilde{A}_n\big)H_n\Big)\varphi_{n,\mathrm{BL}}\,\mathrm{d}s\,, \tag{3.119}$$

which then satisfies

$$\Big(\big(A_n - \widetilde{A}_n\big)H_n + \widetilde{A}_n\big(H_n - \widetilde{H}_n\big)\Big)\varphi_{n,\mathrm{BL}} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\,g_{n,\mathrm{BL}})\,.$$

Let us consider the estimate of $g_{n,\mathrm{BL}}$. We compute the highest order terms in $\widetilde{A}_n\big(H_n - \widetilde{H}_n\big)$ and $\widetilde{A}_n\big(H_n - \widetilde{H}_n\big)$, which are of order $O(|\beta|^3)$ when $n = O(\alpha^{\frac{1}{2}})$:

$$\begin{aligned}\frac{1}{|\beta|^3}\widetilde{A}_n\big(H_n - \widetilde{H}_n\big)\varphi_{n,\mathrm{BL}} &= \frac{1}{r}G'''_{n,\alpha} + \frac{n^2}{|\beta|}\frac{r^2-1}{r^2}G''_{n,\alpha} - \frac{n^2}{r|\beta|^2}G'_{n,\alpha} - \frac{n^4}{|\beta|^3r^2}(r^2-1)G_{n,\alpha}\\ &\quad + \frac{2i\alpha n}{|\beta|^2r}(r-1)G'_{n,\alpha} + \frac{2i\alpha n^3}{|\beta|^3r^2}(r-1)^2(r+1)G_{n,\alpha} + \mathrm{l.o.t}\end{aligned}$$

where all the $G_{n,\alpha}$ are computed at $|\beta|(r-1)$ and the lower order terms are to be understood in terms of $|\beta|$ for $n = O(\alpha^{\frac{1}{2}})$. It is clear from this formula that as long as $n^2 \lesssim \alpha$ then there is a function $R_{n,\alpha,I}(r,\rho)$, uniformly bounded in r and exponentially decaying at infinity in ρ, such that

$$\widetilde{A}_n\big(H_n - \widetilde{H}_n\big)\varphi_{n,\mathrm{BL}} = |\beta|^3 R_{n,\alpha,I}\big(r, |\beta|(r-1)\big)\,.$$

Similarly one has

$$\begin{aligned}\frac{1}{|\beta|^3}\big(A_n - \widetilde{A}_n\big)H_n\varphi_{n,\mathrm{BL}} = \frac{1}{r}G'''_{n,\alpha} - \frac{n^2}{|\beta|r^2}(1-r^2)G''_{n,\alpha} + \frac{2i\alpha n}{|\beta|r^2}(r-1)^2(1+2r)G''_{n,\alpha}\\ - \frac{n^2}{r|\beta|^2}G'_{n,\alpha} + \frac{2i\alpha n^3}{|\beta|^2 r^2}(r-1)^2(1+2r)G_{n,\alpha} + \mathrm{l.o.t}\end{aligned}$$

so again

$$\big(A_n - \widetilde{A}_n\big)H_n\varphi_{n,\mathrm{BL}} = |\beta|^3 R_{n,\alpha,II}\big(r, |\beta|(r-1)\big)$$

with some $R_{n,\alpha,II}(r,\rho)$ which is uniformly bounded in r and exponentially decaying as $\rho \to \infty$. This implies that $g_{n,\mathrm{BL}}$ defined in (3.119) satisfies

$$\|g_{n,\mathrm{BL}}\|_{L^2(\Omega)} \lesssim |\beta|^{\frac{3}{2}}\,.$$

The result (3.100) follows. Proposition 3.3.6 is proved. □

3.4 The nonlinear problem

In this section we construct the solution to the nonlinear problem $(\widetilde{\mathrm{NS}}_\alpha)$

$$\begin{cases} -\Delta v - \alpha(x^\perp\cdot\nabla v - v^\perp) + \nabla q + \alpha U^\perp \mathrm{rot}\, v \;=\; -v^\perp \mathrm{rot}\, v + f\,, & x\in\Omega\,,\\ \mathrm{div}\, v \;=\; 0\,, & x\in\Omega\,,\\ v \;=\; 0\,, & x\in\partial\Omega\,,\end{cases} \qquad (\widetilde{\mathrm{NS}}_\alpha)$$

in the class $v = (\mathcal{P}_0 v, \mathcal{Q}_0 v) \in X = \mathcal{P}_0 W_0^{1,\infty}(\Omega)^2 \times \mathcal{Q}_0 W_0^{1,2}(\Omega)^2$ with a suitable pressure $q \in W^{1,1}_{\mathrm{loc}}(\overline{\Omega})$. As already noted in the introduction, for the solvability of $(\widetilde{\mathrm{NS}}_\alpha)$, the key observation is the decomposition of the nonlinear term $v^\perp \mathrm{rot}\, v$: we have for $v\in X$,

$$\begin{aligned} v^\perp \mathrm{rot}\, v \;&=\; (\mathcal{P}_0 v)^\perp \mathrm{rot}\,\mathcal{Q}_0 v + (\mathcal{Q}_0 v)^\perp \mathrm{rot}\,\mathcal{P}_0 v + (\mathcal{Q}_0 v)^\perp \mathrm{rot}\,\mathcal{Q}_0 v + (\mathcal{P}_0 v)^\perp \mathrm{rot}\,\mathcal{P}_0 v\\ &=\; \mathcal{Q}_0\Big((\mathcal{P}_0 v)^\perp \mathrm{rot}\,\mathcal{Q}_0 v + (\mathcal{Q}_0 v)^\perp \mathrm{rot}\,\mathcal{P}_0 v\Big) + (\mathcal{Q}_0 v)^\perp \mathrm{rot}\,\mathcal{Q}_0 v + (\mathcal{P}_0 v)^\perp \mathrm{rot}\,\mathcal{P}_0 v\,.\end{aligned}$$

Here we have used $\mathcal{P}_0\big((\mathcal{P}_0 v)^\perp \mathrm{rot}\,\mathcal{Q}_0 v + (\mathcal{Q}_0 v)^\perp \mathrm{rot}\,\mathcal{P}_0 v\big) = 0$. Furthermore, since the last term on the right-hand side can be written in a gradient form, the problem $(\widetilde{\mathrm{NS}}_\alpha)$ is in fact reduced to the next system

$$\begin{cases} -\Delta v - \alpha(x^\perp\cdot\nabla v - v^\perp) + \nabla\tilde{q} + \alpha U^\perp \mathrm{rot}\, v \;=\; G(v) + f\,, & x\in\Omega\,,\\ \mathrm{div}\, v \;=\; 0\,, & x\in\Omega\,,\\ v \;=\; 0\,, & x\in\partial\Omega\,.\end{cases} \qquad (\widehat{\mathrm{NS}}_\alpha)$$

Here we have set

$$G(v) \;:=\; -\mathcal{Q}_0\Big((\mathcal{P}_0 v)^\perp \mathrm{rot}\,\mathcal{Q}_0 v + (\mathcal{Q}_0 v)^\perp \mathrm{rot}\,\mathcal{P}_0 v\Big) - (\mathcal{Q}_0 v)^\perp \mathrm{rot}\,\mathcal{Q}_0 v\,, \qquad (3.120)$$

and $\tilde{q} := q + Q$, where $Q = Q(|x|)$ is a radial function satisfying $\nabla Q = -(\mathcal{P}_0 v)^\perp \mathrm{rot}\, \mathcal{P}_0 v$.

Our aim is to prove the existence and uniqueness of solutions (v, q) for $(\widehat{\mathrm{NS}}_\alpha)$ in a suitable subset of X, under some conditions on the external force $f = (\mathcal{P}_0 f, \mathcal{Q}_0 f)$ in $Y = \mathcal{P}_0 L^1(\Omega)^2 \times \mathcal{Q}_0 L^2(\Omega)^2$. The proofs in Subsections 3.4.2 and 3.4.3 rely on the standard Banach fixed point argument, where the estimate of the nonlinearity $G(v)$ in the space Y is important. Thanks to the identity

$$\mathcal{P}_0 G(v) = -\mathcal{P}_0 \Big((\mathcal{Q}_0 v)^\perp \mathrm{rot}\, \mathcal{Q}_0 v \Big) \,,$$

we see that $\mathcal{P}_0 G(v)$ belongs to $L^1(\Omega)^2$, which is the same summability as the space Y. In order to control the L^2-norm of $\mathcal{Q}_0 G(v)$ in the iteration scheme, we introduce the closed subspace X_0 of X equipped with the norm $\|\cdot\|_{X_0}$:

$$\begin{aligned} X_0 \ := \Bigg\{ v \in X \ \Bigg| \ & \mathrm{div}\, \mathcal{P}_0 v \ = \ \mathrm{div}\, \mathcal{Q}_0 v \ = \ 0 \ \text{ in } \Omega \,, \\ & \|v\|_{X_0} \ := \ \|\mathcal{P}_0 v\|_{L^\infty(\Omega)} + \|\nabla \mathcal{P}_0 v\|_{L^\infty(\Omega)} \\ & \quad + \|\mathcal{Q}_0 v\|_{L^2(\Omega)} + \|\nabla \mathcal{Q}_0 v\|_{L^2(\Omega)} + \sum_{|n| \ge 1} \|\mathcal{P}_n v\|_{L^\infty(\Omega)} < \infty \Bigg\} \,. \end{aligned} \tag{3.121}$$

Indeed, we can easily establish an a priori estimate for

$$\mathcal{Q}_0 G(v) = -\mathcal{Q}_0 \Big((\mathcal{P}_0 v)^\perp \mathrm{rot}\, \mathcal{Q}_0 v + (\mathcal{Q}_0 v)^\perp \mathrm{rot}\, \mathcal{P}_0 v + (\mathcal{Q}_0 v)^\perp \mathrm{rot}\, \mathcal{Q}_0 v \Big)$$

in $L^2(\Omega)^2$ for $v \in X_0$; see (3.140) below for the estimate of $\big\| \mathcal{Q}_0 \big((\mathcal{Q}_0 v)^\perp \mathrm{rot}\, \mathcal{Q}_0 v \big) \big\|_{L^2(\Omega)}$.

After proving Theorem 3.1.1 and 3.1.2, we revisit Theorem 3.1.2 in order to study the qualitative behavior of solutions. By fixing the external force $f \in Y$, we consider the fast rotation limit $|\alpha| \to \infty$ for the solution $(v, q) = (v^{(\alpha)}, q^{(\alpha)})$ to $(\widetilde{\mathrm{NS}}_\alpha)$. The results are summarized in Theorem 3.1.3, which will be proved in Subsection 3.4.4.

3.4.1 Useful estimates on (S_α)

Before we give the proofs of Theorems 3.1.1 and 3.1.2, let us restate the main estimates for the linearized problem (S_α) for general α, which will be used throughout this section, as well as some specific estimates corresponding to $|\alpha| \gg 1$.

The case of general α

Let $\alpha \in \mathbb{R} \setminus \{0\}$ and $f \in Y$, and let $(v, q) \in X \times W^{1,1}_{\mathrm{loc}}(\overline{\Omega})$ be the unique solution to (S_α) given in Proposition 3.3.1. For notational simplicity, we denote by v^{low} and v^{high}, respectively, the low and high frequency parts of $\mathcal{Q}_0 v$:

$$v^{\mathrm{low}} \ := \sum_{1 \le |n| < 1 + \sqrt{2|\alpha|}} \mathcal{P}_n v \,, \qquad v^{\mathrm{high}} \ := \sum_{|n| \ge 1 + \sqrt{2|\alpha|}} \mathcal{P}_n v \,. \tag{3.122}$$

From the estimates (3.42) and (3.46) in Proposition 3.3.1, we have

$$\|v^{\mathrm{low}}\|_{L^2(\Omega)} \le \frac{C}{|\alpha|^{\frac{1}{2}}} \big(1 + \frac{1}{|\alpha|^{\frac{1}{2}}}\big) \|\mathcal{Q}_0 f\|_{L^2(\Omega)} \,, \tag{3.123}$$

$$\|v^{\mathrm{high}}\|_{L^2(\Omega)} \le \frac{C}{|\alpha|} \|\mathcal{Q}_0 f\|_{L^2(\Omega)} \,, \tag{3.124}$$

and from (3.45) and (3.49) in the same proposition, we have the estimates for the derivatives.

$$\|\nabla v^{\mathrm{low}}\|_{L^2(\Omega)} \le C\big(1 + \frac{1}{|\alpha|^{\frac{1}{2}}}\big)\|\mathcal{Q}_0 f\|_{L^2(\Omega)}\,, \tag{3.125}$$

$$\|\nabla v^{\mathrm{high}}\|_{L^2(\Omega)} \le \frac{C}{|\alpha|^{\frac{1}{2}}}\|\mathcal{Q}_0 f\|_{L^2(\Omega)}\,. \tag{3.126}$$

For the fixed point argument in the space X_0, the estimate of $\sum_{|n|\ge 1}\|\mathcal{P}_n v\|_{L^\infty(\Omega)}$ is also needed. In the case $0 < |\alpha| < 1$, (3.44) and (3.48) in Proposition 3.3.1, and the Hölder inequality for sequences lead to

$$\begin{aligned}\sum_{|n|\ge 1}\|\mathcal{P}_n v\|_{L^\infty(\Omega)} &\le \frac{C}{|\alpha|^{\frac{3}{4}}}\sum_{|n|\ge 1}\frac{1}{|n|^{\frac{3}{4}}}\|\mathcal{P}_n f\|_{L^2(\Omega)}\\ &\le \frac{C}{|\alpha|^{\frac{3}{4}}}\Big(\sum_{|n|\ge 1}\frac{1}{|n|^{\frac{3}{2}}}\Big)^{\frac{1}{2}}\|\mathcal{Q}_0 f\|_{L^2(\Omega)}\,,\end{aligned}$$

which implies

$$\sum_{|n|\ge 1}\|\mathcal{P}_n v\|_{L^\infty(\Omega)} \le \frac{C}{|\alpha|^{\frac{3}{4}}}\|\mathcal{Q}_0 f\|_{L^2(\Omega)}\,. \tag{3.127}$$

In the case $|\alpha| \ge 1$ we have from (3.44),

$$\sum_{1\le |n| < 1+\sqrt{2|\alpha|}}\|\mathcal{P}_n v\|_{L^\infty(\Omega)} \le \frac{C}{|\alpha|^{\frac{1}{4}}}\|\mathcal{Q}_0 f\|_{L^2(\Omega)}\,, \tag{3.128}$$

and from (3.48) with the Hölder inequality,

$$\sum_{|n|\ge 1+\sqrt{2|\alpha|}}\|\mathcal{P}_n v\|_{L^\infty(\Omega)} \le \frac{C}{|\alpha|^{\frac{1}{2}}}\|\mathcal{Q}_0 f\|_{L^2(\Omega)}\,. \tag{3.129}$$

The case of large $|\alpha|$

Now let us consider the special case when $|\alpha| \gg 1$. Thanks to the boundary layer analysis in Proposition 3.3.2, we have a better estimate for the linearized problem (S_α) in terms of decay in the parameter $|\alpha n|$. Given the parameter κ defined in Proposition 3.3.2 we assume that $|\alpha|$ is large enough, as required in Proposition 3.3.2, and let us truncate frequencies, similarly to (3.122), as follows:

$$v^{\mathrm{low},\kappa} := \sum_{1\le |n|\le \kappa|\alpha|^{\frac{1}{2}}}\mathcal{P}_n v\,, \qquad v^{\mathrm{high},\kappa} := \sum_{|n|>\kappa|\alpha|^{\frac{1}{2}}}\mathcal{P}_n v\,. \tag{3.130}$$

Then thanks to (3.78) we know that

$$\|v^{\mathrm{low},\kappa}\|_{L^2(\Omega)} \le \frac{C}{|\alpha|^{\frac{2}{3}}}\|\mathcal{Q}_0 f\|_{L^2(\Omega)}\,, \tag{3.131}$$

$$\|\nabla v^{\mathrm{low},\kappa}\|_{L^2(\Omega)} \le \frac{C}{|\alpha|^{\frac{1}{3}}}\|\mathcal{Q}_0 f\|_{L^2(\Omega)}\,, \tag{3.132}$$

and

$$\sum_{1\le|n|\le\kappa|\alpha|^{\frac12}} \|\mathcal{P}_n v\|_{L^\infty(\Omega)} \le \frac{C}{|\alpha|^{\frac12}}\Big(\sum_{1\le|n|\le\kappa|\alpha|^{\frac12}} \frac{1}{|n|}\Big)^{\frac12}\|\mathcal{Q}_0 f\|_{L^2(\Omega)}$$
$$\le \frac{C(\log|\alpha|)^{\frac12}}{|\alpha|^{\frac12}}\|\mathcal{Q}_0 f\|_{L^2(\Omega)}\,. \tag{3.133}$$

Note that the constant C depends on κ, which is fixed. One can see that the decay in terms of $|\alpha|$ compared with (3.123), (3.125) and (3.128) are improved.

Similarly thanks to (3.46), (3.49), (3.50) and (3.52) there holds

$$\|v^{\mathrm{high},\kappa}\|_{L^2(\Omega)} \le \frac{C}{|\alpha|}\|\mathcal{Q}_0 f\|_{L^2(\Omega)}\,, \tag{3.134}$$

$$\|\nabla v^{\mathrm{high},\kappa}\|_{L^2(\Omega)} \le \frac{C}{|\alpha|^{\frac12}}\|\mathcal{Q}_0 f\|_{L^2(\Omega)}\,. \tag{3.135}$$

Finally from (3.48), (3.51) and the Hölder inequality, we derive

$$\sum_{|n|\ge\kappa|\alpha|^{\frac12}} \|\mathcal{P}_n v\|_{L^\infty(\Omega)} \le \frac{C}{|\alpha|^{\frac12}}\|\mathcal{Q}_0 f\|_{L^2(\Omega)}\,. \tag{3.136}$$

3.4.2 Proof for general α

In this subsection we prove Theorem 3.1.1, by means of a fixed point argument. The solutions to ($\widehat{\mathrm{NS}}_\alpha$) will be found in the closed convex set $\mathcal{B}_{\vec\delta,\epsilon}$ of X_0 defined as follows:

$$\begin{aligned}\mathcal{B}_{\vec\delta,\epsilon} &:= \mathcal{B}_{(\delta_1,\delta_2,\delta_3,\delta_4),\epsilon}\\ &:= \Big\{h\in X_0 \;\Big|\; \|\mathcal{P}_0 h\|_{L^\infty(\Omega)} + \|\nabla\mathcal{P}_0 h\|_{L^\infty(\Omega)} \le \epsilon|\alpha|^{\delta_1}\,,\\ &\qquad \|\mathcal{Q}_0 h\|_{L^2(\Omega)} \le \epsilon|\alpha|^{\delta_2}\,, \quad \|\nabla\mathcal{Q}_0 h\|_{L^2(\Omega)} \le \epsilon|\alpha|^{\delta_3}\,, \quad \sum_{|n|\ge1}\|\mathcal{P}_n h\|_{L^\infty(\Omega)} \le \epsilon|\alpha|^{\delta_4}\Big\}\,,\end{aligned}$$

where we have set $\vec\delta := (\delta_1,\delta_2,\delta_3,\delta_4)$, and the numbers $\delta_1,\ldots,\delta_4$ and the positive number ϵ will be chosen later. For any $h\in\mathcal{B}_{\vec\delta,\epsilon}$, let (v_h,q_h) be the unique solution constructed in Proposition 3.3.1 to the linear system

$$\begin{cases} -\Delta v_h - \alpha(x^\perp\cdot\nabla v_h - v_h^\perp) + \nabla q_h + \alpha U^\perp \mathrm{rot}\, v_h = G(h) + f\,, & x\in\Omega\,,\\ \mathrm{div}\, v_h = 0\,, & x\in\Omega\,,\\ v_h = 0\,, & x\in\partial\Omega\,,\end{cases} \tag{3.137}$$

where the function G is defined in (3.120).

Let us start with the estimate of $G(h)$ in the space Y. The first two terms in the right-hand side of (3.120) with v replaced by h can be estimated as

$$\begin{aligned}&\|\mathcal{Q}_0\big((\mathcal{P}_0 h)^\perp \mathrm{rot}\,\mathcal{Q}_0 h\big)\|_{L^2(\Omega)} + \|\mathcal{Q}_0\big((\mathcal{Q}_0 h)^\perp \mathrm{rot}\,\mathcal{P}_0 h\big)\|_{L^2(\Omega)}\\ &= \Big(\sum_{|n|\ge1}\|(\mathcal{P}_0 h)^\perp \mathrm{rot}\,\mathcal{P}_n h\|^2_{L^2(\Omega)}\Big)^{\frac12} + \Big(\sum_{|n|\ge1}\|(\mathcal{P}_n h)^\perp \mathrm{rot}\,\mathcal{P}_0 h\|^2_{L^2(\Omega)}\Big)^{\frac12}\\ &\le \|\mathcal{P}_0 h\|_{L^\infty(\Omega)}\|\nabla\mathcal{Q}_0 h\|_{L^2(\Omega)} + \|\nabla\mathcal{P}_0 h\|_{L^\infty(\Omega)}\|\mathcal{Q}_0 h\|_{L^2(\Omega)}\,.\end{aligned} \tag{3.138}$$

For the last term in the right-hand side of (3.120) with v replaced by h, we observe that

$$\mathcal{P}_n\big((\mathcal{Q}_0 h)^\perp \mathrm{rot}\, \mathcal{Q}_0 h\big) \;=\; \sum_{k\in\mathbb{Z}\setminus\{0,n\}} (\mathcal{P}_k h)^\perp \mathrm{rot}\, \mathcal{P}_{n-k} h\,.$$

Then, applying the Hölder inequality we have

$$\begin{aligned}\|\mathcal{P}_0\big((\mathcal{Q}_0 h)^\perp \mathrm{rot}\, \mathcal{Q}_0 h\big)\|_{L^1(\Omega)} &\le \sum_{k\in\mathbb{Z}\setminus\{0\}} \|\mathcal{P}_k h\|_{L^2(\Omega)} \|\mathcal{P}_{-k}\nabla h\|_{L^2(\Omega)} \\ &\le \|\mathcal{Q}_0 h\|_{L^2(\Omega)} \|\nabla \mathcal{Q}_0 h\|_{L^2(\Omega)}\,,\end{aligned} \tag{3.139}$$

and the Young inequality for sequences implies that

$$\begin{aligned}\|\mathcal{Q}_0\big((\mathcal{Q}_0 h)^\perp \mathrm{rot}\, \mathcal{Q}_0 h\big)\|_{L^2(\Omega)} &= \big(\sum_{|n|\ge 1} \|\mathcal{P}_n (\mathcal{Q}_0 h)^\perp \mathrm{rot}\, \mathcal{Q}_0 h\|_{L^2(\Omega)}^2\big)^{\frac12} \\ &\le \Big(\sum_{|n|\ge 1}\big(\sum_{k\in\mathbb{Z}\setminus\{0,n\}} \|\mathcal{P}_k h\|_{L^\infty(\Omega)} \|\nabla \mathcal{P}_{n-k} h\|_{L^2(\Omega)}\big)^2\Big)^{\frac12} \\ &\le \big(\sum_{|n|\ge 1} \|\mathcal{P}_n h\|_{L^\infty(\Omega)}\big)\, \|\nabla \mathcal{Q}_0 h\|_{L^2(\Omega)}\,.\end{aligned} \tag{3.140}$$

Next we estimate the difference $G(h^{(1)}) - G(h^{(2)})$ for $h^{(1)}, h^{(2)} \in X$. Setting $\mathbf{h} = (h^{(1)}, h^{(2)})$ for simplicity, we define the function $H(\mathbf{h})$ on $\mathcal{B}_{\vec{\delta},\epsilon} \times \mathcal{B}_{\vec{\delta},\epsilon}$ as

$$\begin{aligned} H(\mathbf{h}) &:= G(h^{(1)}) - G(h^{(2)}) \\ &= \mathcal{Q}_0\Big((\mathcal{P}_0 h^{(1)} - \mathcal{P}_0 h^{(2)})^\perp \mathrm{rot}\, \mathcal{Q}_0 h^{(1)} + (\mathcal{P}_0 h^{(2)})^\perp \mathrm{rot}\,(\mathcal{Q}_0 h^{(1)} - \mathcal{Q}_0 h^{(2)}) \\ &\qquad + (\mathcal{Q}_0 h^{(1)} - \mathcal{Q}_0 h^{(2)})^\perp \mathrm{rot}\, \mathcal{P}_0 h^{(1)} + (\mathcal{Q}_0 h^{(2)})^\perp \mathrm{rot}\,(\mathcal{P}_0 h^{(1)} - \mathcal{P}_0 h^{(2)})\Big) \\ &\quad + (\mathcal{Q}_0 h^{(1)} - \mathcal{Q}_0 h^{(2)})^\perp \mathrm{rot}\, \mathcal{Q}_0 h^{(1)} + (\mathcal{Q}_0 h^{(2)})^\perp \mathrm{rot}\,(\mathcal{Q}_0 h^{(1)} - \mathcal{Q}_0 h^{(2)})\,.\end{aligned} \tag{3.141}$$

The following estimates on $H(\mathbf{h})$ are obtained in the same way as (3.138)–(3.140):

$$\|\mathcal{P}_0 H(\mathbf{h})\|_{L^1(\Omega)} \le \big(\|\nabla \mathcal{Q}_0 h^{(1)}\|_{L^2(\Omega)} + \|\mathcal{Q}_0 h^{(2)}\|_{L^2(\Omega)}\big)\, \|h^{(1)} - h^{(2)}\|_{X_0}\,, \tag{3.142}$$

$$\begin{aligned}\|\mathcal{Q}_0 H(\mathbf{h})\|_{L^2(\Omega)} &\le \Big(\|\nabla \mathcal{Q}_0 h^{(1)}\|_{L^2(\Omega)} + \|\mathcal{Q}_0 h^{(2)}\|_{L^2(\Omega)} + \|\nabla \mathcal{P}_0 h^{(1)}\|_{L^\infty(\Omega)} \\ &\qquad + \|\mathcal{P}_0 h^{(2)}\|_{L^\infty(\Omega)} + \sum_{|n|\ge 1} \|\mathcal{P}_n h^{(2)}\|_{L^\infty(\Omega)}\Big)\, \|h^{(1)} - h^{(2)}\|_{X_0}\,.\end{aligned} \tag{3.143}$$

From (3.138)–(3.140), (3.142)–(3.143), and the definition of $\mathcal{B}_{\vec{\delta},\epsilon}$, we obtain the following estimates on $G(h)$ and $H(\mathbf{h})$ in Y.

$$\|\mathcal{P}_0 G(h)\|_{L^1(\Omega)} \le \epsilon^2 |\alpha|^{\delta_2+\delta_3}\,, \tag{3.144}$$

$$\|\mathcal{Q}_0 G(h)\|_{L^2(\Omega)} \le \epsilon^2 (|\alpha|^{\delta_1+\delta_2} + |\alpha|^{\delta_1+\delta_3} + |\alpha|^{\delta_3+\delta_4})\,, \tag{3.145}$$

$$\|\mathcal{P}_0 H(\mathbf{h})\|_{L^1(\Omega)} \le \epsilon (|\alpha|^{\delta_2} + |\alpha|^{\delta_3})\, \|h^{(1)} - h^{(2)}\|_{X_0}\,, \tag{3.146}$$

$$\|\mathcal{Q}_0 H(\mathbf{h})\|_{L^2(\Omega)} \le \epsilon (|\alpha|^{\delta_1} + |\alpha|^{\delta_2} + |\alpha|^{\delta_3} + |\alpha|^{\delta_4})\, \|h^{(1)} - h^{(2)}\|_{X_0}\,. \tag{3.147}$$

Now let us define the mapping $\Phi : \mathcal{B}_{\vec{\delta},\epsilon} \to X_0$ by setting $\Phi[h] := v_h$, where v_h is the unique solution to (3.137). Our aim is to show that

(i) Φ is Lipschitz continuous on $\mathcal{B}_{\vec{\delta},\epsilon}$ in the topology of X_0. Namely, there exists $\tau \in (0,1)$ such that $\|\Phi[h^{(1)}] - \Phi[h^{(2)}]\|_{X_0} \le \tau \|h^{(1)} - h^{(2)}\|_{X_0}$ for any $h^{(1)}, h^{(2)} \in \mathcal{B}_{\vec{\delta},\epsilon}$,

(ii) Φ is a mapping from $\mathcal{B}_{\vec{\delta},\epsilon}$ into $\mathcal{B}_{\vec{\delta},\epsilon}$, if the pair $(\vec{\delta},\epsilon)$ and the external force $f = (\mathcal{P}_0 f, \mathcal{Q}_0 f) \in Y$ satisfy a suitable condition.

For convenience in the following proof, let K_1 and K_2 denote the largest constant C (larger than 1 without loss of generality) appearing in (3.41), (3.123)–(3.127), and (3.41), (3.123)–(3.126), (3.128)–(3.129), respectively.

We first show (i). For any $\mathbf{h} = (h^{(1)}, h^{(2)}) \in \mathcal{B}_{\vec{\delta},\epsilon} \times \mathcal{B}_{\vec{\delta},\epsilon}$, we observe that the differences $u_{\mathbf{h}} := \Phi[h^{(1)}] - \Phi[h^{(2)}]$ and $p_{\mathbf{h}} := q_{h^{(1)}} - q_{h^{(2)}}$ solve the following system:

$$\begin{cases} -\Delta u_{\mathbf{h}} - \alpha(x^\perp \cdot \nabla u_{\mathbf{h}} - u_{\mathbf{h}}^\perp) + \nabla p_{\mathbf{h}} + \alpha U^\perp \mathrm{rot}\, u_{\mathbf{h}} \,=\, H(\mathbf{h})\,, & x \in \Omega\,, \\ \mathrm{div}\, u_{\mathbf{h}} \,=\, 0\,, & x \in \Omega\,, \\ u_{\mathbf{h}} \,=\, 0\,, & x \in \partial\Omega\,, \end{cases}$$

where $H(\mathbf{h})$ is defined in (3.141). We consider the case $0 < |\alpha| < 1$. Then for any $h^{(1)}, h^{(2)}$ in X_0, by (3.41) and (3.123)–(3.127) combined with (3.146)–(3.147), we see that

$$\begin{aligned} &\|\Phi[h^{(1)}] - \Phi[h^{(2)}]\|_{X_0} \\ &= \|\mathcal{P}_0 u_{\mathbf{h}}\|_{L^\infty(\Omega)} + \|\nabla \mathcal{P}_0 u_{\mathbf{h}}\|_{L^\infty(\Omega)} + \|\mathcal{Q}_0 u_{\mathbf{h}}\|_{L^2(\Omega)} + \|\nabla \mathcal{Q}_0 u_{\mathbf{h}}\|_{L^2(\Omega)} + \sum_{|n|\ge 1} \|\mathcal{P}_n u_{\mathbf{h}}\|_{L^\infty(\Omega)} \\ &\le K_1 \|\mathcal{P}_0 H(\mathbf{h})\|_{L^1(\Omega)} + \frac{7K_1}{|\alpha|} \|\mathcal{Q}_0 H(\mathbf{h})\|_{L^2(\Omega)} \\ &\le 8K_1 \epsilon \, (|\alpha|^{\delta_1 - 1} + |\alpha|^{\delta_2 - 1} + |\alpha|^{\delta_3 - 1} + |\alpha|^{\delta_4 - 1}) \, \|h^{(1)} - h^{(2)}\|_{X_0}\,. \end{aligned}$$

Hence, if we choose the pair $(\vec{\delta},\epsilon)$ to satisfy

$$\delta_j \ge 1 \quad \text{for} \quad j \,=\, 1\ldots 4\,, \qquad 0 < \epsilon < \frac{1}{32K_1}\,, \qquad \text{when} \quad 0 < |\alpha| < 1\,, \tag{3.148}$$

then the mapping Φ is Lipschitz continuous on the set $\mathcal{B}_{\vec{\delta},\epsilon}$. For the case $|\alpha| \ge 1$, from (3.41), (3.123)–(3.126), and (3.128)–(3.129) combined with (3.146)–(3.147), we have

$$\begin{aligned} \|\Phi[h^{(1)}] - \Phi[h^{(2)}]\|_{X_0} &\le K_2 \|\mathcal{P}_0 H(\mathbf{h})\|_{L^1(\Omega)} + 8K_2 \|\mathcal{Q}_0 H(\mathbf{h})\|_{L^2(\Omega)} \\ &\le 9K_1 \epsilon \, (|\alpha|^{\delta_1} + |\alpha|^{\delta_2} + |\alpha|^{\delta_3} + |\alpha|^{\delta_4}) \, \|h^{(1)} - h^{(2)}\|_{X_0}\,. \end{aligned} \tag{3.149}$$

Then we obtain the next condition

$$\delta_j \le 0 \quad \text{for} \quad j \,=\, 1\ldots 4\,, \qquad 0 < \epsilon < \frac{1}{36K_2}\,, \qquad \text{when} \quad |\alpha| \ge 1\,, \tag{3.150}$$

for the Lipschitz continuity of Φ on $\mathcal{B}_{\vec{\delta},\epsilon}$. We have shown (i) provided the pair $(\vec{\delta},\epsilon)$ satisfies the conditions (3.148) or (3.150).

Next we prove (ii). In the case $0 < |\alpha| < 1$, the estimates (3.41) and (3.144) imply

$$\begin{aligned} \|\mathcal{P}_0 \Phi[h]\|_{L^\infty(\Omega)} + \|\nabla \mathcal{P}_0 \Phi[h]\|_{L^\infty(\Omega)} &\le K_1 \big(\|(\mathcal{P}_0 G(h))_\theta\|_{L^1(\Omega)} + \|(\mathcal{P}_0 f)_\theta\|_{L^1(\Omega)} \big) \\ &\le K_1 \big(\epsilon^2 |\alpha|^{\delta_2 + \delta_3} + \|(\mathcal{P}_0 f)_\theta\|_{L^1(\Omega)} \big)\,, \end{aligned} \tag{3.151}$$

for any $h \in X_0$. From (3.123)–(3.124), (3.125)–(3.126), and (3.145) we have,

$$\begin{aligned}\|\mathcal{Q}_0\Phi[h]\|_{L^2(\Omega)} &\leq \frac{3K_1}{|\alpha|}\big(\|\mathcal{Q}_0G(h)\|_{L^2(\Omega)} + \|\mathcal{Q}_0 f\|_{L^2(\Omega)}\big)\\ &\leq 3K_1\Big(\epsilon^2\big(|\alpha|^{\delta_1+\delta_2-1} + |\alpha|^{\delta_1+\delta_3-1} + |\alpha|^{\delta_3+\delta_4-1}\big) + |\alpha|^{-1}\|\mathcal{Q}_0 f\|_{L^2(\Omega)}\Big)\,,\\ \|\nabla\mathcal{Q}_0\Phi[h]\|_{L^2(\Omega)} &\leq \frac{3K_1}{|\alpha|^{\frac{1}{2}}}\big(\|\mathcal{Q}_0G(h)\|_{L^2(\Omega)} + \|\mathcal{Q}_0 f\|_{L^2(\Omega)}\big)\\ &\leq 3K_1\Big(\epsilon^2\big(|\alpha|^{\delta_1+\delta_2-\frac{1}{2}} + |\alpha|^{\delta_1+\delta_3-\frac{1}{2}} + |\alpha|^{\delta_3+\delta_4-\frac{1}{2}}\big) + |\alpha|^{-\frac{1}{2}}\|\mathcal{Q}_0 f\|_{L^2(\Omega)}\Big)\,,\end{aligned}$$

and

$$\begin{aligned}\sum_{|n|\geq 1}\|\mathcal{P}_n\Phi[h]\|_{L^\infty(\Omega)} &\leq \frac{K_1}{|\alpha|^{\frac{3}{4}}}\big(\|\mathcal{Q}_0G(h)\|_{L^2(\Omega)} + \|\mathcal{Q}_0 f\|_{L^2(\Omega)}\big)\\ &\leq K_1\Big(\epsilon^2\big(|\alpha|^{\delta_1+\delta_2-\frac{3}{4}} + |\alpha|^{\delta_1+\delta_3-\frac{3}{4}} + |\alpha|^{\delta_3+\delta_4-\frac{3}{4}}\big) + |\alpha|^{-\frac{3}{4}}\|\mathcal{Q}_0 f\|_{L^2(\Omega)}\Big)\,.\end{aligned}$$

Hence, recalling the condition (3.148), if we choose the pair $(\vec{\delta}, \epsilon)$ and $f \in Y$ to satisfy

$$\delta_1 = \delta_2 = \delta_3 = \delta_4 = 1\,, \qquad 0 < \epsilon < \frac{1}{32K_1}\,, \qquad \text{when} \quad 0 < |\alpha| < 1\,, \tag{3.152}$$

and

$$\|(\mathcal{P}_0 f)_\theta\|_{L^1(\Omega)} \leq \frac{\epsilon}{2K_1}|\alpha|\,, \quad \|\mathcal{Q}_0 f\|_{L^2(\Omega)} \leq \frac{\epsilon}{6K_1}|\alpha|^2\,, \quad \text{when} \quad 0 < |\alpha| < 1\,, \tag{3.153}$$

then Φ defines a mapping from $\mathcal{B}_{\vec{\delta},\epsilon}$ into itself.
In the case $|\alpha| \geq 1$, we have (3.151) with K_1 replaced by K_2, and moreover, from (3.123)–(3.124), (3.125)–(3.126), and (3.128)–(3.129), along with (3.145), we have

$$\begin{aligned}\|\mathcal{Q}_0\Phi[h]\|_{L^2(\Omega)} &\leq \frac{3K_2}{|\alpha|^{\frac{1}{2}}}\big(\|\mathcal{Q}_0G(h)\|_{L^2(\Omega)} + \|\mathcal{Q}_0 f\|_{L^2(\Omega)}\big)\\ &\leq 3K_2\Big(\epsilon^2\big(|\alpha|^{\delta_1+\delta_2-\frac{1}{2}} + |\alpha|^{\delta_1+\delta_3-\frac{1}{2}} + |\alpha|^{\delta_3+\delta_4-\frac{1}{2}}\big) + |\alpha|^{-\frac{1}{2}}\|\mathcal{Q}_0 f\|_{L^2(\Omega)}\Big)\,,\end{aligned} \tag{3.154}$$

$$\begin{aligned}\|\nabla\mathcal{Q}_0\Phi[h]\|_{L^2(\Omega)} &\leq 3K_2\big(\|\mathcal{Q}_0G(h)\|_{L^2(\Omega)} + \|\mathcal{Q}_0 f\|_{L^2(\Omega)}\big)\\ &\leq 3K_2\Big(\epsilon^2\big(|\alpha|^{\delta_1+\delta_2} + |\alpha|^{\delta_1+\delta_3} + |\alpha|^{\delta_3+\delta_4}\big) + \|\mathcal{Q}_0 f\|_{L^2(\Omega)}\Big)\,,\end{aligned} \tag{3.155}$$

and

$$\begin{aligned}\sum_{|n|\geq 1}\|\mathcal{P}_n\Phi[h]\|_{L^\infty(\Omega)} &\leq \frac{2K_2}{|\alpha|^{\frac{1}{4}}}\big(\|\mathcal{Q}_0G(h)\|_{L^2(\Omega)} + \|\mathcal{Q}_0 f\|_{L^2(\Omega)}\big)\\ &\leq 2K_2\Big(\epsilon^2\big(|\alpha|^{\delta_1+\delta_2-\frac{1}{4}} + |\alpha|^{\delta_1+\delta_3-\frac{1}{4}} + |\alpha|^{\delta_3+\delta_4-\frac{1}{4}}\big) + |\alpha|^{-\frac{1}{4}}\|\mathcal{Q}_0 f\|_{L^2(\Omega)}\Big)\,.\end{aligned} \tag{3.156}$$

Then recalling (3.150), if the pair $(\vec{\delta}, \epsilon)$ and $f \in Y$ satisfy

$$\delta_1 = \delta_2 = \delta_3 = \delta_4 = 0\,, \qquad 0 < \epsilon < \frac{1}{36K_2}\,, \qquad \text{when} \quad |\alpha| \geq 1\,, \tag{3.157}$$

and

$$\|(\mathcal{P}_0 f)_\theta\|_{L^1(\Omega)} \le \frac{\epsilon}{2K_2}\,, \qquad \|\mathcal{Q}_0 f\|_{L^2(\Omega)} \le \frac{\epsilon}{6K_2}\,, \qquad \text{when} \quad |\alpha| \ge 1\,, \tag{3.158}$$

then we see that Φ defines a mapping $\Phi : \mathcal{B}_{\vec{\delta},\epsilon} \to \mathcal{B}_{\vec{\delta},\epsilon}$.

Now we have shown that the mapping Φ defines a contraction on $\mathcal{B}_{\vec{\delta},\epsilon}$ under the conditions (3.152) and (3.153) when $0 < |\alpha| < 1$, and under (3.157) and (3.158) when $|\alpha| \ge 1$. Then there is a unique fixed point of Φ in the closed convex set $\mathcal{B}_{\vec{\delta},\epsilon}$. Hence we finally obtain a unique solution to the nonlinear problem $(\widehat{\mathrm{NS}}_\alpha)$ in $\mathcal{B}_{\vec{\delta},\epsilon}$.

The estimates in Theorem 3.1.1 are obtained as follows. We consider only the case $0 < |\alpha| < 1$, in particular the estimates (3.7)–(3.10), since the case $|\alpha| \ge 1$ can be handled similarly. Let v denote the unique fixed point of Φ in $\mathcal{B}_{\vec{\delta},\epsilon}$. Applying the linear estimates (3.123) and (3.124) to $(\widehat{\mathrm{NS}}_\alpha)$ along with (3.138)–(3.140) we see that

$$\begin{aligned}
\|\mathcal{Q}_0 v\|_{L^2(\Omega)} &\le \frac{3K_1}{|\alpha|}\big(\|\mathcal{Q}_0 G(v)\|_{L^2(\Omega)} + \|\mathcal{Q}_0 f\|_{L^2(\Omega)}\big) \\
&\le \frac{3K_1}{|\alpha|}\Big(\|\mathcal{P}_0 v\|_{L^\infty(\Omega)}\|\nabla\mathcal{Q}_0 v\|_{L^2(\Omega)} + \|\nabla\mathcal{P}_0 v\|_{L^\infty(\Omega)}\|\mathcal{Q}_0 v\|_{L^2(\Omega)} \\
&\qquad + \big(\sum_{|n|\ge 1}\|\mathcal{P}_n v\|_{L^\infty(\Omega)}\big)\|\mathcal{Q}_0 v\|_{L^2(\Omega)} + \|\mathcal{Q}_0 f\|_{L^2(\Omega)}\Big) \\
&\le 3K_1\epsilon\big(\|\nabla\mathcal{Q}_0 v\|_{L^2(\Omega)} + 2\|\mathcal{Q}_0 v\|_{L^2(\Omega)}\big) + \frac{3K_1}{|\alpha|}\|\mathcal{Q}_0 f\|_{L^2(\Omega)}\,.
\end{aligned}$$

In the last inequality we have applied the bounds derived from the assumption $v \in \mathcal{B}_{\vec{\delta},\epsilon}$ with $\delta_j = 1$ for all j. Since $0 < 6K_1\epsilon < \frac{1}{5}$ under the choice of ϵ in (3.152), we obtain

$$\|\mathcal{Q}_0 v\|_{L^2(\Omega)} \le \frac{15}{4}K_1\epsilon\|\nabla\mathcal{Q}_0 v\|_{L^2(\Omega)} + \frac{15}{4}\frac{K_1}{|\alpha|}\|\mathcal{Q}_0 f\|_{L^2(\Omega)}\,. \tag{3.159}$$

On the other hand, if we apply (3.125) and (3.126) to $(\widehat{\mathrm{NS}}_\alpha)$, then we have by the same way

$$\begin{aligned}
\|\nabla\mathcal{Q}_0 v\|_{L^2(\Omega)} &\le \frac{3K_1}{|\alpha|^{\frac{1}{2}}}\big(\|\mathcal{Q}_0 G(v)\|_{L^2(\Omega)} + \|\mathcal{Q}_0 f\|_{L^2(\Omega)}\big) \\
&\le 6K_1\epsilon|\alpha|^{\frac{1}{2}}\|\mathcal{Q}_0 v\|_{L^2(\Omega)} + 3K_1\epsilon|\alpha|^{\frac{1}{2}}\|\nabla\mathcal{Q}_0 v\|_{L^2(\Omega)} + \frac{3K_1}{|\alpha|^{\frac{1}{2}}}\|\mathcal{Q}_0 f\|_{L^2(\Omega)}\,.
\end{aligned}$$

Inserting (3.159) in the above inequality, and by the smallness of $K_1\epsilon$ again, we see that

$$\|\nabla\mathcal{Q}_0 v\|_{L^2(\Omega)} \le \frac{C}{|\alpha|^{\frac{1}{2}}}\|\mathcal{Q}_0 f\|_{L^2(\Omega)}\,,$$

which implies the estimate (3.10). We can obtain (3.8) by (3.159) combined with (3.10) with the condition $0 < |\alpha| < 1$. For the estimate (3.9), by (3.127), (3.138) and (3.140), and the condition $v \in \mathcal{B}_{\vec{\delta},\epsilon}$ with $\delta_j = 1$ for all j we have

$$\begin{aligned}
\sum_{|n|\ge 1}\|\mathcal{P}_n v\|_{L^\infty(\Omega)} &\le \frac{K_1}{|\alpha|^{\frac{3}{4}}}\big(\|\mathcal{Q}_0 G(v)\|_{L^2(\Omega)} + \|\mathcal{Q}_0 f\|_{L^2(\Omega)}\big) \\
&\le K_1\epsilon|\alpha|^{\frac{1}{4}}\big(\|\nabla\mathcal{Q}_0 v\|_{L^2(\Omega)} + 2\|\mathcal{Q}_0 v\|_{L^2(\Omega)}\big) + \frac{K_1}{|\alpha|^{\frac{3}{4}}}\|\mathcal{Q}_0 f\|_{L^2(\Omega)}\,,
\end{aligned}$$

which combined with (3.8) and (3.10) leads to (3.9). The estimate (3.7) follows from (3.41) and the nonlinear estimate (3.139) with h replaced by v. This completes the proof of Theorem 3.1.1. □

3.4.3 Proof for large $|\alpha|$

In this subsection we prove Theorem 3.1.2. We adopt the same notation as in the proof of Theorem 3.1.1 of the previous subsection. Let K_3 denote the largest constant of C (larger than 1 without loss of generality) appearing in (3.41) and (3.131)–(3.136) for convenience.

We show that the mapping Φ is a contraction on the set $\mathcal{B}_{\vec{\delta},\epsilon}$. By the Lipschitz continuity of Φ on $\mathcal{B}_{\vec{\delta},\epsilon}$, the estimate (3.149) is improved in terms of the decay in $|\alpha|$ into

$$\begin{aligned}\|\Phi[h^{(1)}] - \Phi[h^{(2)}]\|_{X_0} &\le K_3\|(\mathcal{P}_0 H(\mathbf{h}))_\theta\|_{L^1(\Omega)} + \frac{6K_3}{|\alpha|^{\frac13}}\|\mathcal{Q}_0 H(\mathbf{h})\|_{L^2(\Omega)}\\ &\le 7K_3\epsilon\,(|\alpha|^{\delta_1-\frac13} + |\alpha|^{\delta_2} + |\alpha|^{\delta_3} + |\alpha|^{\delta_4-\frac13})\,\|h^{(1)} - h^{(2)}\|_{X_0}\,.\end{aligned}$$

Hence, if we choose the pair $(\vec{\delta},\epsilon)$ to satisfy

$$\delta_1 \le \frac13\,,\quad \delta_2 \le 0\,,\quad \delta_3 \le 0\,,\quad \delta_4 \le \frac13\,,\quad 0<\epsilon<\frac{1}{28K_3}\,, \tag{3.160}$$

then the mapping Φ is Lipschitz continuous on $\mathcal{B}_{\vec{\delta},\epsilon}$.

Next we check that Φ is a mapping from $\mathcal{B}_{\vec{\delta},\epsilon}$ into $\mathcal{B}_{\vec{\delta},\epsilon}$. We have (3.151) with K_1 replaced by K_3, and we see that (3.154)–(3.156) are improved in terms of decay in $|\alpha|$ respectively to

$$\begin{aligned}\|\mathcal{Q}_0\Phi[h]\|_{L^2(\Omega)} &\le \frac{2K_3}{|\alpha|^{\frac23}}\big(\|\mathcal{Q}_0 G(h)\|_{L^2(\Omega)} + \|\mathcal{Q}_0 f\|_{L^2(\Omega)}\big)\\ &\le 2K_3\Big(\epsilon^2\big(|\alpha|^{\delta_1+\delta_2-\frac23} + |\alpha|^{\delta_1+\delta_3-\frac23} + |\alpha|^{\delta_3+\delta_4-\frac23}\big) + |\alpha|^{-\frac23}\|\mathcal{Q}_0 f\|_{L^2(\Omega)}\Big)\,,\\ \|\nabla\mathcal{Q}_0\Phi[h]\|_{L^2(\Omega)} &\le \frac{2K_3}{|\alpha|^{\frac13}}\big(\|\mathcal{Q}_0 G(h)\|_{L^2(\Omega)} + \|\mathcal{Q}_0 f\|_{L^2(\Omega)}\big)\\ &\le 2K_3\Big(\epsilon^2\big(|\alpha|^{\delta_1+\delta_2-\frac13} + |\alpha|^{\delta_1+\delta_3-\frac13} + |\alpha|^{\delta_3+\delta_4-\frac13}\big) + |\alpha|^{-\frac13}\|\mathcal{Q}_0 f\|_{L^2(\Omega)}\Big)\,,\end{aligned}$$

and

$$\begin{aligned}\sum_{|n|\ge1}\|\mathcal{P}_n\Phi[h]\|_{L^\infty(\Omega)} &\le \frac{2K_3(\log|\alpha|)^{\frac12}}{|\alpha|^{\frac12}}\big(\|\mathcal{Q}_0 G(h)\|_{L^2(\Omega)} + \|\mathcal{Q}_0 f\|_{L^2(\Omega)}\big)\\ &\le 2K_3\Big(\epsilon^2(\log|\alpha|)^{\frac12}\big(|\alpha|^{\delta_1+\delta_2-\frac12} + |\alpha|^{\delta_1+\delta_3-\frac12} + |\alpha|^{\delta_3+\delta_4-\frac12}\big)\\ &\qquad + (\log|\alpha|)^{\frac12}|\alpha|^{-\frac12}\|\mathcal{Q}_0 f\|_{L^2(\Omega)}\Big)\,.\end{aligned}$$

Then, under the condition (3.160), if we choose the pair $(\vec{\delta},\epsilon)$ and $f\in Y$ to satisfy

$$\delta_1 = \delta_4 = \frac13\,,\quad \delta_2 = \delta_3 = 0\,,\qquad 0<\epsilon<\frac{1}{28K_3}\,, \tag{3.161}$$

and

$$\|(\mathcal{P}_0 f)_\theta\|_{L^1(\Omega)} \le \frac{\epsilon}{2K_3}|\alpha|^{\frac13}\,,\quad \|\mathcal{Q}_0 f\|_{L^2(\Omega)} \le \frac{\epsilon}{6K_2}|\alpha|^{\frac13}\,, \tag{3.162}$$

then Φ defines a mapping $\Phi : \mathcal{B}_{\vec{\delta},\epsilon} \to \mathcal{B}_{\vec{\delta},\epsilon}$. Now we have shown that Φ is a contraction on $\mathcal{B}_{\vec{\delta},\epsilon}$ if we assume the conditions (3.161) and (3.162). Hence there exists a unique fixed point v of Φ in $\mathcal{B}_{\vec{\delta},\epsilon}$. The estimates (3.20)–(3.23) can be obtained in the same way as in the proof of Theorem 3.1.1. This completes the proof of Theorem 3.1.2. □

3.4.4 Proof of Theorem 3.1.3

This subsection is devoted to the proof of Theorem 3.1.3. For a given $f \in Y$, let us take $\alpha \in \mathbb{R}$ large enough to satisfy both the condition (3.19) in Theorem 3.1.2 and the assumption in Proposition 3.3.2. Then, from Theorem 3.1.2 we see that there exists a solution $(v^{(\alpha)}, q^{(\alpha)})$ in $X \times W^{1,1}_{\rm loc}(\overline{\Omega})$ to $(\widetilde{\rm NS}_\alpha)$ satisfying the estimates (3.20)–(3.23). Hence, the proof will be completed as soon as we show all the estimates in Theorem 3.1.3 and the decomposition (3.25) of $v^{(\alpha)}$. Note that $v^{(\alpha)}$ also solves the system $(\widehat{\rm NS}_\alpha)$, which is introduced in the beginning of this section, with a suitable new pressure $\tilde{q}^{(\alpha)} \in W^{1,1}_{\rm loc}(\overline{\Omega})$.

We deduce the estimate (3.24) from the triangle inequality

$$\begin{aligned} \|v^{(\alpha)} - v_0^{\rm linear}\|_{L^\infty(\Omega)} &\le \|\mathcal{Q}_0 v^{(\alpha)}\|_{L^\infty(\Omega)} + \|\mathcal{P}_0 v^{(\alpha)} - v_0^{\rm linear}\|_{L^\infty(\Omega)} \\ &\le \sum_{|n|\ge 1} \|\mathcal{P}_n v^{(\alpha)}\|_{L^\infty(\Omega)} + \|\mathcal{P}_0 v^{(\alpha)} - v_0^{\rm linear}\|_{L^\infty(\Omega)} \,. \end{aligned} \tag{3.163}$$

Since $v := \mathcal{P}_0 v^{(\alpha)} - v_0^{\rm linear} \in W^{1,\infty}_0(\Omega)^2$ is a solution to the next system

$$\begin{cases} -\Delta v - \alpha(x^\perp \cdot \nabla v - v^\perp) + \nabla q + \alpha U^\perp {\rm rot}\, v \;=\; -\mathcal{P}_0\Big((\mathcal{Q}_0 v)^\perp {\rm rot}\, \mathcal{Q}_0 v\Big)\,, & x \in \Omega\,, \\ {\rm div}\, v \;=\; 0\,, \qquad x \in \Omega\,, \\ v \;=\; 0\,, \qquad x \in \partial\Omega\,, \end{cases}$$

with some pressure $q \in W^{1,1}_{\rm loc}(\overline{\Omega})$, we have from (3.41) and (3.139) with h replaced by $v^{(\alpha)}$,

$$\|\mathcal{P}_0 v^{(\alpha)} - v_0^{\rm linear}\|_{L^\infty(\Omega)} \le C \|\mathcal{Q}_0 v^{(\alpha)}\|_{L^2(\Omega)} \|\nabla \mathcal{Q}_0 v^{(\alpha)}\|_{L^2(\Omega)} \,. \tag{3.164}$$

Then, by (3.163)–(3.164) along with (3.21)–(3.23) we obtain the estimate (3.24).

The decomposition (3.25) and the related estimates follow from the results in Proposition 3.3.2. Indeed, we know that $(v^{(\alpha)}, \tilde{q}^{(\alpha)})$ solves $(\widehat{\rm NS}_\alpha)$, and we have the next estimate of $\|\mathcal{P}_n G(v^{(\alpha)})\|_{L^2(\Omega)}$, which is uniform both in $|n| \ge 1$ and $|\alpha|$, combined with estimates (3.20)-(3.23):

$$\begin{aligned} &\|\mathcal{P}_n G(v^{(\alpha)})\|_{L^2(\Omega)} \le \|\mathcal{Q}_0 G(v^{(\alpha)})\|_{L^2(\Omega)} \\ &\le \big\|\mathcal{Q}_0\big((\mathcal{P}_0 v^{(\alpha)})^\perp {\rm rot}\, \mathcal{Q}_0 v^{(\alpha)}\big)\big\|_{L^2(\Omega)} + \big\|\mathcal{Q}_0\big((\mathcal{Q}_0 v^{(\alpha)})^\perp {\rm rot}\, \mathcal{P}_0 v^{(\alpha)}\big)\big\|_{L^2(\Omega)} \\ &\quad + \big\|\mathcal{Q}_0\big((\mathcal{Q}_0 v^{(\alpha)})^\perp {\rm rot}\, \mathcal{Q}_0 v^{(\alpha)}\big)\big\|_{L^2(\Omega)} \\ &\le \|\mathcal{P}_0 v^{(\alpha)}\|_{L^\infty(\Omega)} \|\nabla \mathcal{Q}_0 v^{(\alpha)}\|_{L^2(\Omega)} + \|\nabla \mathcal{P}_0 v^{(\alpha)}\|_{L^\infty(\Omega)} \|\mathcal{Q}_0 v^{(\alpha)}\|_{L^2(\Omega)} \\ &\quad + \Big(\sum_{|n|\ge 1} \|\mathcal{P}_n v^{(\alpha)}\|_{L^\infty(\Omega)}\Big) \|\nabla \mathcal{Q}_0 v^{(\alpha)}\|_{L^2(\Omega)} \,, \end{aligned}$$

where the nonlinear estimates (3.138) and (3.140) with h replaced by $v^{(\alpha)}$ are applied. The proof of Theorem 3.1.3 is complete. □

3.5 Appendix

3.5.1 Solving the boundary layer equation: proof of Lemma 3.3.7

In this section we prove Lemma 3.3.7. All the results concerning the Airy function can be found for instance in [1], Chapter 10. Let us first recall that a solution to the Airy equation

$$\frac{\mathrm{d}^2 f(\rho)}{\mathrm{d}\rho^2} - \rho f(\rho) = 0 \quad \text{in } \mathbb{R}$$

is given by the Airy function Ai, which can be extended as an entire analytic function on $\mathbb{C}$ satisfying

$$\frac{\mathrm{d}^2 f(z)}{\mathrm{d}z^2} - z f(z) = 0 \quad \text{in } \mathbb{C}\,. \tag{3.165}$$

It is the inverse Fourier transform of

$$\xi \mapsto \exp\big(\frac{i\xi^3}{3}\big)$$

and satisfies

$$\mathrm{Ai}(0) = \frac{1}{3^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)}\,, \quad \mathrm{Ai}'(0) = -\frac{1}{3^{\frac{1}{3}}\Gamma\left(\frac{1}{3}\right)}\,,$$

where Γ is the Gamma function. Moreover

$$\mathrm{Ai}(z) \sim_{|z|\to\infty} z^{-\frac{1}{4}} \exp\big(-\frac{2}{3}z^{\frac{3}{2}}\big)\,, \quad |\arg z| < \pi - \epsilon\,, \quad \epsilon > 0\,. \tag{3.166}$$

The results (3.115) and (3.117) are easy consequences of (3.166). Indeed we can write

$$|C_{0,n,\alpha}|^{-1} = \Big|\int_0^\infty e^{\frac{2|n|}{|\beta|}t} \int_t^\infty e^{-\frac{|n|}{|\beta|}s} \mathrm{Ai}(c_- s + \lambda^2)\, \mathrm{d}s\, \mathrm{d}t\Big|\,,$$

where $\lambda = \frac{|n|c_+}{|\beta|}$ with $|\beta| = (2|\alpha n|)^{\frac{1}{3}}$ and $c_\pm = \frac{\sqrt{3}\pm i}{2}$ (hence $\frac{in|\beta|c_-}{2\alpha} = \lambda^2$), and therefore $|C_{0,n,\alpha}|^{-1}$ is bounded uniformly in the set $\{(n,\alpha) \in \mathbb{Z}\times\mathbb{R} \mid |\alpha| \geq 1\,,\ 1 \leq |n| \leq |\alpha|^{\frac{1}{2}}\}$ thanks to (3.166). The result (3.117) is obtained in the same way. As to (3.116), it is known (see for instance [1] page 449) that

$$\int_0^\infty \mathrm{Ai}(s)\, \mathrm{d}s = \frac{1}{3}\,,$$

so the result follows by continuity of the map $\mu \mapsto \displaystyle\int_0^\infty e^{-\mu s}\mathrm{Ai}(s+\mu^2)\,\mathrm{d}s$. This concludes the proof of Lemma 3.3.7. □

3.5.2 Proof of the interpolation inequality (3.65)

We may assume that $g \in W^{1,2}((1,\infty); r\,\mathrm{d}r)$ is nontrivial. Let $\delta \in (0,1]$ be a fixed number which will be determined later. Then we have

$$\begin{aligned}
\|g\|^2_{L^2(\Omega)} &= 2\pi \int_1^\infty |g|^2 r\,\mathrm{d}r \\
&\leq C\int_1^{1+\delta} r\,\mathrm{d}r \|g\|^2_{L^\infty((1,\infty))} + \frac{C}{\delta}\int_{1+\delta}^2 \frac{r^2-1}{r^2}|g|^2 r\,\mathrm{d}r + C\int_2^\infty \frac{r^2-1}{r^2}|g|^2 r\,\mathrm{d}r \\
&\leq C\delta\|g\|^2_{L^\infty((1,\infty))} + \frac{C}{\delta}\|\frac{\sqrt{r^2-1}}{r}g\|^2_{L^2(\Omega)} + C\|\frac{\sqrt{r^2-1}}{r}g\|^2_{L^2(\Omega)}\,.
\end{aligned} \tag{3.167}$$

Let us take

$$\delta = \frac{\|\frac{\sqrt{r^2-1}}{r}g\|_{L^2(\Omega)}}{\|g\|_{L^\infty((1,\infty))} + \|\frac{\sqrt{r^2-1}}{r}g\|_{L^2(\Omega)}} .$$

Then

$$\|g\|_{L^2(\Omega)}^2 \leq C\|\frac{\sqrt{r^2-1}}{r}g\|_{L^2(\Omega)}\|g\|_{L^\infty((1,\infty))} + C\|\frac{\sqrt{r^2-1}}{r}g\|_{L^2(\Omega)}^2 . \tag{3.168}$$

The estimate (3.168) combined with the standard interpolation inequality

$$\|g\|_{L^\infty((1,\infty))} \leq C\|\partial_r g\|_{L^2((1,\infty))}^{\frac{1}{2}}\|g\|_{L^2((1,\infty))}^{\frac{1}{2}} \leq C\|\partial_r g\|_{L^2(\Omega)}^{\frac{1}{2}}\|g\|_{L^2(\Omega)}^{\frac{1}{2}}$$

yields

$$\|g\|_{L^2(\Omega)}^2 \leq C\|\frac{\sqrt{r^2-1}}{r}g\|_{L^2(\Omega)}^{\frac{4}{3}}\|\partial_r g\|_{L^2(\Omega)}^{\frac{2}{3}} + C\|\frac{\sqrt{r^2-1}}{r}g\|_{L^2(\Omega)}^2 .$$

The proof of (3.65) is complete.

Chapter 4

Note on the stability of planar stationary flows in an exterior domain without symmetry

Abstract In the last two chapters, we have discussed the existence of the stationary solutions to the Navier-Stokes equations in a two-dimensional exterior domain. The stability of these stationary solutions in time evolution is a difficult problem even if we assume their smallness, and indeed there are still many problems left open. In this chapter we consider the asymptotic stability of two-dimensional exterior stationary flows, particularly without assuming the symmetry of the exterior domain. Under the smallness condition on initial perturbations, we show the stability of the small stationary flow whose leading profile at spatial infinity is given by the rotating flow $\alpha\frac{x^\perp}{|x|^2}$, $x^\perp = (-x_2, x_1)^\top$, with $|\alpha| \ll 1$. Especially, we prove the L^p-L^q estimates to the semigroup associated with the linearized equations. At the end of this chapter, we will provide some future works related to the main result.

4.1 Introduction

In this chapter we consider the perturbed Stokes equations for viscous incompressible flows in a two-dimensional exterior domain Ω with a smooth boundary.

$$\begin{cases} \partial_t v - \Delta v + V \cdot \nabla v + v \cdot \nabla V + \nabla q \, = \, 0 \,, & t > 0 \,, \ x \in \Omega \,, \\ \operatorname{div} v \, = \, 0 \,, & t \geq 0 \,, \ x \in \Omega \,, \\ v|_{\partial\Omega} \, = \, 0 \,, & t > 0 \,, \\ v|_{t=0} \, = \, v_0 \,, & x \in \Omega \,. \end{cases} \tag{PS}$$

Here the unknown functions $v = v(t,x) = (v_1(t,x), v_2(t,x))^\top$ and $q = q(t,x)$ are respectively the velocity field and the pressure field of the fluid, and $v_0 = v_0(x) = (v_{0,1}(x), v_{0,2}(x))^\top$ is a given initial velocity field. The given vector field $V = V(x) = (V_1(x), V_2(x))^\top$ is assumed to be time-independent and decay in the scale-critical order $V(x) = O(|x|^{-1})$ at spatial infinity. We use the same notations as in Chapters 2 and 3 for spatial differential operators with respect to $x = (x_1, x_2)^\top$, and ∂_t denotes the partial derivative in the time t. The domain Ω is assumed to be contained by the exterior radius-$\frac{1}{2}$ disk $\{x \in \mathbb{R}^2 \mid |x| > \frac{1}{2}\}$.

We have been focusing on the existence of the stationary solutions to the two-dimensional exterior Navier-Stokes equations in Chapters 2 and 3. The stability of the stationary solutions decaying in the order $O(|x|^{-1})$ at spatial infinity is a challenging issue even if the smallness of the solutions is assumed, and the difficulty is brought from the absence of the Hardy inequality, which will be explained below. The aim of this chapter is, under suitable conditions on the vector field $V = V(x)$ and the domain Ω, to investigate the time-decay estimates to the equations (PS) which is related to the stability analysis of V. The equations (PS) have been studied as the linearization of the Navier-Stokes equations around a stationary solution V. In the three-dimensional case, Borchers and Miyakawa [5] establishes the L^p-L^q estimates to (PS) for the small stationary Navier-Stokes flow V satisfying $V(x) = O(|x|^{-1})$ as $|x| \to \infty$. This result is extended to the case when V belongs to the Lorenz space $L^{3,\infty}(\Omega)$ by Kozono and Yamazaki [41]. We also refer to the whole-space result by Hishida and Schonbek [37] considering the time-dependent $V = V(t,x)$ in the scale-critical space $L^\infty(0,\infty; L^{3,\infty}(\mathbb{R}^3))$, where the L^p-L^q estimates are obtained for the evolution operator associated with the linearized equations around $V(t,x)$.

For the two-dimensional problem as in (PS), the analysis becomes quite complicated and there is no general result especially for the time-decay estimate so far. The difficulty arises from the unavailability of the Hardy inequality in the form

$$\|\frac{f}{|x|}\|_{L^2(\Omega)} \le C\|\nabla f\|_{L^2(\Omega)}\,, \qquad f \in \dot{W}^{1,2}_0(\Omega) \,=\, \overline{C_0^\infty(\Omega)}^{\|\nabla f\|_{L^2(\Omega)}}\,, \tag{4.1}$$

where $C_0^\infty(\Omega)$ is the set of smooth and compactly supported functions in Ω. The validity of this bound is well known for three-dimensional exterior domains, and the results mentioned in the above essentially rely on the inequality (4.1). One can recover the Hardy inequality in the two-dimensional case if the factor $|x|^{-1}$ in the left-hand side of (4.1) is replaced with a logarithmic correction $|x|^{-1}\log(e+|x|)^{-1}$, but this inequality has only a narrow application in our scale-critical framework. Another way to recover the inequality (4.1) is to impose the symmetry on both Ω and f, and such an inequality is applied in the analysis of (PS) for the case when V is symmetric. Yamazaki [59] proves the L^p-L^q estimates to (PS) with the symmetric Navier-Stokes flow $V(x) = O(|x|^{-1})$, under the symmetry conditions on both the domain and given data. We note that these estimates imply the asymptotic stability of V under symmetric initial L^2-perturbations; see also Galdi and Yamazaki [21].

An important remark is given by Russo [54] concerning the Hardy-type inequality in two-dimensional exterior domains without symmetry. Let us introduce the next scale-critical radial flow $W = W(x)$, which is called the flux carrier.

$$W(x) \,=\, \frac{x}{|x|^2}\,, \qquad x \in \mathbb{R}^2 \setminus \{0\}\,. \tag{4.2}$$

Then, from the existence of a potential to $W(x) = \nabla \log|x|$, one can show that the following Hardy-type inequality holds in the L^2-inner product $\langle\cdot,\cdot\rangle_{L^2(\Omega)}$:

$$|\langle u\cdot\nabla u\,, W\rangle_{L^2(\Omega)}| \le C\|\nabla u\|^2_{L^2(\Omega)}\,, \qquad u \in \dot{W}^{1,2}_{0,\sigma}(\Omega) \,=\, \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\nabla u\|_{L^2(\Omega)}}\,, \tag{4.3}$$

where $C_{0,\sigma}^\infty(\Omega)$ denotes the function space $\{f \in C_0^\infty(\Omega)^2 \mid \operatorname{div} f = 0\}$. Based on the energy method with the application of (4.3), Guillod [27] proves the global L^2 stability of the flux carrier δW when the flux δ is small enough. On the other hand, the validity of the inequality (4.3) essentially depends on the potential property of W. Indeed, as is pointed out in [27], the bound (4.3) breaks down if W is replaced by the rotating flow $U = U(x)$:

$$U(x) \,=\, \frac{x^\perp}{|x|^2}\,, \qquad x^\perp \,=\, (-x_2, x_1)^\top\,, \qquad x \in \mathbb{R}^2 \setminus \{0\}\,. \tag{4.4}$$

Hence, if we consider the problem (PS) with $V = \alpha U$, $\alpha \in \mathbb{R} \setminus \{0\}$, the linearized term $\alpha(U \cdot \nabla v + v \cdot \nabla U)$ can no more be regarded as a perturbation from the Laplacian, and we cannot avoid the difficulty coming from the lack of the Hardy inequality. Maekawa [44] studies the stability of the flow αU in the exterior unit disk. The symmetry of the domain allows us to express the solution to the problem (PS) explicitly through the Dunford integral of the resolvent operator. Based on this representation formula, [44] obtains the L^p-L^q estimates to (PS) with $V = \alpha U$ for small α, and shows the asymptotic L^2 stability of αU if α and initial perturbations are sufficiently small. This result is extended by the same author in [45] for the more general class of V in (PS) including the flow of the form $V = \alpha U + \delta W$ with small α and δ; see [45] for details.

Our first motivation is to generalize the result in [44] to the case when the domain loses symmetry (and the second one is explained in Remark 4.1.2 (3) below). Let us prepare the assumptions on the domain Ω and the stationary vector field V in (PS). We denote by $B_\rho(0)$ the two-dimensional disk of radius $\rho > 0$ centered at the origin.

Assumption 4.1.1 (1) *There is a positive constant $d \in (0, \frac{1}{4})$ such that the complement of the domain Ω satisfies*

$$\overline{B_{1-2d}(0)} \subset \Omega^{\mathrm{c}} \subset \overline{B_{1-d}(0)}\,. \tag{4.5}$$

(2) *Let the constants $\alpha \in (0, 1)$ and $d \in (0, \frac{1}{4})$ in* (4.5) *be sufficiently small. Then the vector field V in* (PS) *satisfies* $\operatorname{div} V = 0$ *in Ω and the asymptotic behavior*

$$V(x) \;=\; \beta U(x) + R(x)\,, \qquad x \in \Omega\,, \tag{4.6}$$

where $U(x)$ is the rotating flow in (4.4)*. The constant β and the remainder $R(x)$ are assumed to satisfy the following conditions with some $\gamma \in (\frac{1}{2}, 1)$ and $\kappa \in (0, 1)$:*

$$\beta \;=\; \alpha + \tilde{\alpha}_d\,, \qquad |\tilde{\alpha}_d| \le Cd\,, \qquad \beta \in (0, 1)\,, \tag{4.7}$$

$$\sup_{x \in \Omega} |x|^{1+\gamma} |R(x)| \le C\beta^\kappa d\,, \tag{4.8}$$

where the constant C depends only on γ.

Remark 4.1.2 (1) Formally taking $d = 0$ in (4.5)–(4.8) we obtain the flow $V = \alpha U$ in the exterior disk $\Omega = \mathbb{R}^2 \setminus \overline{B_1(0)}$, which solves the two-dimensional stationary Navier-Stokes equations (SNS): $-\Delta u + u \cdot \nabla u + \nabla p = f$, $\operatorname{div} u = 0$ in Ω, $u = b$ on $\partial\Omega$, and $u \to 0$ as $|x| \to \infty$ with $f = 0$ and $b = \alpha x^\perp$. The vector field V in (4.6)–(4.8) describes the flow around αU created from a small perturbation to the exterior disk, and hence, one can naturally expect the existence of such solutions to the nonlinear problem (SNS) if f and $b - \alpha x^\perp$ are sufficiently small with respect to $0 < d \ll 1$. Indeed, imposing the symmetry on the domain perturbation in (4.5), we can construct the Navier-Stokes flow V satisfying at least (4.6) and (4.7) for small symmetric given data, based on the energy method and the recovered Hardy-inequality (4.1) thanks to the symmetry of the domain Ω and the remainder R. We refer to Galdi [18], Russo [53], Yamazaki [58], and Pileckas and Russo [52] for the solvability of (SNS) under the symmetry condition. The reader is also referred to Hillairet and Wittwer [32] proving the existence of solutions to (SNS) in the exterior disk with $f = 0$ and $b = \alpha x^\perp + \tilde{b}$ when α is large enough and $\tilde{b}$ is sufficiently small.

(2) The novelty of our assumption is that we do not impose the symmetry either on the domain Ω and the flow V, and it is a crucial assumption for the stability analysis in [21, 59]

to resolve the difficulty related to the lack of the Hardy inequality. While one can realize the exterior disk case in [44] by putting $d = 0$ to (4.5)–(4.8) formally. In this sense, the assumption gives a generalization of the setting in [44] to non-symmetric domain cases.

(3) Another motivation for the assumption on V is explained as follows. In Chapter 2 we introduced a time-periodic planar Navier-Stokes flow (u, p) moving with an obstacle $\Omega^{\rm c}$ rotating around the origin at a constant speed $\alpha \in \mathbb{R} \setminus \{0\}$, and it is observed that (u, p) solves the following nonlinear problem (RNS): $\partial_t u - \Delta u - \alpha(x^{\perp} \cdot \nabla u - u^{\perp}) + u \cdot \nabla u + \nabla p = f$, $\operatorname{div} u = 0$ in Ω, $u = \alpha x^{\perp}$ on $\partial\Omega$, and $u \to 0$ as $|x| \to \infty$. We studied the stationary problem of (RNS) in Chapter 2 and proved the existence and uniqueness of stationary solutions decaying in the order $O(|x|^{-1})$ when α is sufficiently small and f is of a divergence form $f = \operatorname{div} F$ for some F which is small in a scale-critical norm. Moreover, the leading profile at spatial infinity is shown to be $C\frac{x^{\perp}}{|x|^2} + O(|x|^{-1-\gamma})$ for some constant C as long as F satisfies a decay condition $F = O(|x|^{-2-\gamma})$ with $\gamma \in (0, 1)$.

The motivation comes from the stability analysis of the stationary solutions to (RNS). Indeed, one can construct the solutions V to (RNS) satisfying the estimates (4.6), (4.7), and (4.8) with $\kappa = \frac{1-\gamma}{2}$ under the condition on the domain (4.5) (this result can be shown by extending the proof in Chapter 2 but we omit the details). Obviously, letting us denote the linearization to (RNS) around V by (PRS), then the two equations (PS) and (PRS) are different from each other due to the additional term $-\alpha(x^{\perp} \cdot \nabla v - v^{\perp})$ in (PRS). However, if we consider the *resolvent problems* of each equation, there are some common features thanks to the property of the term $\alpha(x^{\perp} \cdot \nabla v - v^{\perp}) = \sum_{n \in \mathbb{Z}} i\alpha n \mathcal{P}_n v$, which is derived from the Fourier expansion of $v|_{\{|x|>1\}}$; see (4.20) and (4.21) in Subsection 4.2.1. In particular, we can reproduce a similar calculation performed in this chapter to the resolvent problem of (PRS), by observing that the appearance of $\sum_{n \in \mathbb{Z}} i\alpha n \mathcal{P}_n v$ in the resolvent equation (restricted on $|x| > 1$) leads to the shifting of the resolvent parameter from $\lambda \in \mathbb{C}$ to $\lambda + in\alpha$ in the n-Fourier mode. Although the stability of the stationary solutions V to (RNS) still remains open, our analysis will contribute to the resolvent estimate of the problem (PRS).

Before stating the main result, let us introduce some notations and basic facts related to the problem (PS). We denote by $L^2_{\sigma}(\Omega)$ the L^2-closure of $C^{\infty}_{0,\sigma}(\Omega)$. The orthogonal projection $\mathbb{P} : L^2(\Omega)^2 \to L^2_{\sigma}(\Omega)$ is called the Helmholtz projection. Then the Stokes operator $\mathbb{A}$ with the domain $D_{L^2}(\mathbb{A}) = L^2_{\sigma}(\Omega) \cap W^{1,2}_0(\Omega)^2 \cap W^{2,2}(\Omega)^2$ is defined as $\mathbb{A} = -\mathbb{P}\Delta$, and it is well known that the Stokes operator is nonnegative and self-adjoint in $L^2_{\sigma}(\Omega)$. Finally we define the perturbed Stokes operator $\mathbb{A}_V$ as

$$\begin{aligned} D_{L^2}(\mathbb{A}_V) &= D_{L^2}(\mathbb{A}) \,, \\ \mathbb{A}_V v &= \mathbb{A} v + \mathbb{P}(V \cdot \nabla v + v \cdot \nabla V) \,. \end{aligned} \tag{4.9}$$

The perturbation theory for sectorial operators implies that $-\mathbb{A}_V$ generates a C_0-analytic semigroup in $L^2_{\sigma}(\Omega)$. We denote this semigroup by $e^{-t\mathbb{A}_V}$. Then our main result is stated as follows. Let d, β, and κ be the constants in Assumption 4.1.1.

Theorem 4.1.3 *There are positive constants β_* and μ_* such that if $\beta \in (0, \beta_*)$ and $d \in (0, \mu_* \beta^2)$ then the following statement holds. Let $q \in (1, 2]$. Then we have*

$$\|e^{-t\mathbb{A}_V} f\|_{L^2(\Omega)} \le \frac{C}{\beta^2} t^{-\frac{1}{q}+\frac{1}{2}} \|f\|_{L^q(\Omega)} \,, \qquad t > 0 \,, \tag{4.10}$$

$$\|\nabla e^{-t\mathbb{A}_V} f\|_{L^2(\Omega)} \le \frac{C}{\beta^2} t^{-\frac{1}{q}} \|f\|_{L^q(\Omega)} \,, \qquad t > 0 \,, \tag{4.11}$$

for $f \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$. Here the constant C is independent of β and depends on q.

The proof of the following result is omitted in this paper, since it is just a reproduction of the argument in [44, Section 4] using the Banach fixed point theorem.

$$v(t) \,=\, e^{-t\mathbb{A}_V} v_0 - \int_0^t e^{-(t-s)\mathbb{A}_V} \mathbb{P}(v\cdot\nabla v)(s)\,\mathrm{d}s\,, \qquad t>0\,. \tag{INS}$$

The proof of the following result is omitted in this chapter, since the argument is quite straightforward using the Banach fixed point theorem.

Theorem 4.1.4 *Let β_* and μ_* be the constants in Theorem 4.1.3. Then there is a positive constant ν_* such that if $\beta \in (0,\beta_*)$, $d \in (0,\mu_*\beta^2)$, and $\|v_0\|_{L^2(\Omega)} \in (0,\nu_*\beta^2)$ then there exists a unique solution $v \in C([0,\infty); L^2_\sigma(\Omega)) \cap C((0,\infty); W^{1,2}_0(\Omega)^2)$ to* (INS) *satisfying*

$$\lim_{t\to\infty} t^{\frac{k}{2}} \|\nabla^k v(t)\|_{L^2(\Omega)} \,=\, 0\,, \qquad k\,=\,0,1\,. \tag{4.12}$$

The proof of Theorem 4.1.3 relies on the resolvent estimate to the perturbed Stokes operator $\mathbb{A}_V$. Since the difference $\mathbb{A}_V - \mathbb{A}$ is relatively compact to $\mathbb{A}$ in $L^2_\sigma(\Omega)$, one can show that the spectrum of $-\mathbb{A}_V$ has the structure $\sigma(-\mathbb{A}_V) = (-\infty,0] \cup \sigma_{\rm disc}(-\mathbb{A}_V)$ in $L^2_\sigma(\Omega)$, where $\sigma_{\rm disc}(-\mathbb{A}_V)$ denotes the set of discrete spectrum of $-\mathbb{A}_V$; see [44, Lemma 2.11 and Proposition 2.12]. By using the identity $v\cdot\nabla v = \frac12\nabla|v|^2 + v^\perp{\rm rot}\,v$ with ${\rm rot}\,v = \partial_1 v_2 - \partial_2 v_1$ and ${\rm rot}\,U = 0$ in $x\in\Omega$, we can write the resolvent problem associated with (PS) as

$$\left\{\begin{aligned} \lambda v - \Delta v + \beta U^\perp {\rm rot}\,v + {\rm div}\,(R\otimes v + v\otimes R) + \nabla q \,&=\, f\,, \quad x\in\Omega\,,\\ {\rm div}\,v \,&=\, 0\,, \quad x\in\Omega\,,\\ v|_{\partial\Omega} \,&=\, 0\,. \end{aligned}\right. \tag{RS}$$

Here $\lambda\in\mathbb{C}$ is the resolvent parameter and we have used the conditions ${\rm div}\,v = {\rm div}\,R = 0$ to derive $R\cdot\nabla v + v\cdot\nabla R = {\rm div}\,(R\otimes v + v\otimes R)$. Hence, the proof of Theorem 4.1.3 is complete as soon as we show that there is a sector Σ included in the resolvent set $\rho(-\mathbb{A}_V)$, and that the following estimates to (RS) hold for $q\in(1,2]$ and $f\in L^2_\sigma(\Omega)\cap L^q(\Omega)^2$:

$$\begin{aligned} \|(\lambda+\mathbb{A}_V)^{-1} f\|_{L^2(\Omega)} &\le \frac{C}{\beta^2}|\lambda|^{-\frac32+\frac1q}\|f\|_{L^q(\Omega)}\,, \quad \lambda\in\Sigma\,,\\ \|\nabla(\lambda+\mathbb{A}_V)^{-1} f\|_{L^2(\Omega)} &\le \frac{C}{\beta^2}|\lambda|^{-1+\frac1q}\|f\|_{L^q(\Omega)}\,, \quad \lambda\in\Sigma\,. \end{aligned} \tag{4.13}$$

Let us prepare the ingredients for the proof of the resolvent estimates (4.13). Our approach is based on the energy method to (RS), and thus one of the most important steps is to obtain the estimate for the term $|\langle \beta U^\perp {\rm rot}\,v\,, v\rangle_{L^2(\Omega)}|$ which enables us to close the energy computation. Again we note that the bound $|\langle \beta U^\perp {\rm rot}\,v\,, v\rangle_{L^2(\Omega)}| \le C\beta\|\nabla v\|^2_{L^2(\Omega)}$ is no longer available contrary to the three-dimensional cases.

Firstly let us examine the next inequality containing the parameter $T\gg 1$:

$$|\langle \beta U^\perp {\rm rot}\,v, v\rangle_{L^2(\Omega)}| \le \frac{\beta}{T}\|\nabla v\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + C\beta\Theta(T)\|\nabla v\|^2_{L^2(\Omega)}\,, \tag{4.14}$$

where the function $\Theta(T)$ satisfies $\Theta(T)\approx \log T$ if $T\gg 1$. This inequality leads to the closed energy computation for (RS), as long as the coefficient $C\beta\Theta(T)$ is small enough so that the second term in the right-hand side of (4.14) can be controlled by the dissipation from the Laplacian in (RS). However, this observation does not give the information about the spectrum of $-\mathbb{A}_V$ near the origin. More precisely, we cannot close the energy computation when the resolvent parameter λ is exponentially small with respect to β, that is, when $0 < |\lambda| \le O(e^{-\frac{1}{\beta}})$. We emphasize that this difficulty is essentially due to the unavailability of the Hardy inequality (4.1) in two-dimensional exterior domains.

To overcome the difficulty for the case $0 < |\lambda| \le O(e^{-\frac{1}{\beta}})$, we rely on the representation formula to the resolvent problem in the exterior unit disk established in [44]. Since the restriction $(v|_{\{|x|>1\}}, q|_{\{|x|>1\}})$ gives a unique solution to the next problem for (w, r):

$$\begin{cases} \lambda w - \Delta w + \beta U^{\perp} \mathrm{rot}\, w + \nabla r \,=\, -\mathrm{div}\,(R \otimes v + v \otimes R) + f\,, \quad |x| > 1\,, \\ \mathrm{div}\, w \,=\, 0\,, \quad |x| > 1\,, \\ w|_{\{|x|=1\}} \,=\, v|_{\{|x|=1\}}\,, \end{cases} \tag{RS$^{\mathrm{ed}}$}$$

we can study the a priori estimates of $w = v|_{\{|x|>1\}}$ based on the solution formula to (RS$^{\mathrm{ed}}$). Then a detailed calculation shows that $|\langle \beta U^{\perp} \mathrm{rot}\, v, v\rangle_{L^2(\{|x|>1\})}|$ satisfies

$$\begin{aligned} &|\langle \beta U^{\perp} \mathrm{rot}\, v, v\rangle_{L^2(\{|x|>1\})}| \\ &\le \frac{C}{\beta^4} \big(\|R \otimes v + v \otimes R\|_{L^2(\Omega)} + \beta \sum_{|n|=1} \|\mathcal{P}_n v\|_{L^\infty(\{|x|=1\})}\big)^2 \\ &\quad + \frac{C}{\beta^4} |\lambda|^{-2+\frac{2}{q}} \|f\|^2_{L^q(\Omega)} + C\beta \|\nabla v\|^2_{L^2(\Omega)}\,, \end{aligned} \tag{4.15}$$

where $\mathcal{P}_n v$ denotes the Fourier n-mode of $v|_{\{|x|\ge 1\}}$; see (4.21) in Subsection 4.2.1 for the definition. Once we obtain (4.15) then the estimate of $|\langle \beta U^{\perp} \mathrm{rot}\, v, v\rangle_{L^2(\Omega)}|$ is derived by using the Poincaré inequality on the bounded domain $\Omega \setminus \{|x| \ge 1\}$. However, in closing the energy computation, we need to be careful about the β-singularity in the coefficients in (4.15). In fact, the first term in the right-hand side of (4.15) has to be controlled by the dissipation as

$$\big(\|R \otimes v + v \otimes R\|_{L^2(\Omega)} + \beta \sum_{|n|=1} \|\mathcal{P}_n v\|_{L^\infty(\{|x|=1\})}\big)^2 \le \frac{C}{\beta^4} (\beta^\kappa d + \beta d^{\frac{1}{2}})^2 \|\nabla v\|^2_{L^2(\Omega)}\,,$$

and then the smallness of $C(\beta^\kappa d + \beta d^{\frac{1}{2}})^2 \beta^{-4} \ll 1$ is required in order to close the energy computation. This condition is achieved by imposing the smallness on the distance d between the domain Ω and the exterior unit disk, which is introduced in Assumption 4.1.1.

Next we pay close attention to the β-dependencies appearing in Theorem 4.1.3. If we consider the limit case $d = 0$ and $V = \alpha U$ in Assumption 4.1.1, then the term

$$\beta U^{\perp} \mathrm{rot}\, v + \mathrm{div}\,(R \otimes v + v \otimes R) \,=\, \alpha U^{\perp} \mathrm{rot}\, v$$

in (RS) has an oscillation effect on the solutions in the exterior disk $\Omega = \{|x| > 1\}$ at least when $\lambda = 0$. Indeed, for the solutions to (RS) with $\lambda = 0$, this effect leads to the faster spatial decay compared with the case $\alpha = 0$ (i.e. the Stokes equations case), and this observation is indeed an important step in [32] to prove the existence of the Navier-Stokes flows around αU in the exterior disk when the rotation α is large, as explained in Remark 4.1.2 (1). However, contrary to the stationary problem, the situation becomes more complicated if we consider the nonstationary problem requiring the analysis of (RS) for nonzero $\lambda \in \mathbb{C} \setminus \{0\}$, since there is an interaction between the two oscillation effects due to the terms λv and $\alpha U^{\perp} \mathrm{rot}\, v$ in (RS). In fact, even in the exterior disk, a detailed analysis to the representation of the resolvent operator suggests the existence of a time-frequency domain, which we call the *nearly-resonance regime*, where the oscillation effect from $\alpha U^{\perp} \mathrm{rot}\, v$ is drastically weakened by the one from λv and the α-singularity appears in the operator norm of the resolvent. The existence of the nearly-resonance regime yields that the stability of the αU-type flows is sensitive under the perturbation of the domain. This is the reason why the distance d between the fluid domain Ω and the exterior disk is assumed to be small depending on $\beta = \alpha + \tilde{\alpha}_d$ in Theorem 4.1.3. Additionally, Lemma 4.5.6 in Appendix 4.5.3 implies that the nearly-resonance regime lies in the annulus $e^{-\frac{c}{\beta^2}} \leq |\lambda| \leq e^{-\frac{c'}{\beta}}$ in the complex plane. As far as the author knows, the existence of such time-frequency domain and the qualitative analysis seem to be new and have not been achieved before.

Finally, let us briefly sketch an extension of the main theorems. Since there are no structural assumptions on the fluid domain in Theorems 4.1.3 and 4.1.4, we can obtain the solutions to (INS) even for the case when the boundary oscillates highly, namely for the *rough boundary* case. Understanding the rough boundary effect on the dynamics of fluid flows is one of the fundamental themes in fluid mechanics, and this problem arises in many real applications such as the flows on surfaces with fine riblets; we refer to Mikelić [46] for an overview on this field. When the boundary is rough, because of the no-slip boundary condition implemented in the semigroup $e^{-t\mathbb{A}_V}$, the solution to (INS) naturally possesses a highly oscillating part near the boundary. This structure is called the *boundary layer* due to the boundary roughness, and the analysis is widely performed in both theoretical and numerical manners. Concerning the mathematical study, the reader is referred to Jäger and Mikelić [38] for the stationary flow, and to Mikelić, Nečasová, and Neuss-Radu [47], and Gérard-Varet, Lacave, Nguyen, and Roussetd [24] for the non-stationary flow under a smooth external force with zero initial data. In particular, the inviscid limit of the rough domain flow is considered in [24] when a suitable relation holds between the viscosity and the boundary oscillation parameter. For the initial-boundary value problem as in (INS), the study of the rough domain flow is more complicated and has a specific difficulty. Indeed, to describe a profile of the solution possessing an oscillating structure near the boundary, we need to rely on a higher order expansion of the solution in the small oscillation parameter. The verification of this expansion especially requires a compatibility condition and regularity of the initial data because of the appearance of an initial layer in the equations. The only result available so far is the perturbed half-plane case [28] by the author where the flow oscillating near the rough boundary is constructed by verifying the higher order expansion for C^1 initial data with a natural compatibility condition.

On the other hand, from the view of the spatial asymptotic behavior, the rough domain flow in [28] has no precise information in general since its profile in the far field is just described by a nonstationary Navier-Stokes flow in the half-plane. In order to obtain a precise asymptotics of a flow in an unbounded domain with a rough boundary, it is natural to consider the perturbed equations around a given stationary solution and to construct actually the perturbation with an oscillating structure near the boundary. However, the analysis on the rough domain flow around a stationary solution seems to be restricted to the stationary (channel flow) case, and there are no results for the nonstationary case as far as the author knows. The results in Theorems 4.1.3 and 4.1.4 provide the first step to extend the result of [28] in this direction to the nonstationary exterior problem (INS). Especially, by applying the ideas in [28], we will be able to construct a profile of the solution to (INS) which behaves like the scale-critical rotating flow $\beta\frac{x^\perp}{|x|^2}$ in the far field, and at the same time possesses a boundary layer structure near the boundary. The analysis of flows with such complex localized structure seems to be new and to have its own interest.

This chapter is organized as follows. In Section 4.2 we recall some basic facts from vector calculus in polar coordinates, and derive the resolvent estimate to (RS) when $|\lambda| \geq O(\beta^2 e^{-\frac{1}{6\beta}})$ by a standard energy method. In Section 4.3 the resolvent problem is discussed for the case $0 < |\lambda| < e^{-\frac{1}{6\beta}}$. In Subsections 4.3.1, 4.3.2, and 4.3.3 we derive the estimates to the problem ($\mathrm{RS}^{\mathrm{ed}}$) by using the representation formula. The results in Subsections 4.3.1–4.3.3 are applied in Subsection 4.3.4, where the resolvent estimate to (RS) is established in the exceptional region $0 < |\lambda| < e^{-\frac{1}{6\beta}}$. Section 4.4 is devoted to the proof of Theorem 4.1.3.

4.2 Preliminaries

This section is devoted to the preliminary analysis on the resolvent problem (RS) and ($\mathrm{RS}^{\mathrm{ed}}$) in the introduction. In Subsections 4.2.1 and 4.2.2 we recall some basic facts from vector calculus in polar coordinates. In Subsection 4.2.3 we show that the resolvent estimates in (4.13) are valid if the resolvent parameter λ satisfies $|\lambda| \geq O(\beta^2 e^{-\frac{1}{6\beta}})$. Throughout this section, let us denote by D the exterior unit disk $\mathbb{R}^2 \setminus \overline{B_1(0)} = \{x \in \mathbb{R}^2 \mid |x| > 1\}$.

4.2.1 Vector calculus in polar coordinates and Fourier series

The results here have some overlaps with the ones in Subsection 3.2.1 of Chapter 3. We restate them nevertheless for reader's convenience since the notations are slightly changed.

We introduce the usual polar coordinates on D. Set

$$x_1 = r\cos\theta\,, \qquad x_2 = r\sin\theta\,, \qquad r = |x| \geq 1\,, \quad \theta \in [0, 2\pi)\,,$$
$$\mathbf{e}_r = \frac{x}{|x|}\,, \qquad \mathbf{e}_\theta = \frac{x^\perp}{|x|} = \partial_\theta \mathbf{e}_r\,.$$

Let $v = (v_1, v_2)^\top$ be a vector field defined on D. Then we set

$$v = v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta\,, \qquad v_r = v \cdot \mathbf{e}_r\,, \qquad v_\theta = v \cdot \mathbf{e}_\theta\,.$$

The following formulas will be used:

$$\operatorname{div} v = \partial_1 v_1 + \partial_2 v_2 = \frac{1}{r}\partial_r(r v_r) + \frac{1}{r}\partial_\theta v_\theta\,, \tag{4.16}$$

$$\operatorname{rot} v = \partial_1 v_2 - \partial_2 v_1 = \frac{1}{r}\partial_r(r v_\theta) - \frac{1}{r}\partial_\theta v_r\,, \tag{4.17}$$

$$|\nabla v|^2 = |\partial_r v_r|^2 + |\partial_r v_\theta|^2 + \frac{1}{r^2}\big(|\partial_\theta v_r - v_\theta|^2 + |v_r + \partial_\theta v_\theta|^2\big)\,, \tag{4.18}$$

and

$$\begin{aligned} -\Delta v = &\Big(-\partial_r\big(\frac{1}{r}\partial_r(r v_r)\big) - \frac{1}{r^2}\partial_\theta^2 v_r + \frac{2}{r^2}\partial_\theta v_\theta\Big)\mathbf{e}_r \\ &+ \Big(-\partial_r\big(\frac{1}{r}\partial_r(r v_\theta)\big) - \frac{1}{r^2}\partial_\theta^2 v_\theta - \frac{2}{r^2}\partial_\theta v_r\Big)\mathbf{e}_\theta\,. \end{aligned} \tag{4.19}$$

The formulas

$$\mathbf{e}_r \cdot \nabla v \;=\; (\partial_r v_r)\mathbf{e}_r + (\partial_r v_\theta)\mathbf{e}_\theta\,, \qquad \mathbf{e}_\theta \cdot \nabla v \;=\; \frac{\partial_\theta v_r - v_\theta}{r}\mathbf{e}_r + \frac{\partial_\theta v_\theta + v_r}{r}\mathbf{e}_\theta$$

imply the following equality:

$$\begin{aligned} x^\perp \cdot \nabla v - v^\perp &= |x|\big(\mathbf{e}_\theta \cdot \nabla v\big) - \big(v_r \mathbf{e}_r^\perp + v_\theta \mathbf{e}_\theta^\perp\big) \\ &= (\partial_\theta v_r - v_\theta)\,\mathbf{e}_r \;+\; (\partial_\theta v_\theta + v_r)\,\mathbf{e}_\theta - \big(v_r \mathbf{e}_r^\perp + v_\theta \mathbf{e}_\theta^\perp\big) \\ &= \partial_\theta v_r\,\mathbf{e}_r + \partial_\theta v_\theta\,\mathbf{e}_\theta\,. \end{aligned} \tag{4.20}$$

For each $n \in \mathbb{Z}$, we denote by $\mathcal{P}_n$ the projection on the Fourier mode n with respect to the angular variable θ:

$$\mathcal{P}_n v \;=\; v_{r,n}(r)e^{in\theta}\mathbf{e}_r + v_{\theta,n}(r)e^{in\theta}\mathbf{e}_\theta\,, \tag{4.21}$$

where

$$\begin{aligned} v_{r,n}(r) \;&=\; \frac{1}{2\pi}\int_0^{2\pi} v_r(r\cos\theta, r\sin\theta)e^{-in\theta}\,\mathrm{d}\theta\,, \\ v_{\theta,n}(r) \;&=\; \frac{1}{2\pi}\int_0^{2\pi} v_\theta(r\cos\theta, r\sin\theta)e^{-in\theta}\,\mathrm{d}\theta\,. \end{aligned}$$

We also set for $m \in \mathbb{N} \cup \{0\}$,

$$\mathcal{Q}_m v = \sum_{|n|=m+1}^{\infty} \mathcal{P}_n v\,. \tag{4.22}$$

For notational simplicity we often write v_n instead of $\mathcal{P}_n v$. Each $\mathcal{P}_n$ defines an orthogonal projection in $L^2(D)^2$. From (4.18) and (4.21), for $n \in \mathbb{N} \cup \{0\}$ and v in $W^{1,2}(D)^2$ we have

$$\begin{aligned} \|\nabla v\|_{L^2(D)}^2 \;&=\; \sum_{n\in\mathbb{Z}} \|\nabla \mathcal{P}_n v\|_{L^2(D)}^2\,, \\ |\nabla \mathcal{P}_n v|^2 \;&=\; |\partial_r v_{r,n}|^2 + \frac{1+n^2}{r^2}|v_{r,n}|^2 + |\partial_r v_{\theta,n}|^2 + \frac{1+n^2}{r^2}|v_{\theta,n}|^2 - \frac{4n}{r^2}\mathrm{Im}(v_{\theta,n}\overline{v_{r,n}})\,. \end{aligned}$$

In particular, we have

$$|\nabla \mathcal{P}_n v|^2 \geq |\partial_r v_{r,n}|^2 + \frac{(|n|-1)^2}{r^2}|v_{r,n}|^2 + |\partial_r v_{\theta,n}|^2 + \frac{(|n|-1)^2}{r^2}|v_{\theta,n}|^2\,, \tag{4.23}$$

and thus, from the definition of $\mathcal{Q}_m$ in (4.22), we have for $m \in \mathbb{N} \cup \{0\}$,

$$\|\nabla \mathcal{Q}_m v\|_{L^2(D)}^2 \geq \|\partial_r (\mathcal{Q}_m v)_r\|_{L^2(D)}^2 + \|\partial_r (\mathcal{Q}_m v)_\theta\|_{L^2(D)}^2 + m^2 \|\frac{v}{|x|}\|_{L^2(D)}^2\,. \tag{4.24}$$

4.2.2 The Biot-Savart law in polar coordinates

The results here have some overlaps with the ones in Subsection 3.2.2 of Chapter 3. We restate them nevertheless for reader's convenience since the notations are slightly changed.

For a given scalar field ω in D, the streamfunction ψ is formally defined as the solution to the Poisson equation: $-\Delta\psi = \omega$ in D and $\psi = 0$ on ∂D. For $n \in \mathbb{Z}$ and $\omega \in L^2(D)$ we set $\mathcal{P}_n\omega = \mathcal{P}_n\omega(r,\theta)$ and $\omega_n = \omega_n(r)$ as

$$\mathcal{P}_n\omega = \Big(\frac{1}{2\pi}\int_0^{2\pi} \omega(r\cos s, r\sin s)e^{-ins}\,\mathrm{d}s\Big)e^{in\theta}\,, \qquad \omega_n = \big(\mathcal{P}_n\omega\big)e^{-in\theta}\,. \tag{4.25}$$

From the Poisson equation in polar coordinates, we see that each n-Fourier mode of ψ satisfies the following ODE:

$$-\frac{\mathrm{d}\psi_n}{\mathrm{d}r^2} - \frac{1}{r}\frac{\mathrm{d}\psi_n}{\mathrm{d}r} + \frac{n^2}{r^2}\psi_n = \omega_n\,, \quad r > 1\,, \qquad \psi_n(1) = 0\,. \tag{4.26}$$

For $|n| \geq 1$, the solution $\psi_n = \psi_n[\omega_n]$ to (4.26) decaying at spatial infinity is given by

$$\begin{aligned}
\psi_n[\omega_n](r) &= \frac{1}{2|n|}\Big(-\frac{d_n[\omega_n]}{r^{|n|}} + \frac{1}{r^{|n|}}\int_1^r s^{1+|n|}\omega_n(s)\,\mathrm{d}s + r^{|n|}\int_r^\infty s^{1-|n|}\omega_n(s)\,\mathrm{d}s\Big)\,,\\
d_n[\omega_n] &= \int_1^\infty s^{1-|n|}\omega_n(s)\,\mathrm{d}s\,.
\end{aligned}$$

The formula $V_n[\omega_n]$ in the next is called the Biot-Savart law for $\mathcal{P}_n\omega$:

$$\begin{aligned}
V_n[\omega_n] &= V_{r,n}[\omega_n](r)e^{in\theta}\mathbf{e}_r + V_{\theta,n}[\omega_n](r)e^{in\theta}\mathbf{e}_\theta\,,\\
V_{r,n}[\omega_n] &= \frac{in}{r}\psi_n[\omega_n]\,, \qquad V_{\theta,n}[\omega_n] = -\frac{\mathrm{d}}{\mathrm{d}r}\psi_n[\omega_n]\,.
\end{aligned} \tag{4.27}$$

The velocity $V_n[\omega_n]$ is well defined at least when $r^{1-|n|}\omega_n \in L^1((1,\infty))$, and it is straightforward to see that

$$\begin{aligned}
&\operatorname{div} V_n[\omega_n] = 0\,, \quad \operatorname{rot} V_n[\omega_n] = \mathcal{P}_n\omega \quad \text{in } D\,,\\
&\mathbf{e}_r\cdot V_n[\omega_n] = 0 \quad \text{on } \partial D\,.
\end{aligned} \tag{4.28}$$

The condition $r^{1-|n|}\omega_n \in L^1((1,\infty))$ is automatically satisfied when $\omega \in L^2(D)$ and $|n| \geq 2$. When $|n| = 1$, however, the integral in the definition of $\psi_n[\omega_n]$ does not converge absolutely for general $\omega \in L^2(D)$. We can justify this integral for $|n| = 1$ if ω is given in a rotation form $\omega = \operatorname{rot} u$ with some $u \in W^{1,2}(D)^2$, since the integration by parts leads to the convergence of $\lim_{N\to\infty}\int_r^N \omega_n\,\mathrm{d}r$. Hence, for any $v \in L^2_\sigma(D)\cap W^{1,2}(D)^2$, the n-mode $v_n = \mathcal{P}_n v$ can be expressed in terms of its vorticity ω_n by the formula (4.27) when $|n| \geq 1$.

4.2.3 A priori resolvent estimate by energy method

In this subsection we study the energy estimate to the resolvent problem (RS):

$$\begin{cases}
\lambda v - \Delta v + \beta U^\perp \operatorname{rot} v + \operatorname{div}(R\otimes v + v\otimes R) + \nabla q = f\,, & x \in \Omega\,,\\
\operatorname{div} v = 0\,, & x \in \Omega\,,\\
v|_{\partial\Omega} = 0\,.
\end{cases} \tag{RS}$$

Here $\lambda \in \mathbb{C}$ is the resolvent parameter, U is the rotating flow of (4.4) in the introduction, and β and R are defined in Assumption 4.1.1. The first result of this subsection is the a priori estimates to (RS) obtained by the energy method. We recall that D denotes the exterior disk $\{x \in \mathbb{R}^2 \mid |x| > 1\}$, and that γ and κ are the constants in Assumption 4.1.1.

Proposition 4.2.1 *Let $q \in (1,2]$, $f \in L^q(\Omega)^2$, and $\lambda \in \mathbb{C}$. Suppose that $v \in D(\mathbb{A}_V)$ is a solution to* (RS)*. Then there is a constant $\beta_1 \in (0,1)$ depending only on Ω, γ, and κ such that the following estimates hold.*

$$\begin{aligned}
&\mathrm{Re}(\lambda)\|v\|_{L^2(\Omega)}^2 + \frac{3}{4}\|\nabla v\|_{L^2(\Omega)}^2 \\
&\quad \le \beta\Big|\sum_{|n|=1}\big\langle(\mathrm{rot}\, v)_n, \frac{v_{r,n}}{|x|}\big\rangle_{L^2(D)}\Big| + C\|f\|_{L^q(\Omega)}^{\frac{2q}{3q-2}}\|v\|_{L^2(\Omega)}^{\frac{4(q-1)}{3q-2}}\,, \qquad (4.29)\\
&|\mathrm{Im}(\lambda)|\|v\|_{L^2(\Omega)}^2 \le \frac{1}{4}\|\nabla v\|_{L^2(\Omega)}^2 + \beta\Big|\sum_{|n|=1}\big\langle(\mathrm{rot}\, v)_n, \frac{v_{r,n}}{|x|}\big\rangle_{L^2(D)}\Big| \\
&\qquad\qquad + C\|f\|_{L^q(\Omega)}^{\frac{2q}{3q-2}}\|v\|_{L^2(\Omega)}^{\frac{4(q-1)}{3q-2}}\,, \qquad (4.30)
\end{aligned}$$

as long as $\beta \in (0,\beta_1)$. The constant C is independent of β.

Proof: Taking the inner product with v to the first equation of (RS), we find

$$\begin{aligned}
&\mathrm{Re}(\lambda)\|v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2 \\
&\quad = -\beta\mathrm{Re}\langle U^\perp \mathrm{rot}\, v, v\rangle_{L^2(\Omega)} + \mathrm{Re}\langle R\otimes v + v\otimes R, \nabla v\rangle_{L^2(\Omega)} + \mathrm{Re}\langle f, v\rangle_{L^2(\Omega)}\,, \qquad (4.31)\\
&\mathrm{Im}(\lambda)\|v\|_{L^2(\Omega)}^2 \\
&\quad = -\beta\mathrm{Im}\langle U^\perp \mathrm{rot}\, v, v\rangle_{L^2(\Omega)} + \mathrm{Im}\langle R\otimes v + v\otimes R, \nabla v\rangle_{L^2(\Omega)} + \mathrm{Im}\langle f, v\rangle_{L^2(\Omega)}\,. \qquad (4.32)
\end{aligned}$$

After decomposing the domain $\Omega = (\Omega \setminus D) \cup D$, from $U^\perp = -\frac{\mathbf{e_r}}{r}$ on D we have

$$\beta|\langle U^\perp \mathrm{rot}\, v, v\rangle_{L^2(\Omega)}| \le \beta|\langle U^\perp \mathrm{rot}\, v, v\rangle_{L^2(\Omega\setminus D)}| + \beta\big|\big\langle \mathrm{rot}\, v, \frac{v_r}{|x|}\big\rangle_{L^2(D)}\big|\,. \qquad (4.33)$$

Then the Poincare inequality on $\Omega \setminus D$ implies that

$$\beta|\langle U^\perp \mathrm{rot}\, v, v\rangle_{L^2(\Omega\setminus D)}| \le C\beta\|\nabla v\|_{L^2(\Omega)}^2\,, \qquad (4.34)$$

and by applying the Fourier series expansion on D, we see from (4.21) and (4.25) that

$$\begin{aligned}
\big|\big\langle \mathrm{rot}\, v, \frac{v_r}{|x|}\big\rangle_{L^2(D)}\big| &= \Big|\Big(\sum_{|n|=1} + \sum_{n\in\mathbb{Z}\setminus\{\pm1\}}\Big)\big\langle(\mathrm{rot}\, v)_n, \frac{v_{r,n}}{|x|}\big\rangle_{L^2(D)}\Big| \\
&\le \Big|\sum_{|n|=1}\big\langle(\mathrm{rot}\, v)_n, \frac{v_{r,n}}{|x|}\big\rangle_{L^2(D)}\Big| + \sum_{n\in\mathbb{Z}\setminus\{\pm1\}}\|\mathrm{rot}\, v_n\|_{L^2(D)}\Big\|\frac{v_{r,n}}{|x|}\Big\|_{L^2(D)}\,. \qquad (4.35)
\end{aligned}$$

Then the inequality (4.24) ensures that

$$\begin{aligned}
\sum_{n\in\mathbb{Z}\setminus\{\pm1\}}\|\mathrm{rot}\, v_n\|_{L^2(D)}\Big\|\frac{v_{r,n}}{|x|}\Big\|_{L^2(D)} &\le C\sum_{n\in\mathbb{Z}\setminus\{\pm1\}}\|\nabla v_n\|_{L^2(D)}^2 \\
&\le C\|\nabla v\|_{L^2(\Omega)}^2\,. \qquad (4.36)
\end{aligned}$$

Inserting (4.34)–(4.36) into (4.33) we obtain

$$\beta|\langle U^\perp \mathrm{rot}\, v, v\rangle_{L^2(\Omega)}| \le C_1\beta\|\nabla v\|_{L^2(\Omega)}^2 + \beta\Big|\sum_{|n|=1}\big\langle(\mathrm{rot}\, v)_n, \frac{v_{r,n}}{|x|}\big\rangle_{L^2(D)}\Big|\,. \qquad (4.37)$$

Next by (4.8) in Assumption 4.1.1 we have

$$|\langle R \otimes v + v \otimes R, \nabla v \rangle_{L^2(\Omega)}| \le C_2 \beta^\kappa d \, \|\nabla v\|_{L^2(\Omega)}^2 \,, \tag{4.38}$$

where the inequality $\| |x|^{-(1+\gamma)} v \|_{L^2} \le C \|\nabla v\|_{L^2}$ is applied. The constant C_2 depends on $\gamma \in (0,1)$. By the Gagliardo-Nirenberg inequality we see that for $q \in (1,2]$ and $q' = \frac{q}{q-1}$,

$$\begin{aligned} |\langle f, v \rangle_{L^2(\Omega)}| &\le C \|f\|_{L^q(\Omega)} \|u\|_{L^{q'}(\Omega)} \\ &\le C \|f\|_{L^q(\Omega)} \|u\|_{L^2(\Omega)}^{2(1-\frac{1}{q})} \|\nabla u\|_{L^2(\Omega)}^{\frac{2}{q}-1} \\ &\le C \|f\|_{L^q(\Omega)}^{\frac{2q}{3q-2}} \|u\|_{L^2(\Omega)}^{\frac{4(q-1)}{3q-2}} + \frac{1}{8} \|\nabla u\|_{L^2(\Omega)}^2 \,, \end{aligned} \tag{4.39}$$

where the Young inequality is applied in the last line. Now we take β_1 small enough so that

$$C_1 \beta_1 + C_2 \beta_1^\kappa d \le 2 \max\{C_1, C_2\} \beta_1^\kappa \le \frac{1}{8} \tag{4.40}$$

holds for $\kappa \in (0,1)$. Then the assertions (4.29) and (4.30) are proved by inserting (4.37)–(4.39) into (4.31) and (4.32), and using (4.40). This completes the proof. □

As is seen from Proposition 4.2.1, the key object in closing the energy computation is to derive the estimate for the next term appearing in the right-hand sides of (4.29) and (4.30):

$$\Big| \sum_{|n|=1} \langle (\mathrm{rot}\, v)_n, \frac{v_{r,n}}{|x|} \rangle_{L^2(D)} \Big| \,.$$

Note that the Hardy inequality (4.23) cannot be applied to this term. The next proposition shows that this term can be handled if λ in (RS) satisfies $|\lambda| \ge O(\beta^2 e^{-\frac{1}{6\beta}})$.

Proposition 4.2.2 *Let β_1 be the constant in Proposition 4.2.1. Then the following statements hold.*
(1) *Fix a positive number $\beta_2 \in (0, \min\{\frac{1}{12}, \beta_1\})$. Then the set*

$$\mathcal{S}_\beta \,=\, \big\{ \lambda \in \mathbb{C} \mid |\mathrm{Im}(\lambda)| > -\mathrm{Re}(\lambda) + 12 e^{\frac{1}{e}} \beta^2 e^{-\frac{1}{6\beta}} \big\} \tag{4.41}$$

is included in the resolvent $\rho(-\mathbb{A}_V)$ for any $\beta \in (0, \beta_2)$.
(2) *Let $q \in (1,2]$ and $f \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$. Then we have*

$$\begin{aligned} \|(\lambda + \mathbb{A}_V)^{-1} f\|_{L^2(\Omega)} &\le C |\lambda|^{-\frac{3}{2} + \frac{1}{q}} \|f\|_{L^q(\Omega)} \,, \qquad \lambda \in \mathcal{S}_\beta \cap \mathcal{B}_{\frac{1}{2} e^{-\frac{1}{6\beta}}}(0)^{\mathrm{c}} \,, \\ \|\nabla (\lambda + \mathbb{A}_V)^{-1} f\|_{L^2(\Omega)} &\le C |\lambda|^{-1 + \frac{1}{q}} \|f\|_{L^q(\Omega)} \,, \qquad \lambda \in \mathcal{S}_\beta \cap \mathcal{B}_{\frac{1}{2} e^{-\frac{1}{6\beta}}}(0)^{\mathrm{c}} \,, \end{aligned} \tag{4.42}$$

as long as $\beta \in (0, \beta_2)$. Here the constant C is independent of β and $\mathcal{B}_\rho(0) \subset \mathbb{C}$ denotes the disk centered at the origin with radius $\rho > 0$.

Proof: (1) Let us denote the function space $L^q(\Omega)$ by L^q in this proof to simplify notation. Let $|n| = 1$, $\beta \in (0, \beta_2)$, and $v \in D(\mathbb{A}_V)$ solve (RS). Define a function $\Theta = \Theta(T)$ by

$$\Theta(T) \,=\, \int_0^T \frac{1}{\tau} e^{-\frac{1}{\tau}} \, \mathrm{d}\tau \,, \quad T > e \,, \tag{4.43}$$

which satisfies the following lower and upper bounds:

$$e^{-\frac{1}{e}} \log T \le \Theta(T) \le \log T\,, \quad T > e\,, \tag{4.44}$$

which can be easily checked. Then, as is shown in [44, Lemma 3.26], we have

$$\beta \big| \big\langle (\mathrm{rot}\, v)_n, \frac{v_{r,n}}{|x|} \big\rangle_{L^2(D)} \big| \le \frac{\beta}{T} \|v\|_{L^2} \, \|\nabla v\|_{L^2} + \beta \Theta(T) \|\nabla v\|_{L^2}^2\,, \quad T > e\,. \tag{4.45}$$

The proof is done by extending $v \in D(\mathbb{A}_V)$ by zero to the whole space $\mathbb{R}^2$ and using the nondegenerate condition $\{x \in \mathbb{R}^2 \mid |x| \le \frac{1}{2}\} \subset \Omega^{\mathrm{c}}$. Then the Young inequality yields

$$\frac{\beta}{T} \|v\|_{L^2} \|\nabla v\|_{L^2} \le \frac{\beta \Theta(T)}{2} \|\nabla v\|_{L^2}^2 + \frac{\beta}{2T^2 \Theta(T)} \|v\|_{L^2}^2\,. \tag{4.46}$$

Inserting (4.45) and (4.46) into (4.29) and (4.30) in Proposition 4.2.1, we see that

$$\big(\mathrm{Re}(\lambda) - \frac{\beta}{2T^2\Theta(T)}\big) \|v\|_{L^2}^2 + \big(\frac{3}{4} - \frac{3\beta\Theta(T)}{2}\big) \|\nabla v\|_{L^2}^2 \le C \|f\|_{L^q}^{\frac{2q}{3q-2}} \|v\|_{L^2}^{\frac{4(q-1)}{3q-2}}\,, \tag{4.47}$$

$$\big(|\mathrm{Im}(\lambda)| - \frac{\beta}{2T^2\Theta(T)}\big) \|v\|_{L^2}^2 \le \big(\frac{1}{4} + \frac{3\beta\Theta(T)}{2}\big) \|\nabla v\|_{L^2}^2 + C \|f\|_{L^q}^{\frac{2q}{3q-2}} \|v\|_{L^2}^{\frac{4(q-1)}{3q-2}}\,. \tag{4.48}$$

Then (4.47) and (4.48) lead to

$$\begin{aligned} &\big(|\mathrm{Im}(\lambda)| + \mathrm{Re}(\lambda) - \frac{\beta}{T^2\Theta(T)}\big) \|v\|_{L^2}^2 + \big(\frac{1}{2} - 3\beta\Theta(T)\big) \|\nabla v\|_{L^2}^2 \\ &\le C \|f\|_{L^q}^{\frac{2q}{3q-2}} \|v\|_{L^2}^{\frac{4(q-1)}{3q-2}}\,. \end{aligned} \tag{4.49}$$

Now let us take $T = e^{\frac{1}{12\beta}}$. Since $T > e$ by the condition $\beta \in (0, \frac{1}{12})$, from (4.44) we have

$$3\beta\Theta(T) \le 3\beta \log T = \frac{1}{4} \quad \text{and} \quad \frac{\beta}{T^2\Theta(T)} \le \frac{e^{\frac{1}{e}}\beta}{T^2 \log T} = 12 e^{\frac{1}{e}} \beta^2 e^{-\frac{1}{6\beta}}\,. \tag{4.50}$$

By inserting (4.50) into (4.49) we obtain the assertion $\mathcal{S}_\beta \subset \rho(-\mathbb{A}_V)$.

(2) Let $\lambda \in \mathcal{S}_\beta \cap \mathcal{B}_{\frac{1}{2}e^{-\frac{1}{6\beta}}}(0)^{\mathrm{c}}$. If additionally $\lambda \in \{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0\}$ then we have

$$|\mathrm{Im}(\lambda)| \ge \frac{\beta}{T^2\Theta(T)} \quad \text{and} \quad |\mathrm{Im}(\lambda)| \le |\lambda| \le \sqrt{2} |\mathrm{Im}(\lambda)|\,.$$

Then we see from (4.48) and (4.49) that,

$$|\lambda| \|v\|_{L^2}^2 \le \frac{6\sqrt{2}}{8} \|\nabla v\|_{L^2}^2 + 2\sqrt{2} C \|f\|_{L^q}^{\frac{2q}{3q-2}} \|v\|_{L^2}^{\frac{4(q-1)}{3q-2}}\,, \tag{4.51}$$

$$\|\nabla v\|_{L^2}^2 \le 4C \|f\|_{L^q}^{\frac{2q}{3q-2}} \|v\|_{L^2}^{\frac{4(q-1)}{3q-2}}\,, \tag{4.52}$$

where the constant C is independent of β. On the other hand, if additionally $\lambda \in \{z \in \mathbb{C} \mid \mathrm{Re}(z) \ge 0\}$ then we have from (4.50),

$$|\mathrm{Im}(\lambda)| + \mathrm{Re}(\lambda) - \frac{\beta}{T^2\Theta(T)} \ge |\lambda| - 12 e^{\frac{1}{e}} \beta^2 e^{-\frac{1}{6\beta}} \ge \frac{|\lambda|}{2}\,,$$

since $12 e^{\frac{1}{e}} \beta^2 e^{-\frac{1}{6\beta}} \le 24 e^{\frac{1}{e}} \beta^2 |\lambda| \le \frac{|\lambda|}{2}$ holds by $\beta \in (0, \frac{1}{12})$. Then from (4.49) we see that,

$$|\lambda| \|v\|_{L^2}^2 + \frac{1}{2} \|\nabla v\|_{L^2}^2 \le 2C \|f\|_{L^q}^{\frac{2q}{3q-2}} \|v\|_{L^2}^{\frac{4(q-1)}{3q-2}}\,, \tag{4.53}$$

where the constant C is independent of β. The estimates in (4.42) follow from (4.51) and (4.52), and (4.53). This completes the proof of Proposition 4.2.2. □

4.3 Resolvent analysis in region exponentially close to the origin

The resolvent analysis in Proposition 4.2.2 is applicable to the problem (RS) only when the resolvent parameter $\lambda \in \mathbb{C}$ satisfies $|\lambda| \geq e^{-\frac{1}{a\beta}}$ for some $a \in (1, \infty)$, and we have taken $a = 6$ in the proof for simplicity. This restriction is essentially due to the unavailability of the Hardy inequality in two-dimensional exterior domains. In fact, in the proof of Proposition 4.2.2, we rely on the following inequality singular in $T \gg 1$:

$$\Big| \sum_{|n|=1} \big\langle (\mathrm{rot}\, v)_n, \frac{v_{r,n}}{|x|} \big\rangle_{L^2(D)} \Big| \leq \frac{1}{T} \|v\|_{L^2(\Omega)} \, \|\nabla v\|_{L^2(\Omega)} + \log T \|\nabla v\|^2_{L^2(\Omega)} \,,$$

as a substitute for the Hardy inequality, and this leads to the lack of information about the spectrum of $-\mathbb{A}_V$ in the region $0 < |\lambda| \leq O(e^{-\frac{1}{\beta}})$. Here we set $D = \{x \in \mathbb{R}^2 \mid |x| > 1\}$.

To perform the resolvent analysis in the region exponentially close to the origin, we firstly observe that a solution (v, q) to (RS) satisfies the next problem in the exterior disk D:

$$\begin{cases} \lambda w - \Delta w + \beta U^{\perp} \mathrm{rot}\, w + \nabla r \;=\; (-\mathrm{div}\,(R \otimes v + v \otimes R) + f)|_D \,, \quad x \in D \,, \\ \mathrm{div}\, w \;=\; 0 \,, \quad x \in D \,, \\ w|_{\partial D} \;=\; v|_{\partial D} \,. \end{cases} \tag{RS$^{\mathrm{ed}}$}$$

Then thanks to the symmetry, we can use a solution formula to (RS$^{\mathrm{ed}}$) by using polar coordinates, and study the a priori estimate for $w = v|_D$. To make calculation simple, we decompose the linear problem (RS$^{\mathrm{ed}}$) into three parts (RS$^{\mathrm{ed}}_f$), (RS$^{\mathrm{ed}}_{\mathrm{div} F}$), and (RS$^{\mathrm{ed}}_b$), which are respectively introduced in Subsections 4.3.1, 4.3.2, and 4.3.3. Then we derive the estimates to each problem in the corresponding subsections, and finally we collect them in Subsection 4.3.4 in order to establish the resolvent estimate to (RS) when $0 < |\lambda| < e^{-\frac{1}{6\beta}}$.

4.3.1 Problem I: External force f and Dirichlet condition

In this subsection we study the following resolvent problem for $(w, r) = (w^{\mathrm{ed}}_f, r^{\mathrm{ed}}_f)$:

$$\begin{cases} \lambda w - \Delta w + \beta U^{\perp} \mathrm{rot}\, w + \nabla r \;=\; f \,, \quad x \in D \,, \\ \mathrm{div}\, w \;=\; 0 \,, \quad x \in D \,, \\ w|_{\partial D} \;=\; 0 \,. \end{cases} \tag{RS$^{\mathrm{ed}}_f$}$$

Especially, we are interested in the estimates for the ± 1-Fourier mode of w^{ed}_f. Although the L^p-L^q estimates to (RS$^{\mathrm{ed}}_f$) are already proved in [44], we revisit this problem here in order to study the β-dependence in these estimates, and it is one of the most important steps for the energy computation when $0 < |\lambda| < e^{-\frac{1}{6\beta}}$.

Let us recall the representation formula established in [44] for the solution to (RS$^{\mathrm{ed}}_f$) in each Fourier mode. Fix $n \in \mathbb{Z} \setminus \{0\}$ and $\lambda \in \mathbb{C} \setminus \overline{\mathbb{R}_-}$, $\overline{\mathbb{R}_-} = (-\infty, 0]$. Then, by applying the Fourier mode projection $\mathcal{P}_n$ to (RS$^{\mathrm{ed}}_f$) and using the invariant property $\mathcal{P}_n(U^{\perp} \mathrm{rot}\, w) = U^{\perp} \mathrm{rot}\, \mathcal{P}_n w$ in [44, Lemma 2.9], we observe that the n-mode $w_n = \mathcal{P}_n w$ solves

$$\begin{cases} \lambda w_n - \Delta w_n + \beta U^{\perp} \mathrm{rot}\, w_n + \mathcal{P}_n \nabla r \;=\; \mathcal{P}_n f \,, \quad x \in D \,, \\ \mathrm{div}\, w_n \;=\; 0 \,, \quad x \in D \,, \\ w_n|_{\partial D} \;=\; 0 \,. \end{cases} \tag{RS$^{\mathrm{ed}}_{f,n}$}$$

Since the formula in [44] is written in terms of some special functions, we introduce these definitions here. The modified Bessel function of first kind $I_\mu(z)$ of order μ is defined as

$$I_\mu(z) \;=\; \big(\frac{z}{2}\big)^\mu \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(\mu+m+1)} \big(\frac{z}{2}\big)^{2m}\,, \qquad z \in \mathbb{C} \setminus \overline{\mathbb{R}_-}\,, \tag{4.54}$$

where $z^\mu = e^{\mu \operatorname{Log} z}$ and $\operatorname{Log} z$ denotes the principal branch to the logarithm of $z \in \mathbb{C} \setminus \overline{\mathbb{R}_-}$, and the function $\Gamma(z)$ in (4.54) denotes the Gamma function. Next we define the modified Bessel function of second kind $K_\mu(z)$ of order $\mu \notin \mathbb{Z}$ in the following manner:

$$K_\mu(z) \;=\; \frac{\pi}{2} \frac{I_{-\mu}(z) - I_\mu(z)}{\sin \mu\pi}\,, \qquad z \in \mathbb{C} \setminus \overline{\mathbb{R}_-}\,. \tag{4.55}$$

It is classical that $K_\mu(z)$ and $I_\mu(z)$ are linearly independent solutions to the ODE

$$-\frac{\mathrm{d}^2\omega}{\mathrm{d}z^2} - \frac{1}{z}\frac{\mathrm{d}\omega}{\mathrm{d}z} + \big(1 + \frac{\mu^2}{z^2}\big)\omega \;=\; 0\,, \tag{4.56}$$

and that their the Wronskian is z^{-1}. Applying the rotation operator rot to the first equation of ($\mathrm{RS}^{\mathrm{ed}}_{f,n}$), we find that $\omega = (\mathrm{rot}\, w)_n = (\mathrm{rot}\, w_n) e^{-in\theta}$ satisfies the ODE

$$-\frac{\mathrm{d}^2\omega}{\mathrm{d}r^2} - \frac{1}{r}\frac{\mathrm{d}\omega}{\mathrm{d}r} + \big(\lambda + \frac{n^2 + in\beta}{r^2}\big)\omega \;=\; (\mathrm{rot}\, f)_n\,, \qquad r > 1\,. \tag{4.57}$$

Hence, if we set

$$\mu_n \;=\; \mu_n(\beta) \;=\; (n^2 + in\beta)^{\frac{1}{2}}\,, \qquad \mathrm{Re}(\mu_n) > 0\,, \tag{4.58}$$

then $K_{\mu_n}(\sqrt{\lambda} r)$ and $I_{\mu_n}(\sqrt{\lambda} r)$ give linearly independent solutions to the homogeneous equation of (4.57) and their Wronskian is r^{-1}. Here and in the following we always take the square root $\sqrt{z}$ so that $\mathrm{Re}(\sqrt{z}) > 0$ for $z \in \mathbb{C} \setminus \overline{\mathbb{R}_-}$. Furthermore, we set

$$F_n(\sqrt{\lambda};\beta) \;=\; \int_1^\infty s^{1-|n|} K_{\mu_n}(\sqrt{\lambda} s)\,\mathrm{d}s\,, \qquad \lambda \in \mathbb{C} \setminus \overline{\mathbb{R}_-}\,, \tag{4.59}$$

and denote by $\mathcal{Z}(F_n)$ the set of the zeros of $F_n(\sqrt{\lambda};\beta)$ lying in $\mathbb{C} \setminus \overline{\mathbb{R}_-}$;

$$\mathcal{Z}(F_n) \;=\; \{z \in \mathbb{C} \setminus \overline{\mathbb{R}_-} \mid F_n(\sqrt{z};\beta) \;=\; 0\}\,. \tag{4.60}$$

Let $\lambda \in \mathbb{C} \setminus (\overline{\mathbb{R}_-} \cup \mathcal{Z}(F_n))$. Then, from the argument in [44, Section 3], we have the following representation formula for $w^{\mathrm{ed}}_{f,n}$ solving ($\mathrm{RS}^{\mathrm{ed}}_{f,n}$):

$$w^{\mathrm{ed}}_{f,n} \;=\; -\frac{c_{n,\lambda}[f_n]}{F_n(\sqrt{\lambda};\beta)} V_n[K_{\mu_n}(\sqrt{\lambda}\,\cdot\,)] + V_n[\Phi_{n,\lambda}[f_n]]\,. \tag{4.61}$$

Here $V_n[\,\cdot\,]$ is the Biot-Savart law in (4.27) and the function $\Phi_{n,\lambda}[f_n]$ is defined as

$$\begin{aligned}
\Phi_{n,\lambda}[f_n](r) \;=\; &-K_{\mu_n}(\sqrt{\lambda} r)\bigg(\int_1^r I_{\mu_n}(\sqrt{\lambda} s)\,\big(\mu_n f_{\theta,n}(s) + in f_{r,n}(s)\big)\,\mathrm{d}s \\
&\qquad + \sqrt{\lambda}\int_1^r s I_{\mu_n+1}(\sqrt{\lambda} s)\, f_{\theta,n}(s)\,\mathrm{d}s\bigg) \\
&+ I_{\mu_n}(\sqrt{\lambda} r)\bigg(\int_r^\infty K_{\mu_n}(\sqrt{\lambda} s)\,\big(\mu_n f_{\theta,n}(s) - in f_{r,n}(s)\big)\,\mathrm{d}s \\
&\qquad + \sqrt{\lambda}\int_r^\infty s K_{\mu_n-1}(\sqrt{\lambda} s)\, f_{\theta,n}(s)\,\mathrm{d}s\bigg)\,,
\end{aligned} \tag{4.62}$$

while the constant $c_{n,\lambda}[f_n]$ is defined as

$$c_{n,\lambda}[f_n] \;=\; \int_1^\infty s^{1-|n|}\Phi_{n,\lambda}[f_n](s)\,\mathrm{d}s\,. \tag{4.63}$$

Moreover, the vorticity $\operatorname{rot} w^{\mathrm{ed}}_{f,n}$ is represented as

$$\operatorname{rot} w^{\mathrm{ed}}_{f,n} \;=\; -\frac{c_{n,\lambda}[f_n]}{F_n(\sqrt{\lambda};\beta)}K_{\mu_n}(\sqrt{\lambda}r)e^{in\theta} + \Phi_{n,\lambda}[f_n](r)e^{in\theta}\,. \tag{4.64}$$

We shall estimate $w^{\mathrm{ed}}_{f,n}$ and $\operatorname{rot} w^{\mathrm{ed}}_{f,n}$, represented respectively as in (4.61) and (4.64), when $|n| = 1$ in the following two subsections. Our main tools for the proof are the asymptotic analysis of $\mu_n = \mu_n(\beta)$ for small β in Appendix 4.5.1, and the detailed estimates to the modified Bessel functions in Appendix 4.5.2. Before going into details, let us state the estimate of $F_n(\sqrt{\lambda};\beta)$ in a region exponentially close to the origin with respect to β. We denote by Σ_ϕ the sector $\{z \in \mathbb{C}\setminus\{0\} \mid |\arg z| < \phi\}$, $\phi \in (0,\pi)$, in the complex plane $\mathbb{C}$, and by $\mathcal{B}_\rho(0) \subset \mathbb{C}$ the disk centered at the origin with radius $\rho > 0$.

Proposition 4.3.1 *Let $|n| = 1$. Then for any $\epsilon \in (0,\frac{\pi}{2})$ there is a positive constant β_0 depending only on ϵ such that as long as $\beta \in (0,\beta_0)$ and $\lambda \in \Sigma_{\pi-\epsilon}\cap\mathcal{B}_{e^{-\frac{1}{6\beta}}}(0)$ we have*

$$\frac{1}{|F_n(\sqrt{\lambda};\beta)|} \le C|\lambda|^{\frac{\mathrm{Re}(\mu_n)}{2}}\,, \tag{4.65}$$

where the constant C depends only on ϵ. In particular, we have $\mathcal{Z}(F_n)\cap\mathcal{B}_{e^{-\frac{1}{6\beta}}}(0) = \emptyset$.

Proof: The assertion follows from Lemma 4.5.6 in Appendix 4.5.3, since we have $e^{-\frac{1}{6\beta}} < \beta^4$ for any $\beta \in (0,1)$. See Appendix 4.5.3 for the proof of Lemma 4.5.6. □

Estimates of the velocity solving ($\mathrm{RS}^{\mathrm{ed}}_{f,n}$) with $|n| = 1$

In this subsection we derive the estimates for the solution $w^{\mathrm{ed}}_{f,n}$ to ($\mathrm{RS}^{\mathrm{ed}}_{f,n}$) which is now represented as (4.61). The novelty of the following result is the investigation on the β-singularity appearing in each estimate. Let β_0 be the constant in Proposition 4.3.1.

Theorem 4.3.2 *Let $|n| = 1$ and $1 \le q < p \le \infty$ or $1 < q \le p < \infty$. Fix $\epsilon \in (0,\frac{\pi}{2})$. Then there is a positive constant $C = C(q,p,\epsilon)$ independent of β such that the following statement holds. Let $f \in C_0^\infty(D)^2$ and $\beta \in (0,\beta_0)$. Then for $\lambda \in \Sigma_{\pi-\epsilon}\cap\mathcal{B}_{e^{-\frac{1}{6\beta}}}(0)$ we have*

$$\|w^{\mathrm{ed}}_{f,n}\|_{L^p(D)} \le \frac{C}{\beta^2}|\lambda|^{-1+\frac{1}{q}-\frac{1}{p}}\|f\|_{L^q(D)}\,, \tag{4.66}$$

$$\Big\|\frac{w^{\mathrm{ed}}_{f,n}}{|x|}\Big\|_{L^2(D)} \le \frac{C}{\beta}\Big(\frac{1}{\beta^2} + |\log\mathrm{Re}(\sqrt{\lambda})|^{\frac{1}{2}}\Big)|\lambda|^{-1+\frac{1}{q}}\|f\|_{L^q(D)}\,. \tag{4.67}$$

Moreover, (4.66) *and* (4.67) *hold all for $f \in L^q(D)^2$.*

Remark 4.3.3 The logarithmic factor $|\log \mathrm{Re}(\sqrt{\lambda})|$ in (4.67) cannot be removed in our analysis. This singularity might prevent us from closing the energy computation in view of the scaling, however, we observe that it is resolved by considering the following products:

$$\big|\big\langle \omega^{\mathrm{ed}\,(1)}_{f,n}, \frac{(w^{\mathrm{ed}}_{f,r})_n}{|x|}\big\rangle_{L^2(D)}\big|\,, \quad \big|\big\langle \omega^{\mathrm{ed}\,(1)}_{\mathrm{div}F,n}, \frac{(w^{\mathrm{ed}}_{f,r})_n}{|x|}\big\rangle_{L^2(D)}\big|\,, \quad \big|\big\langle \omega^{\mathrm{ed}}_{b,n}, \frac{(w^{\mathrm{ed}}_{f,r})_n}{|x|}\big\rangle_{L^2(D)}\big|\,.$$

Here the vorticities $\omega^{\mathrm{ed}\,(1)}_{f,n}$, $\omega^{\mathrm{ed}\,(1)}_{\mathrm{div}F,n}$, $\omega^{\mathrm{ed}}_{b,n}$ will be introduced respectively in Subsections 4.3.1, 4.3.2, and 4.3.3. This is a key observation in proving Proposition 4.3.23 in Subsection 4.3.4, where the estimate for $\big\langle (\mathrm{rot}\, v)_n, \frac{v_{r,n}}{|x|}\big\rangle_{L^2(D)}$ is established when $0 < |\lambda| < e^{-\frac{1}{6\beta}}$.

We postpone the proof of Theorem 4.3.2 at the end of this subsection, and focus on the term $V_n[\Phi_{n,\lambda}[f_n]]$ in (4.61) for the time being. In order to estimate $V_n[\Phi_{n,\lambda}[f_n]]$, taking into account the definition of $V_n[\,\cdot\,]$ in (4.27), firstly we study the following two integrals

$$\frac{1}{r^{|n|}}\int_1^r s^{1+|n|}\Phi_{n,\lambda}[f_n](s)\,\mathrm{d}s\,, \qquad r^{|n|}\int_r^\infty s^{1-|n|}\Phi_{n,\lambda}[f_n](s)\,\mathrm{d}s\,.$$

Let us recall the decompositions for them used in [44] which are useful in calculations. To state the result we define the functions $g^{(1)}_n(r)$ and $g^{(2)}_n(r)$ by

$$g^{(1)}_n(r) \,=\, \mu_n f_{\theta,n}(r) + in f_{r,n}(r)\,, \qquad g^{(2)}_n(r) \,=\, \mu_n f_{\theta,n}(r) - in f_{r,n}(r)\,,$$

and fix a resolvent parameter $\lambda \in \mathbb{C}\setminus\overline{\mathbb{R}_-}$.

Lemma 4.3.4 ([44, Lemmas 3.6 and 3.9]) *Let $n \in \mathbb{Z}\setminus\{0\}$ and $f \in C_0^\infty(D)^2$. Then we have*

$$\frac{1}{r^{|n|}}\int_1^r s^{1+|n|}\Phi_{n,\lambda}[f_n](s)\,\mathrm{d}s \,=\, \sum_{l=1}^{9} J^{(1)}_l[f_n](r)\,,$$

where

$$\begin{aligned}
J^{(1)}_1[f_n](r) &= -\frac{1}{r^{|n|}}\int_1^r I_{\mu_n}(\sqrt{\lambda}\tau)\, g^{(1)}_n(\tau)\int_\tau^r s^{1+|n|}K_{\mu_n}(\sqrt{\lambda}s)\,\mathrm{d}s\,\mathrm{d}\tau\,,\\
J^{(1)}_2[f_n](r) &= -\frac{\mu_n+|n|}{r^{|n|}}\int_1^r \tau I_{\mu_n+1}(\sqrt{\lambda}\tau)\, f_{\theta,n}(\tau)\int_\tau^r s^{|n|}K_{\mu_n-1}(\sqrt{\lambda}s)\,\mathrm{d}s\,\mathrm{d}\tau\,,\\
J^{(1)}_3[f_n](r) &= \frac{1}{r^{|n|}}\int_1^r K_{\mu_n}(\sqrt{\lambda}\tau)\, g^{(2)}_n(\tau)\int_1^\tau s^{1+|n|}I_{\mu_n}(\sqrt{\lambda}s)\,\mathrm{d}s\,,\\
J^{(1)}_4[f_n](r) &= \frac{\mu_n-|n|}{r^{|n|}}\int_1^r \tau K_{\mu_n-1}(\sqrt{\lambda}\tau)\, f_{\theta,n}(\tau)\int_1^\tau s^{|n|}I_{\mu_n+1}(\sqrt{\lambda}s)\,\mathrm{d}s\,,\\
J^{(1)}_5[f_n](r) &= \frac{1}{r^{|n|}}\bigg(\int_r^\infty K_{\mu_n}(\sqrt{\lambda}s)\, g^{(2)}_n(s)\,\mathrm{d}s\bigg)\bigg(\int_1^r s^{1+|n|}I_{\mu_n}(\sqrt{\lambda}s)\,\mathrm{d}s\bigg)\,,\\
J^{(1)}_6[f_n](r) &= \frac{\mu_n-|n|}{r^{|n|}}\bigg(\int_r^\infty sK_{\mu_n-1}(\sqrt{\lambda}s)\, f_{\theta,n}(s)\,\mathrm{d}s\bigg)\bigg(\int_1^r s^{|n|}I_{\mu_n+1}(\sqrt{\lambda}s)\,\mathrm{d}s\bigg)\,,\\
J^{(1)}_7[f_n](r) &= rK_{\mu_n-1}(\sqrt{\lambda}r)\int_1^r sI_{\mu_n+1}(\sqrt{\lambda}s)\, f_{\theta,n}(s)\,\mathrm{d}s\,,\\
J^{(1)}_8[f_n](r) &= rI_{\mu_n+1}(\sqrt{\lambda}r)\int_r^\infty sK_{\mu_n-1}(\sqrt{\lambda}s)\, f_{\theta,n}(s)\,\mathrm{d}s\,,
\end{aligned}$$

$$J_9^{(1)}[f_n](r) \;=\; -\frac{I_{\mu_n+1}(\sqrt{\lambda})}{r^{|n|}} \int_1^\infty sK_{\mu_n-1}(\sqrt{\lambda}s)\, f_{\theta,n}(s)\,\mathrm{d}s\,,$$

and

$$r^{|n|} \int_r^\infty s^{1-|n|}\Phi_{n,\lambda}[f_n](s)\,\mathrm{d}s \;=\; \sum_{l=10}^{17} J_l^{(1)}[f_n](r)\,,$$

where

$$\begin{aligned}
J_{10}^{(1)}[f_n](r) \;&=\; -r^{|n|}\bigg(\int_1^r I_{\mu_n}(\sqrt{\lambda}s)\, g_n^{(1)}(s)\,\mathrm{d}s\bigg)\bigg(\int_r^\infty s^{1-|n|}K_{\mu_n}(\sqrt{\lambda}s)\,\mathrm{d}s\bigg)\,,\\
J_{11}^{(1)}[f_n](r) \;&=\; -r^{|n|}\int_r^\infty I_{\mu_n}(\sqrt{\lambda}\tau)\, g_n^{(1)}(\tau)\int_\tau^\infty s^{1-|n|}K_{\mu_n}(\sqrt{\lambda}s)\,\mathrm{d}s\,\mathrm{d}\tau\,,\\
J_{12}^{(1)}[f_n](r) \;&=\; -(\mu_n-|n|)r^{|n|}\bigg(\int_1^r sI_{\mu_n+1}(\sqrt{\lambda}s)\, f_{\theta,n}(s)\,\mathrm{d}s\bigg)\bigg(\int_r^\infty s^{-|n|}K_{\mu_n-1}(\sqrt{\lambda}s)\,\mathrm{d}s\bigg)\,,\\
J_{13}^{(1)}[f_n](r) \;&=\; -(\mu_n-|n|)r^{|n|}\int_r^\infty \tau I_{\mu_n+1}(\sqrt{\lambda}\tau)\, f_{\theta,n}(\tau)\int_\tau^\infty s^{-|n|}K_{\mu_n-1}(\sqrt{\lambda}s)\,\mathrm{d}s\,\mathrm{d}\tau\,,\\
J_{14}^{(1)}[f_n](r) \;&=\; r^{|n|}\int_r^\infty K_{\mu_n}(\sqrt{\lambda}\tau)\, g_n^{(2)}(\tau)\int_r^\tau s^{1-|n|}I_{\mu_n}(\sqrt{\lambda}s)\,\mathrm{d}s\,\mathrm{d}\tau\,,\\
J_{15}^{(1)}[f_n](r) \;&=\; (\mu_n+|n|)r^{|n|}\int_r^\infty \tau K_{\mu_n-1}(\sqrt{\lambda}\tau)\, f_{\theta,n}(\tau)\int_r^\tau s^{-|n|}I_{\mu_n+1}(\sqrt{\lambda}s)\,\mathrm{d}s\,\mathrm{d}\tau\,,\\
J_{16}^{(1)}[f_n](r) \;&=\; -rK_{\mu_n-1}(\sqrt{\lambda}r)\int_1^r sI_{\mu_n+1}(\sqrt{\lambda}s)\, f_{\theta,n}(s)\,\mathrm{d}s\,,\\
J_{17}^{(1)}[f_n](r) \;&=\; -rI_{\mu_n+1}(\sqrt{\lambda}r)\int_r^\infty sK_{\mu_n-1}(\sqrt{\lambda}s)\, f_{\theta,n}(s)\,\mathrm{d}s\,.
\end{aligned}$$

Remark 4.3.5 (1) The estimate to the term $J_9^{(1)}[f_n]$ is not needed in the following analysis thanks to the cancellation $J_9^{(1)}[f_n](r) - r^{-|n|}J_{17}^{(1)}[f_n](1) = 0$ in the Biot-Savart law $V_n[\Phi_{n,\lambda}[f_n]]$. This fact will be used in the proof of Proposition 4.3.10.
(2) Note that $J_7^{(1)}[f_n] = -J_{16}^{(1)}[f_n]$ and $J_8^{(1)}[f_n] = -J_{17}^{(1)}[f_n]$ hold. Therefore we will skip the derivation of the estimates for $J_{16}^{(1)}[f_n]$ and $J_{17}^{(1)}[f_n]$ in Lemma 4.3.7.
(3) We can express the constant $c_{n,\lambda}[f_n]$ in (4.63) in terms of $J_l^{(1)}[f_n](r)$ as $c_{n,\lambda}[f_n] = \sum_{l=11,13,14,15,17} J_l^{(1)}[f_n](1)$.

The estimates to $J_l^{(1)}[f_n]$, $l \in \{1,\ldots,8\}$, in Lemma 4.3.4 are given as follows.

Lemma 4.3.6 *Let $|n| = 1$ and $q \in [1,\infty)$, and let $\lambda \in \Sigma_{\pi-\epsilon} \cap \mathcal{B}_1(0)$ for some $\epsilon \in (0, \frac{\pi}{2})$. Then there is a positive constant $C = C(q,\epsilon)$ independent of β such that the following statements hold.*
(1) *Let $f \in C_0^\infty(D)^2$. Then for $l \in \{1,\ldots,8\}$ we have*

$$|J_l^{(1)}[f_n](r)| \le \frac{C}{\beta} r^{3-\frac{2}{q}} \|f\|_{L^q(D)}\,, \qquad 1 \le r < \mathrm{Re}(\sqrt{\lambda})^{-1}\,. \tag{4.68}$$

On the other hand, for $l \in \{1,\ldots,6\}$ we have

$$|J_l^{(1)}[f_n](r)| \le \frac{C}{\beta} |\lambda|^{-1} r^{1-\frac{2}{q}} \|f\|_{L^q(D)}\,, \qquad r \ge \mathrm{Re}(\sqrt{\lambda})^{-1}\,, \tag{4.69}$$

while for $l \in \{7,8\}$ we have

$$|J_l^{(1)}[f_n](r)| \le C|\lambda|^{-1+\frac{1}{2q}} r^{1-\frac{1}{q}} \|f\|_{L^q(D)}\,, \qquad r \ge \mathrm{Re}(\sqrt{\lambda})^{-1}\,. \tag{4.70}$$

(2) *Let $f \in C_0^\infty(D)^2$. Then for $l \in \{7,8\}$ we have*

$$\|r^{-1} J_l^{(1)}[f_n]\|_{L^\infty(D)} \le \frac{C}{\beta} |\lambda|^{-1} \|f\|_{L^\infty(D)}\,, \tag{4.71}$$

$$\|r^{-1} J_l^{(1)}[f_n]\|_{L^1(D)} \le \frac{C}{\beta} |\lambda|^{-1} \|f\|_{L^1(D)}\,. \tag{4.72}$$

Proof: (1) (i) Estimate of $J_1^{(1)}[f_n]$: For $1 \le r < \mathrm{Re}(\sqrt{\lambda})^{-1}$, by (4.154) for $k=0$ in Lemma 4.5.2 and (4.157) for $k=0$ in Lemma 4.5.3 in Appendix 4.5.2, we find

$$\begin{aligned} |J_1^{(1)}[f_n](r)| &\le r^{-1} \int_1^r |I_{\mu_n}(\sqrt{\lambda}\tau)\, g_n^{(1)}(\tau)| \left| \int_\tau^r s^2 K_{\mu_n}(\sqrt{\lambda}s)\,\mathrm{d}s \right| \mathrm{d}\tau \\ &\le C r \int_1^r |f_n(\tau)|\tau\,\mathrm{d}\tau\,, \end{aligned}$$

which leads to the estimate (4.68). For $r \ge \mathrm{Re}(\sqrt{\lambda})^{-1}$, by (4.154) and (4.156) for $k=0$ in Lemma 4.5.2 and (4.158) and (4.159) for $k=0$ in Lemma 4.5.3, we have

$$\begin{aligned} |J_1^{(1)}[f_n](r)| &\le r^{-1} \Big(\int_1^{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}} + \int_{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}}^r \Big) |I_{\mu_n}(\sqrt{\lambda}\tau)\, g_n^{(1)}(\tau)| \left| \int_\tau^r s^2 K_{\mu_n}(\sqrt{\lambda}s)\,\mathrm{d}s \right| \mathrm{d}\tau \\ &\le C|\lambda|^{-1} r^{-1} \int_1^{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}} |f_n(\tau)|\tau\,\mathrm{d}\tau + C\,|\lambda|^{-1} r^{-1} \int_{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}}^r |f_n(\tau)|\tau\,\mathrm{d}\tau\,, \end{aligned}$$

which implies the estimate (4.69).
(ii) Estimate of $J_2^{(1)}[f_n]$: The proof is parallel to that for $J_1^{(1)}[f_n]$ using the results in Lemmas 4.5.2 and 4.5.3 for $k=1$. We omit the details here.
(iii) Estimate of $J_3^{(1)}[f_n]$: For $1 \le r < \mathrm{Re}(\sqrt{\lambda})^{-1}$, by (4.150) and (4.152) in Lemma 4.5.2 and (4.162) for $k=0$ in Lemma 4.5.4, we see that

$$\begin{aligned} |J_3^{(1)}[f_n](r)| &\le r^{-1} \int_1^r |K_{\mu_n}(\sqrt{\lambda}\tau)\, g_n^{(2)}(\tau)| \int_1^\tau |s^2 I_{\mu_n}(\sqrt{\lambda}s)|\,\mathrm{d}s\,\mathrm{d}\tau \\ &\le C\, r \int_1^r |f_n(\tau)|\tau\,\mathrm{d}\tau\,. \end{aligned}$$

Thus we have (4.68). For $r \ge \mathrm{Re}(\sqrt{\lambda})^{-1}$, by (4.150), (4.152), and (4.155) for $k=0$ in Lemma 4.5.2 and (4.162) and (4.163) for $k=0$ in Lemma 4.5.4 we have

$$\begin{aligned} |J_3^{(1)}[f_n](r)| &\le r^{-1} \Big(\int_1^{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}} + \int_{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}}^r \Big) |K_{\mu_n}(\sqrt{\lambda}\tau)\, g_n^{(2)}(\tau)| \int_1^\tau |s^2 I_{\mu_n}(\sqrt{\lambda}s)|\,\mathrm{d}s\,\mathrm{d}\tau \\ &\le C\,|\lambda|^{-1} r^{-1} \int_1^{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}} |f_n(\tau)|\tau\,\mathrm{d}\tau + C\,|\lambda|^{-1} r^{-1} \int_{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}}^r |f_n(\tau)|\tau\,\mathrm{d}\tau\,, \end{aligned}$$

which leads to (4.69).
(iv) Estimate of $J_4^{(1)}[f_n]$: The proof is parallel to that for $J_3^{(1)}[f_n]$ using the results in Lemmas 4.5.2 and 4.5.4 for $k=1$, and we omit here.
(v) Estimates of $J_5^{(1)}[f_n]$ and $J_6^{(1)}[f_n]$: We give a proof only for $J_5^{(1)}[f_n]$ since the proof for $J_6^{(1)}[f_n]$ is similar. For $1 \le r < \mathrm{Re}(\sqrt{\lambda})^{-1}$, by (4.150), (4.152), and (4.155) for $k=0$ in Lemma 4.5.2 and (4.162) for $k=0$ in Lemma 4.5.4 we observe that

$$
\begin{aligned}
|J_5^{(1)}[f_n](r)| &\le r^{-1}\int_1^r |s^2 I_{\mu_n}(\sqrt{\lambda}s)|\,\mathrm{d}s \biggl(\int_r^{\frac{1}{\mathrm{Re}\,(\sqrt{\lambda})}} + \int_{\frac{1}{\mathrm{Re}\,(\sqrt{\lambda})}}^{\infty}\biggr)|K_{\mu_n}(\sqrt{\lambda}s)\, g_n^{(2)}(s)|\,\mathrm{d}s \\
&\le C\, r^{\mathrm{Re}(\mu_n)+2}\int_r^{\frac{1}{\mathrm{Re}\,(\sqrt{\lambda})}} s^{-(\mathrm{Re}(\mu_n)+1)}|f_n(s)|s\,\mathrm{d}s \\
&\quad + C\,|\lambda|^{\frac{\mathrm{Re}(\mu_n)}{2}-\frac14} r^{\mathrm{Re}(\mu_n)+2}\int_{\frac{1}{\mathrm{Re}\,(\sqrt{\lambda})}}^{\infty} s^{-\frac32}e^{-\mathrm{Re}(\sqrt{\lambda})s}|f_n(s)|s\,\mathrm{d}s\,.
\end{aligned}
$$

Then a direct calculation shows (4.68). For $r \ge \mathrm{Re}(\sqrt{\lambda})^{-1}$, by (4.155) for $k=0$ in Lemma 4.5.2 and (4.163) for $k=0$ in Lemma 4.5.4 we have

$$
\begin{aligned}
|J_5^{(1)}[f_n](r)| &\le r^{-1}\int_1^r |s^2 I_{\mu_n}(\sqrt{\lambda}s)|\,\mathrm{d}s \int_r^{\infty} |K_{\mu_n}(\sqrt{\lambda}s)\, g_n^{(2)}(s)|\,\mathrm{d}s \\
&\le C\,|\lambda|^{-1} r^{-\frac12} e^{\mathrm{Re}\,(\sqrt{\lambda})r}\int_r^{\infty} s^{-\frac12}e^{-\mathrm{Re}(\sqrt{\lambda})s}|f_n(s)|s\,\mathrm{d}s\,,
\end{aligned}
$$

which implies (4.69).
(vi) Estimate of $J_7^{(1)}[f_n]$: For $1 \le r < \mathrm{Re}(\sqrt{\lambda})^{-1}$, by (4.151), (4.153), and (4.154) for $k=1$ in Lemma 4.5.2 we find

$$
\begin{aligned}
|J_7^{(1)}[f_n](r)| &\le |rK_{\mu_n-1}(\sqrt{\lambda}r)|\int_1^r |I_{\mu_n+1}(\sqrt{\lambda}s)\, f_{\theta,n}(s)s|\,\mathrm{d}s \\
&\le C\beta^{-1} r\int_1^r |f_n(s)|s\,\mathrm{d}s\,.
\end{aligned}
\tag{4.73}
$$

Thus we have (4.68). For $r \ge \mathrm{Re}(\sqrt{\lambda})^{-1}$, by (4.154)–(4.156) for $k=1$ in Lemma 4.5.2 we have

$$
\begin{aligned}
|J_7^{(1)}[f_n](r)| &\le |rK_{\mu_n-1}(\sqrt{\lambda}r)|\biggl(\int_1^{\frac{1}{\mathrm{Re}\,(\sqrt{\lambda})}} + \int_{\frac{1}{\mathrm{Re}\,(\sqrt{\lambda})}}^{r}\biggr)|I_{\mu_n+1}(\sqrt{\lambda}s)\, f_{\theta,n}(s)s|\,\mathrm{d}s \\
&\le C\,|\lambda|^{-\frac14} r^{\frac12} e^{-\mathrm{Re}\,(\sqrt{\lambda})r}\int_1^{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}} |f_n(s)|s\,\mathrm{d}s \\
&\quad + C\,|\lambda|^{-\frac12} r^{\frac12} e^{-\mathrm{Re}\,(\sqrt{\lambda})r}\int_{\frac{1}{\mathrm{Re}\,(\sqrt{\lambda})}}^{r} s^{-\frac12}e^{\mathrm{Re}\,(\sqrt{\lambda})s}|f_n(s)|s\,\mathrm{d}s\,,
\end{aligned}
\tag{4.74}
$$

which leads to (4.70).
(vii) Estimate of $J_8^{(1)}[f_n]$: For $1 \le r < \mathrm{Re}(\sqrt{\lambda})^{-1}$, by the results in Lemmas 4.5.2 for

$k = 1$ we find

$$|J_8^{(1)}[f_n](r)| \leq |rI_{\mu_n+1}(\sqrt{\lambda}r)| \bigg(\int_r^{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}} + \int_{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}}^\infty \bigg) |K_{\mu_n-1}(\sqrt{\lambda}s)\, f_{\theta,n}(s)s| \,\mathrm{d}s$$

$$\leq C\beta^{-1}|\lambda| r^3 \int_r^{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}} |f_n(s)|s \,\mathrm{d}s + C|\lambda|^{\frac{3}{4}} r^3 \int_{\frac{1}{\mathrm{Re}\,(\sqrt{\lambda})}}^\infty s^{-\frac{1}{2}} e^{-\mathrm{Re}(\sqrt{\lambda})s} |f_n(s)|s \,\mathrm{d}s \,, \tag{4.75}$$

which implies (4.68). For $r \geq \mathrm{Re}(\sqrt{\lambda})^{-1}$, by Lemma 4.5.2 for $k = 1$ again we have

$$|J_8^{(1)}[f_n](r)| \leq |rI_{\mu_n+1}(\sqrt{\lambda}r)| \int_r^\infty |K_{\mu_n-1}(\sqrt{\lambda}s)\, f_{\theta,n}(s)s| \,\mathrm{d}s$$

$$\leq C|\lambda|^{-\frac{1}{2}} r^{\frac{1}{2}} e^{\mathrm{Re}\,(\sqrt{\lambda})r} \int_r^\infty s^{-\frac{1}{2}} e^{-\mathrm{Re}\,(\sqrt{\lambda})s} |f_n(s)|s \,\mathrm{d}s \,, \tag{4.76}$$

which leads to (4.70). Hence we obtain the assertion (1) of Lemma 4.3.6.
(2) The estimate (4.71) follows from (4.73)–(4.76) in the above. For the proof of (4.72), one can reproduce the calculation performed in [44, Lemma 3.7] using the results in Lemma 4.5.2, and hence we omit the details here. This completes the proof of Lemma 4.3.6. □

The next lemma summarizes the estimates to $J_l^{(1)}[f_n](r)$, $l \in \{10, \ldots, 17\}$, in Lemma 4.3.4. We skip the proofs for $J_{16}^{(1)}[f_n]$ and $J_{17}^{(1)}[f_n]$ as is mentioned in Remark 4.3.5 (2).

Lemma 4.3.7 *Let* $|n| = 1$ *and* $q \in [1, \infty)$*, and let* $\lambda \in \Sigma_{\pi-\epsilon} \cap \mathcal{B}_1(0)$ *for some* $\epsilon \in (0, \frac{\pi}{2})$*. Then there is a positive constant* $C = C(q, \epsilon)$ *independent of* β *such that the following statements hold.*
(1) *Let* $f \in C_0^\infty(D)^2$*. Then for* $l \in \{10, \ldots, 17\}$ *we have*

$$|J_l^{(1)}[f_n](r)| \leq \frac{C}{\beta} |\lambda|^{-1+\frac{1}{q}} r \|f\|_{L^q(D)} \,, \qquad 1 \leq r < \mathrm{Re}(\sqrt{\lambda})^{-1} \,. \tag{4.77}$$

On the other hand, for $l \in \{10, \ldots, 15\}$ *we have*

$$|J_l^{(1)}[f_n](r)| \leq C|\lambda|^{-1} r^{1-\frac{2}{q}} \|f\|_{L^q(D)} \,, \qquad r \geq \mathrm{Re}(\sqrt{\lambda})^{-1} \,, \tag{4.78}$$

while for $l \in \{16, 17\}$ *we have*

$$|J_l^{(1)}[f_n](r)| \leq C|\lambda|^{-1+\frac{1}{2q}} r^{1-\frac{1}{q}} \|f\|_{L^q(D)} \,, \qquad r \geq \mathrm{Re}(\sqrt{\lambda})^{-1} \,. \tag{4.79}$$

(2) *Let* $f \in C_0^\infty(D)^2$*. Then for* $l \in \{16, 17\}$ *we have*

$$\|r^{-1} J_l^{(1)}[f_n]\|_{L^\infty(D)} \leq \frac{C}{\beta} |\lambda|^{-1} \|f\|_{L^\infty(D)} \,, \tag{4.80}$$

$$\|r^{-1} J_l^{(1)}[f_n]\|_{L^1(D)} \leq \frac{C}{\beta} |\lambda|^{-1} \|f\|_{L^1(D)} \,. \tag{4.81}$$

Proof: (1) (i) Estimate of $J_{10}^{(1)}[f_n]$: For $1 \leq r < \mathrm{Re}(\sqrt{\lambda})^{-1}$, by (4.154) for $k = 0$ in Lemma 4.5.2 and (4.160) for $k = 0$ in Lemma 4.5.3 in Appendix 4.5.2, we find

$$|J_{10}^{(1)}[f_n](r)| \leq r \bigg| \int_r^\infty K_{\mu_n}(\sqrt{\lambda}s) \,\mathrm{d}s \bigg| \int_1^r |I_{\mu_n}(\sqrt{\lambda}s)\, g_n^{(1)}(s)| \,\mathrm{d}s$$

$$\leq C\beta^{-1} r \int_1^r |f_n(s)|s \,\mathrm{d}s \,,$$

which implies (4.77). For $r \geq \mathrm{Re}(\sqrt{\lambda})^{-1}$, by (4.154) and (4.156) for $k = 0$ in Lemma 4.5.2 and (4.161) for $k = 0$ in Lemma 4.5.3, we have

$$
\begin{aligned}
|J_{10}^{(1)}[f_n](r)| &\leq r \int_r^\infty |K_{\mu_n}(\sqrt{\lambda}s)| \, \mathrm{d}s \bigg(\int_1^{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}} + \int_{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}}^r \bigg) |I_{\mu_n}(\sqrt{\lambda}s) \, g_n^{(1)}(s)| \, \mathrm{d}s \\
&\leq C \, |\lambda|^{-\frac{1}{4}} r^{\frac{1}{2}} e^{-\mathrm{Re}(\sqrt{\lambda})r} \int_1^{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}} |f_n(s)| s \, \mathrm{d}s \\
&\quad + C \, |\lambda|^{-1} r^{\frac{1}{2}} e^{-\mathrm{Re}(\sqrt{\lambda})r} \int_{\frac{1}{\mathrm{Re}\,(\sqrt{\lambda})}}^r s^{-\frac{3}{2}} e^{\mathrm{Re}(\sqrt{\lambda})s} |f_n(s)| s \, \mathrm{d}s \,,
\end{aligned}
$$

which leads to (4.78).

(ii) Estimate of $J_{11}^{(1)}[f_n]$: For $1 \leq r < \mathrm{Re}(\sqrt{\lambda})^{-1}$, by (4.154) and (4.156) for $k = 0$ in Lemma 4.5.2 and (4.160) and (4.161) for $k = 0$ in Lemma 4.5.3, we see that

$$
\begin{aligned}
|J_{11}^{(1)}[f_n](r)| &\leq r \bigg(\int_r^{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}} + \int_{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}}^\infty \bigg) |I_{\mu_n}(\sqrt{\lambda}\tau) \, g_n^{(1)}(\tau)| \bigg| \int_\tau^\infty K_{\mu_n}(\sqrt{\lambda}s) \, \mathrm{d}s \bigg| \, \mathrm{d}\tau \\
&\leq C\beta^{-1} r \int_r^{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}} |f_n(\tau)| \tau \, \mathrm{d}\tau + C \, |\lambda|^{-1} r \int_{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}}^\infty \tau^{-2} |f_n(\tau)| \tau \, \mathrm{d}\tau \,,
\end{aligned}
$$

which implies (4.77). For $r \geq \mathrm{Re}(\sqrt{\lambda})^{-1}$, by (4.156) for $k = 0$ in Lemma 4.5.2 and (4.161) for $k = 0$ in Lemma 4.5.3, we have

$$
\begin{aligned}
|J_{11}^{(1)}[f_n](r)| &\leq r \int_r^\infty |I_{\mu_n}(\sqrt{\lambda}\tau) \, g_n^{(1)}(\tau)| \int_\tau^\infty |K_{\mu_n}(\sqrt{\lambda}s)| \, \mathrm{d}s \, \mathrm{d}\tau \\
&\leq C \, |\lambda|^{-1} r \int_r^\infty \tau^{-2} |f_n(\tau)| \tau \, \mathrm{d}\tau \,,
\end{aligned}
$$

which leads to (4.78).

(iii) Estimates of $J_{12}^{(1)}[f_n]$ and $J_{13}^{(1)}[f_n]$: The proof for $J_{12}^{(1)}[f_n]$ is parallel to that for $J_{10}^{(1)}[f_n]$ using the bound $|\mu_n - 1| \leq C\beta$ and the results in Lemmas 4.5.2 and 4.5.3 for $k = 1$. The proof for $J_{13}^{(1)}[f_n]$ is similar to that for $J_{11}^{(1)}[f_n]$. Thus we omit the details here.

(iv) Estimate of $J_{14}^{(1)}[f_n]$: For $1 \leq r < \mathrm{Re}(\sqrt{\lambda})^{-1}$, by (4.150), (4.152), and (4.155) for $k = 0$ in Lemma 4.5.2 and (4.164) and (4.165) for $k = 0$ in Lemma 4.5.4, we observe that

$$
\begin{aligned}
|J_{14}^{(1)}[f_n](r)| &\leq r \bigg(\int_r^{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}} + \int_{\frac{1}{\mathrm{Re}\,(\sqrt{\lambda})}}^\infty \bigg) |K_{\mu_n}(\sqrt{\lambda}\tau) \; g_n^{(2)}(\tau)| \int_r^\tau |I_{\mu_n}(\sqrt{\lambda}s)| \, \mathrm{d}s \, \mathrm{d}\tau \\
&\leq Cr \int_r^{\frac{1}{\mathrm{Re}\,(\sqrt{\lambda})}} |f_n(\tau)| \tau \, \mathrm{d}\tau + C|\lambda|^{-1} r \int_{\frac{1}{\mathrm{Re}\,(\sqrt{\lambda})}}^\infty \tau^{-2} |f_n(\tau)| \tau \, \mathrm{d}\tau \,,
\end{aligned}
$$

which implies (4.77). For $r \geq \mathrm{Re}(\sqrt{\lambda})^{-1}$, by (4.155) in Lemma 4.5.2 and (4.166) in Lemma 4.5.4 for $k = 0$ we have

$$
\begin{aligned}
|J_{14}^{(1)}[f_n](r)| &\leq r \int_r^\infty |K_{\mu_n}(\sqrt{\lambda}\tau) \; g_n^{(2)}(\tau)| \int_r^\tau |I_{\mu_n}(\sqrt{\lambda}s)| \, \mathrm{d}s \, \mathrm{d}\tau \\
&\leq C|\lambda|^{-1} r \int_r^\infty \tau^{-2} |f_n(\tau)| \tau \, \mathrm{d}\tau \,,
\end{aligned}
$$

which leads to (4.78).
(v) Estimate of $J_{15}^{(1)}[f_n]$: The proof is parallel to that for $J_{14}^{(1)}[f_n]$ using Lemmas 4.5.2 and 4.5.4 for $k = 1$, and thus we omit here. This completes the proof of Lemma 4.3.7. $\square$

Lemmas 4.3.6 and 4.3.7 lead to the next important estimates that we shall need in the proof of Proposition 4.3.10 below. Let $c_{n,\lambda}[f_n]$ be the constant in (4.63).

Corollary 4.3.8 *Let $|n| = 1$ and $1 \leq q < p \leq \infty$ or $1 < q \leq p < \infty$, and let $\lambda \in \Sigma_{\pi-\epsilon} \cap \mathcal{B}_1(0)$ for some $\epsilon \in (0, \frac{\pi}{2})$. Then there is a positive constant $C = C(q, p, \epsilon)$ independent of β such that the following statement holds. Let $f \in C_0^\infty(D)^2$. Then for $l \in \{1, \ldots, 17\} \setminus \{9\}$ we have*

$$|c_{n,\lambda}[f_n]| \leq \frac{C}{\beta} |\lambda|^{-1+\frac{1}{q}} \|f\|_{L^q(D)}\,, \tag{4.82}$$

$$\|r^{-1} J_l^{(1)}[f_n]\|_{L^p(D)} \leq \frac{C}{\beta} |\lambda|^{-1+\frac{1}{q}-\frac{1}{p}} \|f\|_{L^q(D)}\,, \tag{4.83}$$

$$\|r^{-2} J_l^{(1)}[f_n]\|_{L^2(D)} \leq \frac{C}{\beta} |\lambda|^{-1+\frac{1}{q}} |\log \mathrm{Re}(\sqrt{\lambda})|^{\frac{1}{2}} \|f\|_{L^q(D)}\,. \tag{4.84}$$

Proof: (i) Estimate of $c_{n,\lambda}[f_n]$: Remark 4.3.5 (3) ensures that

$$|c_{n,\lambda}[f_n]| \leq \sum_{l=11,13,14,15,17} |J_l^{(1)}[f_n](1)|\,.$$

Then the estimate (4.82) follows from putting $r = 1$ to (4.77) in Lemma 4.3.7.
(ii) Estimate of $r^{-1} J_l^{(1)}[f_n]$: If $l \in \{1, \ldots, 17\} \setminus \{7, 8, 9, 16, 17\}$, then it is easy to see from the pointwise estimates in Lemmas 4.3.6 and 4.3.7 that

$$\sup_{r \geq 1} r^{\frac{2}{q}} |r^{-1} J_l^{(1)}[f_n](r)| \leq C\beta^{-1} |\lambda|^{-1} \|f\|_{L^q(D)}\,, \qquad 1 \leq q < \infty\,.$$

Thus by the Marcinkiewicz interpolation theorem we have (4.83) for the case $1 < p = q < \infty$. Moreover, again from Lemmas 4.3.6 and 4.3.7 one can see that

$$\sup_{r \geq 1} |r^{-1} J_l^{(1)}[f_n](r)| \leq C\beta^{-1} \|f\|_{L^1(D)}\,, \tag{4.85}$$

which leads to (4.83) for the case $1 < p \leq \infty$ and $q = 1$. Hence finally we have (4.83) for $1 \leq q < p \leq \infty$ and $1 < q \leq p < \infty$ by the Marcinkiewicz interpolation theorem again.
If $l \in \{7, 8, 16, 17\}$, from (4.71), (4.72), (4.80), and (4.81) we have (4.83) for the case $1 \leq p = q \leq \infty$ by the interpolation argument. Moreover, (4.68), (4.70), (4.77), and (4.79) lead to the estimate in the form (4.85) for $l \in \{7, 8, 16, 17\}$. Thus we obtain (4.83) for the case $1 \leq p \leq \infty$ and $q = 1$, and hence (4.83) for $1 \leq q \leq p \leq \infty$ by the Marcinkiewicz interpolation theorem.
(iii) Estimate of $r^{-2} J_l^{(1)}[f_n]$: The assertion (4.84) can be checked easily by a direct calculation using Lemmas 4.3.6 and 4.3.7. We note that the logarithmic factor in (4.84) is due to the estimate (4.77). The proof of Corollary 4.3.8 is complete. $\square$

Now we are in position to prove the main theorem of this subsection. Let us start with the simple proposition about the estimate for the term $V_n[K_{\mu_n}(\sqrt{\lambda}\,\cdot)]$ in (4.61).

Proposition 4.3.9 *Let $|n| = 1$, $p \in (1, \infty]$, and let $\lambda \in \Sigma_{\pi-\epsilon} \cap \mathcal{B}_1(0)$ for some $\epsilon \in (0, \frac{\pi}{2})$. Then there is a positive constant $C = C(p, \epsilon)$ independent of β such that we have*

$$\|V_n[K_{\mu_n}(\sqrt{\lambda}\,\cdot\,)]\|_{L^p(D)} \le \frac{C}{\beta}|\lambda|^{-\frac{\mathrm{Re}(\mu_n)}{2}-\frac{1}{p}}\,, \tag{4.86}$$

$$\|\frac{V_n[K_{\mu_n}(\sqrt{\lambda}\,\cdot\,)]}{|x|}\|_{L^2(D)} \le \frac{C}{\beta^2}|\lambda|^{-\frac{\mathrm{Re}(\mu_n)}{2}}\,. \tag{4.87}$$

Proof: It is easy to see from the definition of $V_n[\,\cdot\,]$ in (4.27) that

$$|V_n[K_{\mu_n}(\sqrt{\lambda}\,\cdot\,)]| \le Cr^{-2}\Big(|F_n(\sqrt{\lambda};\beta)| + \Big|\int_1^r s^2 K_{\mu_n}(\sqrt{\lambda}s)\,\mathrm{d}s\Big|\Big) + C\Big|\int_r^\infty K_{\mu_n}(\sqrt{\lambda}s)\,\mathrm{d}s\Big|\,.$$

By the results in Lemma 4.5.3 for $k = 0$ in Appendix 4.5.2 we have

$$|V_n[K_{\mu_n}(\sqrt{\lambda}\,\cdot\,)](r)| \le C\beta^{-1}|\lambda|^{-\frac{\mathrm{Re}(\mu_n)}{2}} r^{-\mathrm{Re}(\mu_n)+1}\,, \qquad 1 \le r < \mathrm{Re}(\sqrt{\lambda})^{-1}\,, \tag{4.88}$$

$$|V_n[K_{\mu_n}(\sqrt{\lambda}\,\cdot\,)](r)| \le C\beta^{-1}|\lambda|^{-\frac{3}{2}} r^{-2}\,, \qquad r \ge \mathrm{Re}(\sqrt{\lambda})^{-1}\,. \tag{4.89}$$

Then for $p \in [1, \infty]$ we find

$$\sup_{r\ge 1} r^{\frac{2}{p}}|V_n[K_{\mu_n}(\sqrt{\lambda}\,\cdot\,)](r)| \le C\beta^{-1}|\lambda|^{-\frac{\mathrm{Re}(\mu_n)}{2}-\frac{1}{p}}\,.$$

Hence by an interpolation argument (4.86) follows. Moreover, a direct calculation combined with (4.88), (4.89), and $(\mathrm{Re}(\mu_n(\beta)) - 1)^{\frac{1}{2}} \approx O(\beta)$ yield (4.87). This completes the proof. □

The next proposition gives the estimate for the term $V_n[\Phi_{n,\lambda}[f_n]]$ in (4.61).

Proposition 4.3.10 *Let $|n| = 1$ and $1 \le q < p \le \infty$ or $1 < q \le p < \infty$, and let $\lambda \in \Sigma_{\pi-\epsilon} \cap \mathcal{B}_1(0)$ for some $\epsilon \in (0, \frac{\pi}{2})$. Then there is a positive constant $C = C(q, p, \epsilon)$ independent of β such that for $f \in C_0^\infty(D)^2$ we have*

$$\|V_n[\Phi_{n,\lambda}[f_n]]\|_{L^p(D)} \le \frac{C}{\beta}|\lambda|^{-1+\frac{1}{q}-\frac{1}{p}}\|f\|_{L^q(D)}\,, \tag{4.90}$$

$$\|\frac{V_n[\Phi_{n,\lambda}[f_n]]}{|x|}\|_{L^2(D)} \le \frac{C}{\beta}|\lambda|^{-1+\frac{1}{q}}|\log \mathrm{Re}(\sqrt{\lambda})|^{\frac{1}{2}}\|f\|_{L^q(D)}\,. \tag{4.91}$$

Proof: The definition of the Biot-Savart law $V_n[\,\cdot\,]$ in (4.27) leads to the next representations for the radial part $V_{r,n}[\Phi_{n,\lambda}[f_n]]$ and the angular part $V_{\theta,n}[\Phi_{n,\lambda}[f_n]]$ of $V_n[\Phi_{n,\lambda}[f_n]]$:

$$V_{r,n}[\Phi_{n,\lambda}[f_n]] \;=\; -\frac{in}{2r}\Big(\frac{c_{n,\lambda}[f_n]}{r} - \frac{1}{r}\int_1^r s^2\Phi_{n,\lambda}[f_n](s)\,\mathrm{d}s - r\int_r^\infty \Phi_{n,\lambda}[f_n](s)\,\mathrm{d}s\Big)\,,$$

$$V_{\theta,n}[\Phi_{n,\lambda}[f_n]] \;=\; \frac{1}{2r}\Big(\frac{c_{n,\lambda}[f_n]}{r} - \frac{1}{r}\int_1^r s^2\Phi_{n,\lambda}[f_n](s)\,\mathrm{d}s + r\int_r^\infty \Phi_{n,\lambda}[f_n](s)\,\mathrm{d}s\Big)\,,$$

where $c_{n,\lambda}[f_n]$ is defined in (4.63). From Lemma 4.3.4 and Remark 4.3.5 (1) and (3) we see that

$$\begin{aligned} &\frac{c_{n,\lambda}[f_n]}{r} - \frac{1}{r}\int_1^r s^2 \Phi_{n,\lambda}[f_n](s)\,\mathrm{d}s \\ &= r^{-1} \sum_{l=11,13,14,15} J_l^{(1)}[f_n](1) - \sum_{l=1}^{8} r^{-1} J_l^{(1)}[f_n](r)\,. \end{aligned} \tag{4.92}$$

Then, by (4.92) and the decomposition of the integral $r\int_r^\infty \Phi_{n,\lambda}[f_n](s)\,\mathrm{d}s$ in Lemma 4.3.4, we find the following pointwise estimate of $V_n[\Phi_{n,\lambda}[f_n]](r)$:

$$\begin{aligned} &|V_n[\Phi_{n,\lambda}[f_n]](r)| \\ &\le C\Big(r^{-2} \sum_{l=11,13,14,15} |J_l^{(1)}[f_n](1)| + \sum_{l\in\{1,\ldots,17\}\setminus\{9\}} |r^{-1} J_l^{(1)}[f_n](r)|\Big)\,. \end{aligned} \tag{4.93}$$

Thus the assertions (4.90) and (4.91) follow from Corollary 4.3.8. This completes the proof. □

Finally we give a proof of Theorem 4.3.2, which is a direct consequence of Corollary 4.3.8 and Propositions 4.3.9 and 4.3.10.

Proof of Theorem 4.3.2: In view of Proposition 4.3.10, it suffices to show that the first term in the right-hand side of (4.61) satisfies the estimates (4.66) and (4.67). By using Proposition 4.3.1 and (4.82) in Corollary 4.3.8, one can see that (4.66) and (4.67) respectively follow from (4.86) and (4.87) in Proposition 4.3.9. This completes the proof of Theorem 4.3.2. □

Estimates of the vorticity for (RS$^{\mathrm{ed}}_{f,n}$) with $|n| = 1$

This subsection is devoted to the estimate of the vorticity $\omega^{\mathrm{ed}}_{f,n}(r) = (\mathrm{rot}\, w^{\mathrm{ed}}_{f,n})e^{-in\theta}$ with $|n| = 1$, where $w^{\mathrm{ed}}_{f,n}$ solves (RS$^{\mathrm{ed}}_{f,n}$) in Subsection 4.3.1. We recall that $\omega^{\mathrm{ed}}_{f,n}$ is represented as

$$\omega^{\mathrm{ed}}_{f,n}(r) \;=\; -\frac{c_{n,\lambda}[f_n]}{F_n(\sqrt{\lambda};\beta)} K_{\mu_n}(\sqrt{\lambda} r) + \Phi_{n,\lambda}[f_n](r)$$

by (4.64). The main result is stated as follows. Let β_0 be the constant in Proposition 4.3.1.

Theorem 4.3.11 *Let $|n| = 1$, $q \in (1,\infty)$, and $\tilde{q} \in (\max\{1, \frac{q}{2}\}, q]$. Fix $\epsilon \in (0, \frac{\pi}{2})$. Then there is a positive constant $C = C(q,\tilde{q},\epsilon)$ independent of β such that the following statement holds. Let $f \in C_0^\infty(D)^2$ and $\beta \in (0,\beta_0)$. Set*

$$\omega^{\mathrm{ed}\,(1)}_{f,n}(r) \;=\; -\frac{c_{n,\lambda}[f_n]}{F_n(\sqrt{\lambda};\beta)} K_{\mu_n}(\sqrt{\lambda} r)\,, \qquad \omega^{\mathrm{ed}\,(2)}_{f,n}(r) \;=\; \Phi_{n,\lambda}[f_n](r)\,. \tag{4.94}$$

Then for $\lambda \in \Sigma_{\pi-\epsilon} \cap \mathcal{B}_{e^{-\frac{1}{6\beta}}}(0)$ *we have*

$$\|\omega_{f,n}^{\mathrm{ed}\,(1)}\|_{L^2(D)} \le \frac{C}{\beta^2}|\lambda|^{-1+\frac{1}{q}}\|f\|_{L^q(D)}\,, \tag{4.95}$$

$$\|\frac{\omega_{f,n}^{\mathrm{ed}\,(2)}}{|x|}\|_{L^{\tilde q}(D)} \le \frac{C}{\beta}|\lambda|^{-\frac{1}{\tilde q}+\frac{1}{q}}\|f\|_{L^q(D)}\,, \tag{4.96}$$

$$\big|\langle\omega_{f,n}^{\mathrm{ed}\,(1)}, \frac{(w_{f,r}^{\mathrm{ed}})_n}{|x|}\rangle_{L^2(D)}\big| \le \frac{C}{\beta^5}|\lambda|^{-2+\frac{2}{q}}\|f\|_{L^q(D)}^2\,. \tag{4.97}$$

Moreover, (4.95)*,* (4.96)*, and* (4.97) *hold all for* $f \in L^q(D)^2$*.*

Proof: (i) Estimate of $\omega_{f,n}^{\mathrm{ed}\,(1)}$: The estimate (4.95) is a direct consequence of Proposition 4.3.1, (4.82) in Corollary 4.3.8, and (4.167) with $p=2$ in Lemma 4.5.5 in Appendix 4.5.2.
(ii) Estimate of $|x|^{-1}\omega_{f,n}^{\mathrm{ed}\,(2)}$: We decompose $\omega_{f,n}^{\mathrm{ed}\,(2)}$ into $\omega_{f,n}^{\mathrm{ed}\,(2)} = \sum_{l=1}^4 \Phi_{n,\lambda}^{(l)}[f_n]$ by setting

$$\begin{aligned}
\Phi_{n,\lambda}^{(1)}[f_n] &= -K_{\mu_n}(\sqrt{\lambda}r)\int_1^r I_{\mu_n}(\sqrt{\lambda}s)\, g_n^{(1)}(s)\,\mathrm{d}s\,,\\
\Phi_{n,\lambda}^{(2)}[f_n] &= -\sqrt{\lambda}K_{\mu_n}(\sqrt{\lambda}r)\int_1^r sI_{\mu_n+1}(\sqrt{\lambda}s)\, f_{\theta,n}(s)\,\mathrm{d}s\,,\\
\Phi_{n,\lambda}^{(3)}[f_n] &= I_{\mu_n}(\sqrt{\lambda}r)\int_r^\infty K_{\mu_n}(\sqrt{\lambda}s)\, g_n^{(2)}(s)\,\mathrm{d}s\,,\\
\Phi_{n,\lambda}^{(4)}[f_n] &= \sqrt{\lambda}I_{\mu_n}(\sqrt{\lambda}r)\int_r^\infty sK_{\mu_n-1}(\sqrt{\lambda}s)\, f_{\theta,n}(s)\,\mathrm{d}s\,.
\end{aligned}$$

Then the assertion (4.96) follows from the estimates of each term $|x|^{-1}\Phi_{n,\lambda}^{(l)}[f_n]$, $l \in \{1.2.3.4\}$.
(I) Estimates of $|x|^{-1}\Phi_{n,\lambda}^{(1)}[f_n]$ and $|x|^{-1}\Phi_{n,\lambda}^{(2)}[f_n]$: We give a proof only for $|x|^{-1}\Phi_{n,\lambda}^{(2)}[f_n]$ since the proof for $|x|^{-1}\Phi_{n,\lambda}^{(1)}[f_n]$ is similar. The Minkowski inequality leads to

$$\begin{aligned}
\|\frac{\Phi_{n,\lambda}^{(2)}[f_n]}{|x|}\|_{L^{\tilde q}(D)} &= |\lambda|^{\frac12}\bigg(\int_1^\infty\bigg|\int_1^r r^{-1}K_{\mu_n}(\sqrt{\lambda}r)sI_{\mu_n+1}(\sqrt{\lambda}s)\, f_{\theta,n}(s)\,\mathrm{d}s\bigg|^{\tilde q} r\,\mathrm{d}r\bigg)^{\frac{1}{\tilde q}}\\
&\le |\lambda|^{\frac12}\int_1^\infty |sI_{\mu_n+1}(\sqrt{\lambda}s)\, f_{\theta,n}(s)|\bigg(\int_s^\infty |r^{-1}K_{\mu_n}(\sqrt{\lambda}r)|^{\tilde q} r\,\mathrm{d}r\bigg)^{\frac{1}{\tilde q}}\mathrm{d}s\,.
\end{aligned}$$

By (4.154) and (4.156) for $k=1$ in Lemma 4.5.2 and (4.168) and (4.169) in Lemma 4.5.5, we have

$$\|\frac{\Phi_{n,\lambda}^{(2)}[f_n]}{|x|}\|_{L^{\tilde q}(D)} \le C|\lambda|\int_1^{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}} s^{\frac{2}{\tilde q}}|f_n(s)|s\,\mathrm{d}s + C\,|\lambda|^{-\frac{1}{2\tilde q}}\int_{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}}^\infty s^{-2+\frac{1}{\tilde q}}|f_n(s)|s\,\mathrm{d}s\,,$$

which implies (4.96) since $\frac{q}{q-1}(-2+\frac{1}{\tilde q})+2<0$ holds if $\tilde q \in (\max\{1,\frac{q}{2}\},q]$.
(II) Estimates of $|x|^{-1}\Phi_{n,\lambda}^{(3)}[f_n]$ and $|x|^{-1}\Phi_{n,\lambda}^{(4)}[f_n]$: We give a proof only for $|x|^{-1}\Phi_{n,\lambda}^{(4)}[f_n]$. After using the Minkowski inequality in the same way as above, from (4.151), (4.153), and

(4.155) with $k=1$ in Lemma 4.5.2 and (4.170) and (4.171) in Lemma 4.5.5, we have

$$\begin{aligned}\Big\|\frac{\Phi^{(4)}_{n,\lambda}[f_n]}{|x|}\Big\|_{L^{\tilde q}(D)} &\le C|\lambda|^{\frac12}\int_1^\infty \big|sK_{\mu_n-1}(\sqrt{\lambda}s)\,f_{\theta,n}(s)\big|\bigg(\int_1^s |r^{-1}I_{\mu_n}(\sqrt{\lambda}r)|^{\tilde q} r\,\mathrm{d}r\bigg)^{\frac{1}{\tilde q}}\,\mathrm{d}s\\ &\le C\beta^{-1}|\lambda|\int_1^{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}} s^{\frac{2}{\tilde q}}\,|f_n(s)|s\,\mathrm{d}s + C|\lambda|^{-\frac{1}{2\tilde q}}\int_{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}}^\infty s^{-2+\frac{1}{\tilde q}}|f_n(s)|s\,\mathrm{d}s\,,\end{aligned}$$

which leads to (4.96). Hence we obtain the assertion (4.96).
(iii) Estimate of $\big|\langle \omega^{\mathrm{ed}\,(1)}_{f,n}, |x|^{-1}(w^{\mathrm{ed}}_{f,r})_n\rangle_{L^2(D)}\big|$: From (4.61) and (4.94) we see that

$$\begin{aligned}\Big|\Big\langle \omega^{\mathrm{ed}\,(1)}_{f,n}, \frac{(w^{\mathrm{ed}}_{f,r})_n}{|x|}\Big\rangle_{L^2(D)}\Big| \le \Big|\frac{c_{n,\lambda}[f_n]}{F_n(\sqrt{\lambda};\beta)}\Big|^2\Big|\Big\langle K_{\mu_n}(\sqrt{\lambda}\,\cdot\,), \frac{V_{r,n}[K_{\mu_n}(\sqrt{\lambda}\,\cdot\,)]}{|x|}\Big\rangle_{L^2(D)}\Big|\\ + \Big|\frac{c_{n,\lambda}[f_n]}{F_n(\sqrt{\lambda};\beta)}\Big|\,\Big|\Big\langle K_{\mu_n}(\sqrt{\lambda}\,\cdot\,), \frac{V_{r,n}[\Phi_{n,\lambda}[f_n]]}{|x|}\Big\rangle_{L^2(D)}\Big|\,.\end{aligned} \tag{4.98}$$

Then, by Proposition 4.3.1 and (4.82) in Corollary 4.3.8 combined with the results in Lemma 4.5.2 for $k=0$ and (4.88) and (4.89) in the proof Proposition 4.3.9, we have

$$\begin{aligned}&\Big|\frac{c_{n,\lambda}[f_n]}{F_n(\sqrt{\lambda};\beta)}\Big|^2\Big|\Big\langle K_{\mu_n}(\sqrt{\lambda}\,\cdot\,), \frac{V_{r,n}[K_{\mu_n}(\sqrt{\lambda}\,\cdot\,)]}{|x|}\Big\rangle_{L^2(D)}\Big|\\ &\le C\beta^{-3}|\lambda|^{-2+\frac{2}{q}}\|f\|^2_{L^q(D)}\bigg(\int_1^{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}} r^{-\mathrm{Re}(\mu_n)}\,\mathrm{d}r + |\lambda|^{\mathrm{Re}(\mu_n)-\frac12}\int_{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}}^\infty e^{-\mathrm{Re}(\sqrt{\lambda})r}\,\mathrm{d}r\bigg)\,.\end{aligned}$$

By (4.93) in the proof of Proposition 4.3.10 combined with Lemmas 4.3.6 and 4.3.7, we have

$$\begin{aligned}&\Big|\frac{c_{n,\lambda}[f_n]}{F_n(\sqrt{\lambda};\beta)}\Big|\,\Big|\Big\langle K_{\mu_n}(\sqrt{\lambda}\,\cdot\,), \frac{V_{r,n}[\Phi_{n,\lambda}[f_n]]}{|x|}\Big\rangle_{L^2(D)}\Big|\\ &\le C\beta^{-2}|\lambda|^{-2+\frac{2}{q}}\|f\|^2_{L^q(D)}\bigg(\int_1^{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}} r^{-\mathrm{Re}(\mu_n)}\,\mathrm{d}r + |\lambda|^{\frac{\mathrm{Re}(\mu_n)}{2}-\frac14}\int_{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}}^\infty r^{\frac12}e^{-\mathrm{Re}(\sqrt{\lambda})r}\,\mathrm{d}r\bigg)\,.\end{aligned}$$

Hence, by inserting the above two estimates into (4.98), one can check that the assertion (4.97) holds. This completes the proof of Theorem 4.3.11. □

4.3.2 Problem II: External force $\mathrm{div}\,F$ and Dirichlet condition

In this subsection we consider the following resolvent problem for $(w,r)=(w^{\mathrm{ed}}_{\mathrm{div}F}, r^{\mathrm{ed}}_{\mathrm{div}F})$:

$$\left\{\begin{aligned}\lambda w - \Delta w + \beta U^{\perp}\mathrm{rot}\,w + \nabla r \;&=\; \mathrm{div}\,F\,, \quad x\in D\,,\\ \mathrm{div}\,w \;&=\; 0\,, \quad x\in D\,,\\ w|_{\partial D} \;&=\; 0\,.\end{aligned}\right. \tag{$\mathrm{RS}^{\mathrm{ed}}_{\mathrm{div}F}$}$$

In particular, the estimates for the ±1-Fourier mode of $w^{\mathrm{ed}}_{\mathrm{div}F}$ are our interest. Here $F=(F_{ij}(x))_{1\le i,j\le 2}$ is a 2×2 matrix. We recall that the operator div on matrices $G=(G_{ij}(x))_{1\le i,j\le 2}$ is defined as $\mathrm{div}\,G=(\partial_1 G_{11}+\partial_2 G_{12}, \partial_1 G_{21}+\partial_2 G_{22})^\top$. The assumption on F is as follows: let us take the constant $\gamma \in (\frac{1}{2}, 1)$ of Assumption 4.1.1 in the introduction. Fix $\gamma' \in (\frac{1}{2}, \gamma)$. Then we assume that F belongs to the function space $X_{\gamma'}(D)$:

$$X_{\gamma'}(D) \,=\, \{F \in L^2(D)^{2\times 2} \mid |x|^{\gamma'} F \in L^2(D)^{2\times 2}\}\,. \tag{4.99}$$

This definition is motivated from the property of the matrix $R \otimes v + R \otimes v$ appearing in ($\mathrm{RS}^{\mathrm{ed}}$), where R is the function in Assumption 4.1.1 and $v \in D(\mathbb{A}_V)$ is a solution to (RS). In view of the regularity of F, we define the class of solutions to ($\mathrm{RS}^{\mathrm{ed}}_{\mathrm{div}F}$) in each Fourier mode by the weak form. Let $n \in \mathbb{Z} \setminus \{0\}$ and $L^q_\sigma(D)$, $q \in (1, \infty)$, denote the L^q-closure of $C^\infty_{0,\sigma}(D)$, and let $p \in (\frac{2}{\gamma'}, \infty)$. Then a velocity $w_n \in \mathcal{P}_n(L^p_\sigma(D) \cap W^{1,p}_0(D)^2)$ is said to be a weak solution to ($\mathrm{RS}^{\mathrm{ed}}_{\mathrm{div}F}$) replacing $\mathrm{div}\, F$ by $(\mathrm{div}\, F)_n = \mathcal{P}_n \mathrm{div}\, F$ if

$$\begin{aligned}&\lambda\langle w_n, \varphi\rangle_{L^2(D)} + \langle \nabla w_n, \nabla\varphi\rangle_{L^2(D)} + \beta\langle U^{\perp}\mathrm{rot}\, w_n, \varphi\rangle_{L^2(D)} \\ &= -\langle F, \nabla \mathcal{P}_n\varphi\rangle_{L^2(D)}\end{aligned} \tag{$\mathrm{RS}^{\mathrm{ed}}_{\mathrm{div}F,n}$}$$

holds for all $\varphi \in C^\infty_{0,\sigma}(D)^2$. Then the pressure $r \in W^{1,p}_{\mathrm{loc}}(\overline{D})$ is recovered by a standard functional analytic argument; see [56, page 73, Lemma 2.21] for instance. The uniqueness of weak solutions is trivial thanks to the representation formula (4.100) below. In the following we consider the solutions to ($\mathrm{RS}^{\mathrm{ed}}_{\mathrm{div}F,n}$) for given $F \in X_{\gamma'}(D)$.

Let $n \in \mathbb{Z} \setminus \{0\}$. By the solution formula (4.61) in Subsection 4.3.1, at least when $F \in C^\infty_0(D)^{2\times 2}$, we can represent the n-Fourier mode of the solution $w^{\mathrm{ed}}_{\mathrm{div}F}$ to ($\mathrm{RS}^{\mathrm{ed}}_{\mathrm{div}F}$) as

$$w^{\mathrm{ed}}_{\mathrm{div}\,F,n} \,=\, -\frac{c_{n,\lambda}[(\mathrm{div}\,F)_n]}{F_n(\sqrt{\lambda};\beta)} V_n[K_{\mu_n}(\sqrt{\lambda}\,\cdot)] + V_n[\Phi_{n,\lambda}[(\mathrm{div}F)_n]]\,, \tag{4.100}$$

if $\lambda \in \mathbb{C} \setminus \overline{\mathbb{R}_-}$ satisfies $F_n(\sqrt{\lambda};\beta) \neq 0$. Here $c_{n,\lambda}[\,\cdot\,]$, $F_n(\sqrt{\lambda};\beta)$, $V_n[\,\cdot\,]$, and $\Phi_{n,\lambda}[\,\cdot\,]$ are respectively defined in (4.63), (4.59), (4.27), and (4.62). The vorticity is given by

$$\mathrm{rot}\, w^{\mathrm{ed}}_{\mathrm{div}F,n} \,=\, -\frac{c_{n,\lambda}[(\mathrm{div}\,F)_n]}{F_n(\sqrt{\lambda};\beta)} K_{\mu_n}(\sqrt{\lambda}r)e^{in\theta} + \Phi_{n,\lambda}[(\mathrm{div}\,F)_n](r)e^{in\theta}\,. \tag{4.101}$$

We prove the estimates of (4.100) and (4.101) in the next two subsections. Before concluding this subsection, we prepare a useful lemma for the calculation concerning $\Phi_{n,\lambda}[(\mathrm{div}\,F)_n]$.

Lemma 4.3.12 *Let $n \in \mathbb{Z} \setminus \{0\}$ and $F \in C^\infty_0(D)^{2\times 2}$. Then there are functions $\widetilde{F}^{(k)}_n = \widetilde{F}^{(k)}_n(r)$, $k \in \{1, \ldots, 7\}$, each of which is a linear combination containing the n-Fourier mode of the components of $F = (F_{ij})_{1\le i,j\le 2}$, such that $\Phi_{n,\lambda}[(\mathrm{div}\,F)_n]$ is represented as*

$$\begin{aligned}&\Phi_{n,\lambda}[(\mathrm{div}\,F)_n](r) \\ &= -K_{\mu_n}(\sqrt{\lambda}r)\bigg(\int_1^r s^{-1} I_{\mu_n}(\sqrt{\lambda}s)\,\widetilde{F}^{(1)}_n(s)\,\mathrm{d}s \\ &\qquad\qquad + \sqrt{\lambda}\int_1^r I_{\mu_n+1}(\sqrt{\lambda}s)\,\widetilde{F}^{(2)}_n(s)\,\mathrm{d}s - \lambda\int_1^r sI_{\mu_n}(\sqrt{\lambda}s)\,\widetilde{F}^{(3)}_n(s)\,\mathrm{d}s\bigg) \\ &\quad + I_{\mu_n}(\sqrt{\lambda}r)\bigg(\int_r^\infty s^{-1}K_{\mu_n}(\sqrt{\lambda}s)\,\widetilde{F}^{(4)}_n(s)\,\mathrm{d}s \\ &\qquad\qquad + \sqrt{\lambda}\int_r^\infty K_{\mu_n-1}(\sqrt{\lambda}s)\,\widetilde{F}^{(5)}_n(s)\,\mathrm{d}s + \lambda\int_r^\infty sK_{\mu_n}(\sqrt{\lambda}s)\,\widetilde{F}^{(6)}_n(s)\,\mathrm{d}s\bigg) \\ &\quad - \sqrt{\lambda}r\big(K_{\mu_n}(\sqrt{\lambda}r)I_{\mu_n+1}(\sqrt{\lambda}r) + K_{\mu_n-1}(\sqrt{\lambda}r)I_{\mu_n}(\sqrt{\lambda}r)\big)\,\widetilde{F}^{(7)}_n(r)\,.\end{aligned} \tag{4.102}$$

Proof: Let $n \in \mathbb{Z}\setminus\{0\}$. By the definition of $\operatorname{div} F$, there are functions $G_n^{(l)} \in C_0^\infty((1,\infty))$, $l \in \{1,\dots,4\}$, such that the n-Fourier mode $(\operatorname{div} F)_n$ has a representation

$$
\begin{aligned}
(\operatorname{div} F)_n \;&=\; (\operatorname{div} F)_{r,n} e^{in\theta}\mathbf{e}_r + (\operatorname{div} F)_{\theta,n} e^{in\theta}\mathbf{e}_\theta \\
&=\; \big(\partial_r G_n^{(1)}(r) + \frac{1}{r} G_n^{(2)}(r)\big) e^{in\theta}\mathbf{e}_r + \big(\partial_r G_n^{(3)}(r) + \frac{1}{r} G_n^{(4)}(r)\big) e^{in\theta}\mathbf{e}_\theta\,.
\end{aligned}
\tag{4.103}
$$

Then there are functions $H_n^{(m)} \in C_0^\infty((1,\infty))$, $m \in \{1,\dots,4\}$, each of which is a linear combination containing the n-mode of the components of $F = (F_{ij})_{1\le i,j\le 2}$, such that

$$
\mu_n(\operatorname{div} F)_{\theta,n}(r) + in(\operatorname{div} F)_{r,n}(r) \;=\; \partial_r H_n^{(1)}(r) + \frac{1}{r} H_n^{(2)}(r)\,, \tag{4.104}
$$

$$
\mu_n(\operatorname{div} F)_{\theta,n}(r) - in(\operatorname{div} F)_{r,n}(r) \;=\; \partial_r H_n^{(3)}(r) + \frac{1}{r} H_n^{(4)}(r)\,. \tag{4.105}
$$

By inserting (4.103)–(4.105) into the representation of $\Phi[f_n]$ in (4.62) replacing f_n by $(\operatorname{div} F)_n$, and using the next relations of Bessel functions $I_\mu(z)$ and $K_\mu(z)$ (see [1] page 376):

$$
\frac{\mathrm{d}I_\mu}{\mathrm{d}z}(z) \;=\; \frac{\mu}{z} I_\mu(z) + I_{\mu+1}(z)\,, \qquad \frac{\mathrm{d}K_\mu}{\mathrm{d}z}(z) \;=\; -\frac{\mu}{z} K_\mu(z) - K_{\mu-1}(z)\,,
$$

we can obtain the assertion (4.102). We omit the details since the calculations are straightforward using integration by parts. The proof is complete. □

Estimates of the velocity solving ($\mathrm{RS}^{\mathrm{ed}}_{\operatorname{div} F,n}$) with $|n| = 1$

The main result of this subsection is the estimates of $w^{\mathrm{ed}}_{\operatorname{div} F,n}$ represented as in (4.100). Let us recall that β_0 is the constant in Proposition 4.3.1.

Theorem 4.3.13 *Let $|n| = 1$, $\gamma' \in (\frac{1}{2},\gamma)$, and $p \in (\frac{2}{\gamma'},\infty)$. Fix $\epsilon \in (0,\frac{\pi}{2})$. Then there is a positive constant $C = C(\gamma',p,\epsilon)$ independent of β such that the following statement holds. Let $F \in C_0^\infty(D)^{2\times 2}$ and $\beta \in (0,\beta_0)$. Then for $\lambda \in \Sigma_{\pi-\epsilon} \cap \mathcal{B}_{e^{-\frac{1}{6\beta}}}(0)$ we have*

$$
\|w^{\mathrm{ed}}_{\operatorname{div} F,n}\|_{L^p(D)} \le \frac{C}{\beta^2} |\lambda|^{-\frac{1}{p}} \||x|^{\gamma'} F\|_{L^2(D)}\,, \tag{4.106}
$$

$$
\Big\|\frac{w^{\mathrm{ed}}_{\operatorname{div} F,n}}{|x|}\Big\|_{L^2(D)} \le \frac{C}{\beta^3} \||x|^{\gamma'} F\|_{L^2(D)}\,. \tag{4.107}
$$

Moreover, (4.106) *and* (4.107) *hold all for $F \in X_{\gamma'}(D)$ defined in* (4.99).

By following a similar procedure as in Subsection 4.3.1, we give the proof of Theorem 4.3.13 at the end of this subsection. We firstly focus on the term $V_n[\Phi_{n,\lambda}[(\operatorname{div} F)_n]]$ in (4.100). By using Lemma 4.3.12, one can see that the next decomposition holds. Let $\widetilde{F}_n^{(k)}(r)$, $k \in \{1,\dots,7\}$, be the functions in Lemma 4.3.12.

Lemma 4.3.14 *Let $n \in \mathbb{Z}\setminus\{0\}$ and $F \in C_0^\infty(D)^{2\times 2}$. Then we have*

$$
\frac{1}{r^{|n|}} \int_1^r s^{1+|n|} \Phi_{n,\lambda}[(\operatorname{div} F)_n](s)\,\mathrm{d}s \;=\; \sum_{l=1}^{10} J_l^{(2)}[(\operatorname{div} F)_n](r)\,, \tag{4.108}
$$

where

$$
\begin{aligned}
J_1^{(2)}[(\operatorname{div} F)_n](r) &= -\frac{1}{r^{|n|}} \int_1^r \tau^{-1} I_{\mu_n}(\sqrt{\lambda}\tau)\, \widetilde{F}_n^{(1)}(\tau) \int_\tau^r s^{1+|n|} K_{\mu_n}(\sqrt{\lambda}s)\, \mathrm{d}s\, \mathrm{d}\tau\,, \\
J_2^{(2)}[(\operatorname{div} F)_n](r) &= -\frac{\sqrt{\lambda}}{r^{|n|}} \int_1^r I_{\mu_n+1}(\sqrt{\lambda}\tau)\, \widetilde{F}_n^{(2)}(\tau) \int_\tau^r s^{1+|n|} K_{\mu_n}(\sqrt{\lambda}s)\, \mathrm{d}s\, \mathrm{d}\tau\,, \\
J_3^{(2)}[(\operatorname{div} F)_n](r) &= -\frac{\lambda}{r^{|n|}} \int_1^r \tau I_{\mu_n}(\sqrt{\lambda}\tau)\, \widetilde{F}_n^{(3)}(\tau) \int_\tau^r s^{1+|n|} K_{\mu_n}(\sqrt{\lambda}s)\, \mathrm{d}s\, \mathrm{d}\tau\,, \\
J_4^{(2)}[(\operatorname{div} F)_n](r) &= \frac{1}{r^{|n|}} \int_1^r \tau^{-1} K_{\mu_n}(\sqrt{\lambda}\tau)\, \widetilde{F}_n^{(4)}(\tau) \int_1^\tau s^{1+|n|} I_{\mu_n}(\sqrt{\lambda}s)\, \mathrm{d}s\, \mathrm{d}\tau\,, \\
J_5^{(2)}[(\operatorname{div} F)_n](r) &= \frac{1}{r^{|n|}} \bigg(\int_r^\infty \tau^{-1} K_{\mu_n}(\sqrt{\lambda}\tau)\, \widetilde{F}_n^{(4)}(\tau)\, \mathrm{d}\tau \bigg) \bigg(\int_1^r s^{1+|n|} I_{\mu_n}(\sqrt{\lambda}s)\, \mathrm{d}s \bigg)\,, \\
J_6^{(2)}[(\operatorname{div} F)_n](r) &= \frac{\sqrt{\lambda}}{r^{|n|}} \int_1^r K_{\mu_n-1}(\sqrt{\lambda}\tau)\, \widetilde{F}_n^{(5)}(\tau) \int_1^\tau s^{1+|n|} I_{\mu_n}(\sqrt{\lambda}s)\, \mathrm{d}s\, \mathrm{d}\tau\,, \\
J_7^{(2)}[(\operatorname{div} F)_n](r) &= \frac{\sqrt{\lambda}}{r^{|n|}} \bigg(\int_r^\infty K_{\mu_n-1}(\sqrt{\lambda}\tau)\, \widetilde{F}_n^{(5)}(\tau)\, \mathrm{d}\tau \bigg) \bigg(\int_1^r s^{1+|n|} I_{\mu_n}(\sqrt{\lambda}s)\, \mathrm{d}s \bigg)\,, \\
J_8^{(2)}[(\operatorname{div} F)_n](r) &= \frac{\lambda}{r^{|n|}} \int_1^r \tau K_{\mu_n}(\sqrt{\lambda}\tau)\, \widetilde{F}_n^{(6)}(\tau) \int_1^\tau s^{1+|n|} I_{\mu_n}(\sqrt{\lambda}s)\, \mathrm{d}s\, \mathrm{d}\tau\,, \\
J_9^{(2)}[(\operatorname{div} F)_n](r) &= \frac{\lambda}{r^{|n|}} \bigg(\int_r^\infty \tau K_{\mu_n}(\sqrt{\lambda}\tau)\, \widetilde{F}_n^{(6)}(\tau)\, \mathrm{d}\tau \bigg) \bigg(\int_1^r s^{1+|n|} I_{\mu_n}(\sqrt{\lambda}s)\, \mathrm{d}s \bigg)\,, \\
J_{10}^{(2)}[(\operatorname{div} F)_n](r) &= -\frac{\sqrt{\lambda}}{r^{|n|}} \int_1^r s\big(K_{\mu_n}(\sqrt{\lambda}s) I_{\mu_n+1}(\sqrt{\lambda}s) + K_{\mu_n-1}(\sqrt{\lambda}s) I_{\mu_n}(\sqrt{\lambda}s)\big)\, \widetilde{F}_n^{(7)}(s)\, \mathrm{d}s\,.
\end{aligned}
$$

Here $\widetilde{F}_n^{(k)}(r)$*,* $k \in \{1, \ldots, 7\}$*, are the functions in Lemma 4.3.12.*

Proof: The assertion follows by inserting (4.102) in Lemma 4.3.12 into the left-hand side of (4.108), and changing order of integration as $\int_1^r \int_1^s \mathrm{d}\tau\, \mathrm{d}s = \int_1^r \int_\tau^r \mathrm{d}s\, \mathrm{d}\tau$ and $\int_1^r \int_s^\infty \mathrm{d}\tau\, \mathrm{d}s = \int_1^r \int_1^\tau \mathrm{d}s\, \mathrm{d}\tau + \int_r^\infty \mathrm{d}\tau \int_1^r \mathrm{d}s$. This completes the proof. $\square$

The next lemma gives the estimates to $J_l^{(2)}[(\operatorname{div} F)_n]$, $l \in \{1, \ldots, 10\}$, in Lemma 4.3.14.

Lemma 4.3.15 *Let* $|n| = 1$ *and* $\gamma' \in (\frac{1}{2}, \gamma)$*, and let* $\lambda \in \Sigma_{\pi-\epsilon} \cap \mathcal{B}_1(0)$ *for some* $\epsilon \in (0, \frac{\pi}{2})$*. Then there is a positive constant* $C = C(\gamma', \epsilon)$ *independent of* β *such that the following statement holds. Let* $F \in C_0^\infty(D)^{2\times 2}$*. Then for* $l \in \{1, \cdots, 10\}$ *we have*

$$
\big|J_l^{(2)}[(\operatorname{div} F)_n](r)\big| \le \frac{C}{\beta} (|\lambda|^{\frac{1}{2}} r^2 + r^{2-\mathrm{Re}(\mu_n)} + r^{1-\gamma'})\, \||x|^{\gamma'} F\|_{L^2(D)}\,, \qquad 1 \le r < \mathrm{Re}(\sqrt{\lambda})^{-1}\,, \quad (4.109)
$$

$$
\big|J_l^{(2)}[(\operatorname{div} F)_n](r)\big| \le \frac{C}{\beta} (|\lambda|^{-\frac{1}{2}} + r^{1-\gamma'})\, \||x|^{\gamma'} F\|_{L^2(D)}\,, \qquad r \ge \mathrm{Re}(\sqrt{\lambda})^{-1}\,. \quad (4.110)
$$

Proof: (i) Estimate of $J_1^{(2)}[(\operatorname{div} F)_n]$: For $1 \le r < \mathrm{Re}(\sqrt{\lambda})^{-1}$, by (4.154) for $k = 0$ in Lemma 4.5.2 and (4.157) for $k = 0$ in Lemma 4.5.3 in Appendix 4.5.2, we find

$$
|J_1^{(2)}[(\operatorname{div} F)_n](r)| \le C r^{-1} \int_1^r |\tau^{-1} I_{\mu_n}(\sqrt{\lambda}\tau)\, \widetilde{F}_n^{(1)}(\tau)| \bigg| \int_\tau^r s^2 K_{\mu_n}(\sqrt{\lambda}s)\, \mathrm{d}s \bigg| \mathrm{d}\tau
$$

$$\leq C r^{2-\mathrm{Re}(\mu_n)} \int_1^r \tau^{\mathrm{Re}(\mu_n)-2} |F_n(\tau)| \tau \, \mathrm{d}\tau \,,$$

which implies $|J_1^{(2)}[(\operatorname{div} F)_n](r)| \leq C r^{2-\mathrm{Re}(\mu_n)} \| |x|^{\gamma'} F \|_{L^2}$. For $r \geq \mathrm{Re}(\sqrt{\lambda})^{-1}$, by (4.154) and (4.156) for $k = 0$ in Lemma 4.5.2 and (4.158) and (4.159) for $k = 0$ in Lemma 4.5.3, we have

$$\begin{aligned}
&|J_1^{(2)}[(\operatorname{div} F)_n](r)| \\
&\leq C r^{-1} \bigg(\int_1^{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}} + \int_{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}}^r \bigg) |\tau^{-1} I_{\mu_n}(\sqrt{\lambda}\tau) \, \widetilde{F}_n^{(1)}(\tau)| \bigg| \int_\tau^r s^2 K_{\mu_n}(\sqrt{\lambda}s) \, \mathrm{d}s \bigg| \, \mathrm{d}\tau \\
&\leq C \, |\lambda|^{-\frac12} \int_1^{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}} \tau^{-1} |F_n(\tau)| \tau \, \mathrm{d}\tau + C \, |\lambda|^{-\frac12} \int_{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}}^r \tau^{-1} |F_n(\tau)| \tau \, \mathrm{d}\tau \,,
\end{aligned}$$

which leads to $|J_1^{(2)}[(\operatorname{div} F)_n](r)| \leq C |\lambda|^{-\frac12} \| |x|^{\gamma'} F \|_{L^2}$.
(ii) Estimate of $J_2^{(2)}[(\operatorname{div} F)_n]$: In the similar manner as the proof of $J_1^{(2)}[(\operatorname{div} F)_n]$, for $1 \leq r < \mathrm{Re}(\sqrt{\lambda})^{-1}$ we have $|J_2^{(2)}[(\operatorname{div} F)_n](r)| \leq C |\lambda|^{\frac12} r^2 \|F\|_{L^2}$, and for $r \geq \mathrm{Re}(\sqrt{\lambda})^{-1}$ we have $|J_2^{(2)}[(\operatorname{div} F)_n](r)| \leq C |\lambda|^{-\frac12} \|F\|_{L^2}$. We omit since the proof is straightforward.
(iii) Estimate of $J_3^{(2)}[(\operatorname{div} F)_n]$: For $1 \leq r < \mathrm{Re}(\sqrt{\lambda})^{-1}$, we have $|J_3^{(2)}[(\operatorname{div} F)_n](r)| \leq C |\lambda|^{\frac12} r^2 \|F\|_{L^2}$ by same way as the proof of $J_1^{(2)}[f_n]$. For $r \geq \mathrm{Re}(\sqrt{\lambda})^{-1}$, we observe that

$$\begin{aligned}
&|J_3^{(2)}[(\operatorname{div} F)_n](r)| \\
&\leq C \, |\lambda| \, r^{-1} \bigg(\int_1^{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}} + \int_{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}}^r \bigg) |\tau I_{\mu_n}(\sqrt{\lambda}\tau) \, \widetilde{F}_n^{(3)}(\tau)| \bigg| \int_\tau^r s^2 K_{\mu_n}(\sqrt{\lambda}s) \, \mathrm{d}s \bigg| \, \mathrm{d}\tau \\
&\leq C \, |\lambda|^{-\frac12} r^{-1} \int_1^{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}} |F_n(\tau)| \tau \, \mathrm{d}\tau + C \, r^{-1} \int_{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}}^r \tau |F_n(\tau)| \tau \, \mathrm{d}\tau \,.
\end{aligned}$$

Thus we have $|J_3^{(2)}[(\operatorname{div} F)_n](r)| \leq C (|\lambda|^{-\frac12} \|F\|_{L^2} + r^{1-\gamma'} \| |x|^{\gamma'} F \|_{L^2})$.
(iv) Estimate of $J_4^{(2)}[(\operatorname{div} F)_n]$: For $1 \leq r < \mathrm{Re}(\sqrt{\lambda})^{-1}$, by (4.150) and (4.152) in Lemma 4.5.2 and (4.162) for $k = 0$ in Lemma 4.5.4, we find

$$\begin{aligned}
|J_4^{(2)}[(\operatorname{div} F)_n](r)| &\leq r^{-1} \int_1^r |\tau^{-1} K_{\mu_n}(\sqrt{\lambda}\tau) \, \widetilde{F}_n^{(4)}(\tau)| \int_1^\tau |s^2 I_{\mu_n}(\sqrt{\lambda}s)| \, \mathrm{d}s \, \mathrm{d}\tau \\
&\leq C \int_1^r \tau |F_n(\tau)| \tau \, \mathrm{d}\tau \,,
\end{aligned}$$

which implies $|J_4^{(2)}[(\operatorname{div} F)_n](r)| \leq C r^{1-\gamma'} \| |x|^{\gamma'} F \|_{L^2}$. For $r \geq \mathrm{Re}(\sqrt{\lambda})^{-1}$, (4.150), (4.152), and (4.155) for $k = 0$ in Lemma 4.5.2 and (4.162) and (4.163) for $k = 0$ in Lemma 4.5.4 yield

$$\begin{aligned}
&|J_4^{(2)}[(\operatorname{div} F)_n](r)| \\
&\leq C r^{-1} \bigg(\int_1^{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}} + \int_{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}}^r \bigg) |\tau^{-1} K_{\mu_n}(\sqrt{\lambda}\tau) \, \widetilde{F}_n^{(4)}(\tau)| \int_1^\tau |s^2 I_{\mu_n}(\sqrt{\lambda}s)| \, \mathrm{d}s \, \mathrm{d}\tau \\
&\leq C |\lambda|^{-\frac12} r^{-1} \int_1^{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}} |F_n(\tau)| \tau \, \mathrm{d}\tau + C \, |\lambda|^{-\frac12} r^{-1} \int_{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}}^r |F_n(\tau)| \tau \, \mathrm{d}\tau \,,
\end{aligned}$$

which leads to $|J_4^{(2)}[(\operatorname{div} F)_n](r)| \le C|\lambda|^{-\frac{1}{2}}\|F\|_{L^2}$.
(v) Estimate of $J_5^{(2)}[(\operatorname{div} F)_n]$: For $1 \le r < \operatorname{Re}(\sqrt{\lambda})^{-1}$, by the same estimates in Lemmas 4.5.2 and 4.5.4 which have been used in (iv) we find

$$\begin{aligned}
&|J_5^{(2)}[(\operatorname{div} F)_n](r)| \\
&\le r^{-1}\int_1^r |s^2 I_{\mu_n}(\sqrt{\lambda}s)|\,\mathrm{d}s\bigg(\int_r^{\frac{1}{\operatorname{Re}(\sqrt{\lambda})}} + \int_{\frac{1}{\operatorname{Re}(\sqrt{\lambda})}}^\infty\bigg)|\tau^{-1}K_{\mu_n}(\sqrt{\lambda}\tau)\,\widetilde{F}_n^{(4)}(\tau)|\,\mathrm{d}\tau \\
&\le C\,r^{\operatorname{Re}(\mu_n)+2}\int_r^{\frac{1}{\operatorname{Re}(\sqrt{\lambda})}} \tau^{-\operatorname{Re}(\mu_n)-2}|F_n(\tau)|\tau\,\mathrm{d}\tau \\
&\quad + C\,|\lambda|^{\frac{\operatorname{Re}(\mu_n)}{2}-\frac{1}{4}}r^{\operatorname{Re}(\mu_n)+2}\int_{\frac{1}{\operatorname{Re}(\sqrt{\lambda})}}^\infty \tau^{-\frac{5}{2}}e^{-\operatorname{Re}(\sqrt{\lambda})\tau}|F_n(\tau)|\tau\,\mathrm{d}\tau\,,
\end{aligned}$$

and thus we see that $|J_5^{(2)}[(\operatorname{div} F)_n](r)| \le C(r^{1-\gamma'}\||x|^{\gamma'}F\|_{L^2} + |\lambda|^{\frac{1}{2}}r^2\|F\|_{L^2})$ holds. For $r \ge \operatorname{Re}(\sqrt{\lambda})^{-1}$, we have

$$\begin{aligned}
|J_5^{(2)}[(\operatorname{div} F)_n](r)| &\le r^{-1}\int_1^r |s^2 I_{\mu_n}(\sqrt{\lambda}s)|\,\mathrm{d}s\int_r^\infty |\tau^{-1}K_{\mu_n}(\sqrt{\lambda}\tau)\,\widetilde{F}^{(4)}(\tau)|\,\mathrm{d}\tau \\
&\le C\,|\lambda|^{-1}r^{\frac{1}{2}}e^{\operatorname{Re}(\sqrt{\lambda})r}\int_r^\infty \tau^{-\frac{5}{2}}e^{-\operatorname{Re}(\sqrt{\lambda})\tau}|F_n(\tau)|\tau\,\mathrm{d}\tau\,,
\end{aligned}$$

which implies $|J_5^{(2)}[(\operatorname{div} F)_n](r)| \le C|\lambda|^{-\frac{1}{2}}\|F\|_{L^2}$.
(vi) Estimates of $J_l^{(2)}[(\operatorname{div} F)_n]$, $l \in \{6,7,8,9\}$: In the similar manner as the proofs for $J_4^{(2)}[f_n]$ and $J_5^{(2)}[(\operatorname{div} F)_n]$, we see that

$$\begin{aligned}
&|J_l^{(2)}[(\operatorname{div} F)_n](r)| \le C\beta^{-1}|\lambda|^{\frac{1}{2}}r^2\|F\|_{L^2}\,, \quad 1 \le r < \operatorname{Re}(\sqrt{\lambda})^{-1}\,, \\
&|J_l^{(2)}[(\operatorname{div} F)_n](r)| \le C(\beta^{-1}|\lambda|^{-\frac{1}{2}}\|F\|_{L^2} + r^{1-\gamma'}\||x|^{\gamma'}F\|_{L^2})\,, \quad r \ge \operatorname{Re}(\sqrt{\lambda})^{-1}\,,
\end{aligned}$$

for $l \in \{6,7,8,9\}$. We omit the details since the calculations are straightforward.
(vii) Estimate of $J_{10}^{(2)}[(\operatorname{div} F)_n]$: For $1 \le r < \operatorname{Re}(\sqrt{\lambda})^{-1}$, from (4.150)–(4.153), and (4.154) for $k = 0, 1$ in Lemma 4.5.2 we have

$$|J_{10}^{(2)}[(\operatorname{div} F)_n](r)| \le C\beta^{-1}|\lambda|\int_1^r |F_n(s)|s\,\mathrm{d}s\,,$$

which implies $|J_{10}^{(2)}[(\operatorname{div} F)_n](r)| \le C\beta^{-1}|\lambda|^{\frac{1}{2}}r^2\|F\|_{L^2}$. For $r \ge \operatorname{Re}(\sqrt{\lambda})^{-1}$, from (4.150)–(4.153), and (4.154)–(4.156) for $k = 0, 1$ in Lemma 4.5.2 we have

$$|J_{10}^{(2)}[(\operatorname{div} F)_n](r)| \le C\beta^{-1}|\lambda|\,r^{-1}\int_1^{\frac{1}{\operatorname{Re}(\sqrt{\lambda})}} s|F_n(s)|s\,\mathrm{d}s + C\,r^{-1}\int_{\frac{1}{\operatorname{Re}(\sqrt{\lambda})}}^r s^{-1}|F_n(s)|s\,\mathrm{d}s\,,$$

which leads to $|J_{10}^{(2)}[(\operatorname{div} F)_n](r)| \le C(\beta^{-1}|\lambda|^{-\frac{1}{2}}\|F\|_{L^2} + r^{1-\gamma'}\||x|^{\gamma'}F\|_{L^2})$. This completes the proof of Lemma 4.3.15. □

We continue the analysis on $V_n[\Phi_{n,\lambda}[(\operatorname{div} F)_n]]$ in (4.100). The next decomposition is also useful in calculation as is Lemma 4.3.14.

Lemma 4.3.16 *Let $n \in \mathbb{Z} \setminus \{0\}$ and $F \in C_0^\infty(D)^{2\times 2}$. Then we have*

$$r^{|n|} \int_r^\infty s^{1-|n|} \Phi_{n,\lambda}[(\operatorname{div} F)_n](s) \, \mathrm{d}s \;=\; \sum_{l=11}^{20} J_l^{(2)}[(\operatorname{div} F)_n](r)\,, \tag{4.111}$$

where

$$\begin{aligned}
J_{11}^{(2)}[(\operatorname{div} F)_n](r) &= -r^{|n|} \int_1^r \tau^{-1} I_{\mu_n}(\sqrt{\lambda}\tau)\, \widetilde{F}_n^{(1)}(\tau) \int_r^\infty s^{1-|n|} K_{\mu_n}(\sqrt{\lambda}s)\, \mathrm{d}s\, \mathrm{d}\tau\,,\\
J_{12}^{(2)}[(\operatorname{div} F)_n](r) &= -r^{|n|} \int_r^\infty \tau^{-1} I_{\mu_n}(\sqrt{\lambda}\tau)\, \widetilde{F}_n^{(1)}(\tau) \int_\tau^\infty s^{1-|n|} K_{\mu_n}(\sqrt{\lambda}s)\, \mathrm{d}s\, \mathrm{d}\tau\,,\\
J_{13}^{(2)}[(\operatorname{div} F)_n](r) &= -\sqrt{\lambda}\, r^{|n|} \int_1^r I_{\mu_n+1}(\sqrt{\lambda}\tau)\, \widetilde{F}_n^{(2)}(\tau) \int_r^\infty s^{1-|n|} K_{\mu_n}(\sqrt{\lambda}s)\, \mathrm{d}s\, \mathrm{d}\tau\,,\\
J_{14}^{(2)}[(\operatorname{div} F)_n](r) &= -\sqrt{\lambda}\, r^{|n|} \int_r^\infty I_{\mu_n+1}(\sqrt{\lambda}\tau)\, \widetilde{F}_n^{(2)}(\tau) \int_\tau^\infty s^{1-|n|} K_{\mu_n}(\sqrt{\lambda}s)\, \mathrm{d}s\, \mathrm{d}\tau\,,\\
J_{15}^{(2)}[(\operatorname{div} F)_n](r) &= \lambda\, r^{|n|} \int_1^r \tau I_{\mu_n}(\sqrt{\lambda}\tau)\, \widetilde{F}_n^{(3)}(\tau) \int_r^\infty s^{1-|n|} K_{\mu_n}(\sqrt{\lambda}s)\, \mathrm{d}s\, \mathrm{d}\tau\,,\\
J_{16}^{(2)}[(\operatorname{div} F)_n](r) &= \lambda\, r^{|n|} \int_r^\infty \tau I_{\mu_n}(\sqrt{\lambda}\tau)\, \widetilde{F}_n^{(3)}(\tau) \int_\tau^\infty s^{1-|n|} K_{\mu_n}(\sqrt{\lambda}s)\, \mathrm{d}s\, \mathrm{d}\tau\,,\\
J_{17}^{(2)}[(\operatorname{div} F)_n](r) &= r^{|n|} \int_r^\infty \tau^{-1} K_{\mu_n}(\sqrt{\lambda}\tau)\, \widetilde{F}_n^{(4)}(\tau) \int_r^\tau s^{1-|n|} I_{\mu_n}(\sqrt{\lambda}s)\, \mathrm{d}s\, \mathrm{d}\tau\,,\\
J_{18}^{(2)}[(\operatorname{div} F)_n](r) &= \sqrt{\lambda}\, r^{|n|} \int_r^\infty K_{\mu_n-1}(\sqrt{\lambda}\tau)\, \widetilde{F}_n^{(5)}(\tau) \int_r^\tau s^{1-|n|} I_{\mu_n}(\sqrt{\lambda}s)\, \mathrm{d}s\, \mathrm{d}\tau\,,\\
J_{19}^{(2)}[(\operatorname{div} F)_n](r) &= \lambda\, r^{|n|} \int_r^\infty s K_{\mu_n}(\sqrt{\lambda}\tau)\, \widetilde{F}_n^{(6)}(\tau) \int_r^\tau s^{1-|n|} I_{\mu_n}(\sqrt{\lambda}s)\, \mathrm{d}s\, \mathrm{d}\tau\,,\\
J_{20}^{(2)}[(\operatorname{div} F)_n](r) &= -\sqrt{\lambda}\, r^{|n|} \int_r^\infty s\big(K_{\mu_n}(\sqrt{\lambda}s) I_{\mu_n+1}(\sqrt{\lambda}s) + K_{\mu_n-1}(\sqrt{\lambda}s) I_{\mu_n}(\sqrt{\lambda}s)\big)\, \widetilde{F}_n^{(7)}(s)\, \mathrm{d}s\,.
\end{aligned}$$

Here $\widetilde{F}_n^{(k)}(r)$, $k \in \{1,\dots,7\}$, are the functions in Lemma 4.3.12.

Proof: The assertion is a consequence of inserting (4.102) in Lemma 4.3.12 into the left-hand side of (4.111), and changing order of integration as $\int_r^\infty \int_1^s \mathrm{d}\tau\, \mathrm{d}s = \int_1^r \mathrm{d}\tau \int_r^\infty \mathrm{d}s + \int_r^\infty \int_\tau^\infty \mathrm{d}s\, \mathrm{d}\tau$ and $\int_r^\infty \int_s^\infty \mathrm{d}\tau\, \mathrm{d}s = \int_r^\infty \int_r^\tau \mathrm{d}s\, \mathrm{d}\tau$. This completes the proof. □

The next lemma summarizes the estimates to $J_l^{(2)}[f_n]$, $l \in \{11,\dots,20\}$, in Lemma 4.3.16.

Lemma 4.3.17 *Let $|n| = 1$ and $\gamma' \in (\frac{1}{2}, \gamma)$, and let $\lambda \in \Sigma_{\pi-\epsilon} \cap \mathcal{B}_1(0)$ for some $\epsilon \in (0, \frac{\pi}{2})$. Then there is a positive constant $C = C(\gamma', \epsilon)$ independent of β such that the following statement holds. Let $F \in C_0^\infty(D)^{2\times 2}$. Then for $l \in \{11, \cdots, 20\}$ we have*

$$|J_l^{(2)}[(\operatorname{div} F)_n](r)| \le \frac{C}{\beta}\big(|\lambda|^{\frac{1}{2}} r^2 + r^{2-\mathrm{Re}(\mu_n)} + r^{1-\gamma'}\big)\, \||x|^{\gamma'} F\|_{L^2(D)}\,, \qquad 1 \le r < \mathrm{Re}(\sqrt{\lambda})^{-1}\,, \tag{4.112}$$

$$|J_l^{(2)}[(\operatorname{div} F)_n](r)| \le C\big(|\lambda|^{-\frac{1}{2}} + r^{1-\gamma'}\big)\, \||x|^{\gamma'} F\|_{L^2(D)}\,, \qquad r \ge \mathrm{Re}(\sqrt{\lambda})^{-1}\,. \tag{4.113}$$

Proof: (i) Estimate of $J_{11}^{(2)}[(\operatorname{div} F)_n]$: For $1 \le r < \operatorname{Re}(\sqrt{\lambda})^{-1}$, by (4.154) for $k = 0$ in Lemma 4.5.2 and (4.160) for $k = 0$ in Lemma 4.5.3 in Appendix 4.5.2, we find

$$\begin{aligned}|J_{11}^{(2)}[(\operatorname{div} F)_n](r)| &\le r\bigg|\int_r^\infty K_{\mu_n}(\sqrt{\lambda}s)\,\mathrm{d}s\bigg|\int_1^r|\tau^{-1}I_{\mu_n}(\sqrt{\lambda}\tau)\,\widetilde{F}_n^{(1)}(\tau)|\,\mathrm{d}\tau\\ &\le C\beta^{-1}r^{2-\operatorname{Re}(\mu_n)}\int_1^r\tau^{\operatorname{Re}(\mu_n)-2}|F_n(\tau)|\tau\,\mathrm{d}\tau\,,\end{aligned}$$

which implies $|J_{11}^{(2)}[(\operatorname{div} F)_n](r)| \le C\beta^{-1}r^{2-\operatorname{Re}(\mu_n)}\||x|^{\gamma'}F\|_{L^2}$. For $r \ge \operatorname{Re}(\sqrt{\lambda})^{-1}$, by (4.154) and (4.156) for $k = 0$ in Lemma 4.5.2 and (4.161) for $k = 0$ in Lemma 4.5.3, we see that

$$\begin{aligned}&|J_{11}^{(2)}[(\operatorname{div} F)_n](r)| \le r\int_r^\infty|K_{\mu_n}(\sqrt{\lambda}s)|\,\mathrm{d}s\bigg(\int_1^{\frac{1}{\operatorname{Re}(\sqrt{\lambda})}}+\int_{\frac{1}{\operatorname{Re}(\sqrt{\lambda})}}^r\bigg)|\tau^{-1}I_{\mu_n}(\sqrt{\lambda}\tau)\,\widetilde{F}_n^{(1)}(\tau)|\,\mathrm{d}\tau\\ &\le C|\lambda|^{\frac{\operatorname{Re}(\mu_n)}{2}-\frac{3}{4}}r^{\frac{1}{2}}e^{-\operatorname{Re}(\sqrt{\lambda})r}\int_1^{\frac{1}{\operatorname{Re}(\sqrt{\lambda})}}\tau^{\operatorname{Re}(\mu_n)-2}|F_n(\tau)|\tau\,\mathrm{d}\tau\\ &\quad+C|\lambda|^{-1}r^{\frac{1}{2}}e^{-\operatorname{Re}(\sqrt{\lambda})r}\int_{\frac{1}{\operatorname{Re}(\sqrt{\lambda})}}^r\tau^{-\frac{5}{2}}e^{\operatorname{Re}(\sqrt{\lambda})\tau}|F_n(\tau)|\tau\,\mathrm{d}\tau\,.\end{aligned}$$

Thus we have $|J_{11}^{(2)}[(\operatorname{div} F)_n](r)| \le C|\lambda|^{-\frac{1}{2}}\||x|^{\gamma'}F\|_{L^2}$.
(ii) Estimate of $J_{12}^{(2)}[(\operatorname{div} F)_n]$: For $1 \le r < \operatorname{Re}(\sqrt{\lambda})^{-1}$, by (4.154) and (4.156) for $k = 0$ in Lemma 4.5.2 and (4.160) and (4.161) for $k = 0$ in Lemma 4.5.3, we observe that

$$\begin{aligned}&|J_{12}^{(2)}[(\operatorname{div} F)_n](r)| \le r\bigg(\int_r^{\frac{1}{\operatorname{Re}(\sqrt{\lambda})}}+\int_{\frac{1}{\operatorname{Re}(\sqrt{\lambda})}}^\infty\bigg)|\tau^{-1}I_{\mu_n}(\sqrt{\lambda}\tau)\,\widetilde{F}_n^{(1)}(\tau)|\bigg|\int_\tau^\infty K_{\mu_n}(\sqrt{\lambda}s)\,\mathrm{d}s\bigg|\,\mathrm{d}\tau\\ &\le C\beta^{-1}r\int_r^{\frac{1}{\operatorname{Re}(\sqrt{\lambda})}}\tau^{-1}|F_n(\tau)|\tau\,\mathrm{d}\tau+C|\lambda|^{-1}r\int_{\frac{1}{\operatorname{Re}(\sqrt{\lambda})}}^\infty\tau^{-3}|F_n(\tau)|\tau\,\mathrm{d}\tau\,,\end{aligned}$$

which implies $|J_{12}^{(2)}[(\operatorname{div} F)_n](r)| \le C\beta^{-1}r^{1-\gamma'}\||x|^{\gamma'}F\|_{L^2}$. For $r \ge \operatorname{Re}(\sqrt{\lambda})^{-1}$, by (4.156) for $k = 0$ in Lemma 4.5.2 and (4.161) for $k = 0$ in Lemma 4.5.3 we find

$$\begin{aligned}|J_{12}^{(2)}[(\operatorname{div} F)_n](r)| &\le Cr\int_r^\infty|\tau^{-1}I_{\mu_n}(\sqrt{\lambda}\tau)\,\widetilde{F}_n^{(1)}(\tau)|\int_\tau^\infty|K_{\mu_n}(\sqrt{\lambda}s)|\,\mathrm{d}s\,\mathrm{d}\tau\\ &\le C|\lambda|^{-1}r\int_r^\infty\tau^{-3}|F_n(\tau)|\tau\,\mathrm{d}\tau\,,\end{aligned}$$

which leads to $|J_{12}^{(2)}[(\operatorname{div} F)_n](r)| \le C|\lambda|^{-\frac{1}{2}}\|F\|_{L^2}$.
(iii) Estimates of $J_l^{(2)}[(\operatorname{div} F)_n]$, $l \in \{13, 14, 15, 16\}$: In the similar manner as the proofs of $J_{11}^{(2)}[f_n]$ and $J_{12}^{(2)}[(\operatorname{div} F)_n]$, we have

$$\begin{aligned}|J_l^{(2)}[(\operatorname{div} F)_n](r)| &\le C\beta^{-1}(|\lambda|^{\frac{1}{2}}r^2\|F\|_{L^2}+r^{1-\gamma'}\||x|^{\gamma'}F\|_{L^2})\,,\quad 1\le r<\operatorname{Re}(\sqrt{\lambda})^{-1}\,,\\ |J_l^{(2)}[(\operatorname{div} F)_n](r)| &\le C(|\lambda|^{-\frac{1}{2}}\|F\|_{L^2}+r^{1-\gamma'}\||x|^{\gamma'}F\|_{L^2})\,,\quad r\ge\operatorname{Re}(\sqrt{\lambda})^{-1}\,,\end{aligned}$$

for $l \in \{13, 14, 15, 16\}$. We omit the details since the calculations are straightforward.
(iv) Estimates of $J_l^{(2)}[(\operatorname{div} F)_n]$, $l \in \{17, 18, 19\}$: We give a proof only for $J_{19}^{(2)}[(\operatorname{div} F)_n]$ since the proofs for $J_{17}^{(2)}[(\operatorname{div} F)_n]$ and $J_{18}^{(2)}[(\operatorname{div} F)_n]$ are similar. For $1 \le r < \operatorname{Re}(\sqrt{\lambda})^{-1}$,

from (4.150), (4.152), and (4.155) for $k = 0$ in Lemma 4.5.2 and (4.164) and (4.165) for $k = 0$ in Lemma 4.5.4, we observe that

$$|J_{19}^{(2)}[(\operatorname{div} F)_n](r)| \le |\lambda| r \Big(\int_r^{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}} + \int_{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}}^{\infty} \Big) |\tau K_{\mu_n}(\sqrt{\lambda}\tau)\, \widetilde{F}_n^{(6)}(\tau)| \int_r^{\tau} |I_{\mu_n}(\sqrt{\lambda}s)| \,\mathrm{d}s \,\mathrm{d}\tau$$
$$\le C|\lambda| r \int_r^{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}} \tau |F_n(\tau)| \tau \,\mathrm{d}\tau + Cr \int_{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}}^{\infty} \tau^{-1} |F_n(\tau)| \tau \,\mathrm{d}\tau\,,$$

which implies $|J_{19}^{(2)}[(\operatorname{div} F)_n](r)| \le C r^{1-\gamma'} \| |x|^{\gamma'} F \|_{L^2}$. For $r \ge \mathrm{Re}(\sqrt{\lambda})^{-1}$, by (4.155) for $k = 0$ in Lemma 4.5.2 and (4.166) in Lemma 4.5.4 for $k = 0$, we have

$$\begin{aligned} |J_{19}^{(2)}[(\operatorname{div} F)_n](r)| &\le |\lambda|\, r \int_r^{\infty} |\tau K_{\mu_n}(\sqrt{\lambda}\tau)\, \widetilde{F}_n^{(6)}(\tau)| \int_r^{\tau} |I_{\mu_n}(\sqrt{\lambda}s)| \,\mathrm{d}s \,\mathrm{d}\tau \\ &\le C\, r \int_r^{\infty} \tau^{-1} |F_n(\tau)| \tau \,\mathrm{d}\tau\,, \end{aligned}$$

which leads to $|J_{19}^{(2)}[(\operatorname{div} F)_n](r)| \le C r^{1-\gamma'} \| |x|^{\gamma'} F \|_{L^2}$.
(v) Estimate of $J_{20}^{(2)}[(\operatorname{div} F)_n]$: For $1 \le r < \mathrm{Re}(\sqrt{\lambda})^{-1}$, from (4.150)–(4.153), and (4.154)–(4.156) for $k = 0, 1$ in Lemma 4.5.2, we have

$$|J_{20}^{(2)}[(\operatorname{div} F)_n](r)| \le C\beta^{-1} |\lambda|\, r \int_r^{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}} s |F_n(s)| s \,\mathrm{d}s + C\, r \int_{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}}^{\infty} s^{-1} |F_n(s)| s \,\mathrm{d}s\,.$$

Thus we have $|J_{20}^{(2)}[(\operatorname{div} F)_n](r)| \le C\beta^{-1} r^{1-\gamma'} \| |x|^{\gamma'} F \|_{L^2}$. For $r \ge \mathrm{Re}(\sqrt{\lambda})^{-1}$, from (4.155) and (4.156) for $k = 0, 1$ in Lemma 4.5.2, we have

$$|J_{20}^{(2)}[(\operatorname{div} F)_n](r)| \le C\, r \int_r^{\infty} \tau^{-1} |F_n(\tau)| \tau \,\mathrm{d}\tau\,,$$

which implies $|J_{20}^{(2)}[(\operatorname{div} F)_n](r)| \le C r^{1-\gamma'} \| |x|^{\gamma'} F \|_{L^2}$. This completes the proof of Lemma 4.3.17. □

From Lemmas 4.3.15 and 4.3.17 we see that the following estimates hold.

Corollary 4.3.18 *Let $|n| = 1$, $\gamma' \in (\frac{1}{2}, \gamma)$, and $p \in (\frac{2}{\gamma'}, \infty)$, and let $\lambda \in \Sigma_{\pi-\epsilon} \cap \mathcal{B}_1(0)$ for some $\epsilon \in (0, \frac{\pi}{2})$. Then there is a positive constant $C = C(\gamma', p, \epsilon)$ independent of β such that the following statement holds. Let $F \in C_0^\infty(D)^{2\times 2}$. Then for $l \in \{1, \ldots, 20\}$ we have*

$$|c_{n,\lambda}[(\operatorname{div} F)_n]| \le \frac{C}{\beta} \| |x|^{\gamma'} F \|_{L^2(D)}\,, \tag{4.114}$$

$$\| r^{-1} J_l^{(2)}[(\operatorname{div} F)_n] \|_{L^p(D)} \le \frac{C}{\beta} |\lambda|^{-\frac{1}{p}} \| |x|^{\gamma'} F \|_{L^2(D)}\,, \tag{4.115}$$

$$\| r^{-2} J_l^{(2)}[(\operatorname{div} F)_n] \|_{L^2(D)} \le \frac{C}{\beta^2} \| |x|^{\gamma'} F \|_{L^2(D)}\,. \tag{4.116}$$

Here $c_{n,\lambda}[(\operatorname{div} F)_n]$ is the constant in (4.63) *replacing f_n by $(\operatorname{div} F)_n$.*

Proof: (i) Estimate of $c_{n,\lambda}[(\mathrm{div}\,F)_n]$: By the definitions of $J_l^{(2)}[f_n]$ for $l \in \{11, \cdots, 20\}$ in Lemma 4.3.16, we see that $|c_{n,\lambda}[(\mathrm{div}\,F)_n]| \le \sum_{l=12,14,16,17,18,19,20} |J_l^{(2)}[f_n](1)|$. Hence we obtain the estimate (4.114) by putting $r = 1$ to (4.112) in Lemma 4.3.17.
(ii) Estimate of $r^{-1}J_l^{(2)}[(\mathrm{div}\,F)_n]$: By Lemmas 4.3.15 and 4.3.17, for $p \in [\frac{2}{\gamma'}, \infty)$ we have

$$\sup_{r\ge1} r^{\frac{2}{p}} |r^{-1}J_l^{(2)}[(\mathrm{div}\,F)_n](r)| \le C\beta^{-1}|\lambda|^{-\frac{1}{p}} \| |x|^{\gamma'} F\|_{L^2(D)} .$$

Thus by the Marcinkiewicz interpolation theorem we obtain (4.115) for $p \in (\frac{2}{\gamma'}, \infty)$.
(iii) Estimate of $r^{-2}J_l^{(2)}[(\mathrm{div}\,F)_n]$: The assertion (4.116) can be checked easily by using Lemmas 4.3.15 and 4.3.17 and $(\mathrm{Re}(\mu_n) - 1)^{\frac{1}{2}} \approx O(\beta)$. This completes the proof. □

The next proposition gives the estimate for the term $V_n[\Phi_{n,\lambda}[(\mathrm{div}\,F)_n]]$ in (4.100).

Proposition 4.3.19 *Let $|n| = 1$, $\gamma' \in (\frac{1}{2}, \gamma)$, and $p \in (\frac{2}{\gamma'}, \infty)$, and let $\lambda \in \Sigma_{\pi-\epsilon} \cap \mathcal{B}_1(0)$ for some $\epsilon \in (0, \frac{\pi}{2})$. Then there is a positive constant $C = C(\gamma', p, \epsilon)$ independent of β such that for $F \in C_0^\infty(D)^{2\times2}$ we have*

$$\|V_n[\Phi_{n,\lambda}[(\mathrm{div}\,F)_n]]\|_{L^p(D)} \le \frac{C}{\beta}|\lambda|^{-\frac{1}{p}} \| |x|^{\gamma'} F\|_{L^2(D)} , \tag{4.117}$$

$$\|\frac{V_n[\Phi_{n,\lambda}[(\mathrm{div}\,F)_n]]}{|x|}\|_{L^2(D)} \le \frac{C}{\beta^2} \| |x|^{\gamma'} F\|_{L^2(D)} . \tag{4.118}$$

Proof: In the similar manner as the proof of Proposition 4.3.10 we find

$$\begin{aligned}
&|V_n[\Phi_{n,\lambda}[(\mathrm{div}\,F)_n]](r)| \\
&\le C\Big(r^{-2}\sum_{l=1}^{20} |J_l^{(1)}[(\mathrm{div}\,F)_n](1)| + \sum_{l=1}^{20} |r^{-1}J^{(1)}[(\mathrm{div}\,F)_n](r)|\Big) .
\end{aligned}$$

Thus the assertions (4.117) and (4.118) follow from Corollary 4.3.18. The proof is complete. □

From Corollary 4.3.18 and Proposition 4.3.19, Theorem 4.3.13 follows.

Proof of Theorem 4.3.13: (i) Estimate for the case $F \in C_0^\infty(D)^{2\times2}$: It suffices to prove that the first term in the right-hand side of (4.100) has the estimates (4.106) and (4.107) in view of Proposition 4.3.19. By using Proposition 4.3.1 and (4.114) in Corollary 4.3.18, we see that (4.106) and (4.107) respectively follow from (4.86) and (4.87) in Proposition 4.3.9.
(ii) Estimate for the case $F \in X_{\gamma'}(D)$: Let us take sequences $\{G^{(m)}\}_{m=1}^\infty \subset C_0^\infty(D)^{2\times2}$ and $\{w_n^{(m)}\}_{n=1}^\infty \subset \mathcal{P}_n(L^p_\sigma(D) \cap W_0^{1,p}(D)^2)$ such that $\lim_{m\to\infty} \| |x|^{\gamma'}(F - G^{(m)})\|_{L^2(D)} = 0$ and $w_n^{(m)}$ is a (unique) solution to $(\mathrm{RS}^{\mathrm{ed}}_{\mathrm{div}F,n})$ replacing F by $G^{(m)}$. Then, since $w_n^{(m)}$ satisfies (4.106), (4.107), and the estimates in Theorem 4.3.20 below replacing F by $G^{(m)}$, by using $\|\nabla h\|_{L^2(D)} \le C\|\mathrm{rot}\,h\|_{L^2(D)}$ for $h \in L^p_\sigma(D) \cap W_0^{1,p}(D)^2$, we observe that the limit $w_n = \lim_{m\to\infty} w_n^{(m)} \in \mathcal{P}_n(L^p_\sigma(D) \cap W_0^{1,p}(D)^2)$ exists and satisfies (4.106), (4.107), and the estimates in Theorem 4.3.20. Moreover, by taking the limit $m \to \infty$ in $(\mathrm{RS}^{\mathrm{ed}}_{\mathrm{div}F,n})$ replacing F by $G^{(m)}$, we see that w_n gives a weak solution to $(\mathrm{RS}^{\mathrm{ed}}_{\mathrm{div}F,n})$. This completes the proof. □

Estimates of the vorticity for $(\mathrm{RS}^{\mathrm{ed}}_{\mathrm{div}F,n})$ **with** $|n|=1$

In this subsection we estimate the vorticity $\omega^{\mathrm{ed}}_{\mathrm{div}F,n}(r)=(\mathrm{rot}\, w^{\mathrm{ed}}_{\mathrm{div}F,n})e^{-in\theta}$ with $|n|=1$, where $\mathrm{rot}\, w^{\mathrm{ed}}_{\mathrm{div}F,n}$ is represented as (4.101). We take the constant β_0 in Proposition 4.3.1.

Theorem 4.3.20 *Let $|n|=1$, $\gamma'\in(\frac12,\gamma)$, $p\in[2,\infty)$, and $q\in(1,\infty)$. Fix $\epsilon\in(0,\frac{\pi}{2})$. Then there is a positive constant $C=C(\gamma',p,q,\epsilon)$ independent of β such that the following statement holds. Let $F\in C_0^\infty(D)^{2\times2}$, $f\in L^q(D)^2$, and $\beta\in(0,\beta_0)$. Set*

$$\omega^{\mathrm{ed}\,(1)}_{\mathrm{div}F,n}(r)\;=\;-\frac{c_{n,\lambda}[(\mathrm{div}\,F)_n]}{F_n(\sqrt{\lambda};\beta)}K_{\mu_n}(\sqrt{\lambda}r)\,,\qquad \omega^{\mathrm{ed}\,(2)}_{\mathrm{div}F,n}(r)\;=\;\Phi_{n,\lambda}[(\mathrm{div}\,F)_n](r)\,. \tag{4.119}$$

Then for $\lambda\in\Sigma_{\pi-\epsilon}\cap\mathcal{B}_{e^{-\frac{1}{6\beta}}}(0)$ we have

$$\|\omega^{\mathrm{ed}\,(1)}_{\mathrm{div}F,n}\|_{L^p(D)}\le\frac{C}{\beta(p\mathrm{Re}(\mu_n)-2)^{\frac1p}}\||x|^{\gamma'}F\|_{L^2(D)}\,, \tag{4.120}$$

$$\|\omega^{\mathrm{ed}\,(2)}_{\mathrm{div}F,n}\|_{L^p(D)}+\beta\Big\|\frac{\omega^{\mathrm{ed}\,(2)}_{\mathrm{div}F,n}}{|x|}\Big\|_{L^1(D)}\le\frac{C}{\beta}\||x|^{\gamma'}F\|_{L^2(D)}\,, \tag{4.121}$$

$$\Big|\Big\langle\omega^{\mathrm{ed}\,(1)}_{\mathrm{div}F,n},\frac{(w^{\mathrm{ed}}_{f,r})_n}{|x|}\Big\rangle_{L^2(D)}\Big|\le\frac{C}{\beta^5}|\lambda|^{-1+\frac1q}\|f\|_{L^q(D)}\||x|^{\gamma'}F\|_{L^2(D)}\,. \tag{4.122}$$

Moreover, (4.120)*,* (4.121)*, and* (4.122) *hold all for $F\in X_{\gamma'}(D)$ defined in* (4.99) *by a density argument as in the proof of Theorem 4.3.13 above.*

Proof: (i) Estimate of $\omega^{\mathrm{ed}\,(1)}_{\mathrm{div}F,n}$: The estimate (4.120) is a direct consequence of Proposition 4.3.1, (4.114) in Corollary 4.3.18, and (4.167) in Lemma 4.5.5 in Appendix 4.5.2.
(ii) Estimates of $\omega^{\mathrm{ed}\,(2)}_{\mathrm{div}F,n}$ and $|x|^{-1}\omega^{\mathrm{ed}\,(2)}_{\mathrm{div}F,n}$: Firstly we decompose $\omega^{\mathrm{ed}\,(2)}_{\mathrm{div}F,n}$ into $\omega^{\mathrm{ed}\,(2)}_{\mathrm{div}F,n}=\sum_{l=1}^7\Phi^{(l)}_{n,\lambda}[(\mathrm{div}\,F)_n]$ as in Lemma 4.3.12. Then the assertion (4.121) follows from the estimates of each term $\Phi^{(l)}_{n,\lambda}[(\mathrm{div}\,F)_n]$, $l\in\{1,\dots,7\}$.
(I) Estimates of $\Phi^{(l)}_{n,\lambda}[(\mathrm{div}\,F)_n]$, $l\in\{1,2,3\}$: We give a proof only for $\Phi^{(3)}_{n,\lambda}[(\mathrm{div}\,F)_n]$ since the proofs for $\Phi^{(1)}_{n,\lambda}[(\mathrm{div}\,F)_n]$ and $\Phi^{(2)}_{n,\lambda}[(\mathrm{div}\,F)_n]$ are similar. The Minkowski inequality and the Fubini theorem lead to

$$\begin{aligned}&\|\Phi^{(3)}_{n,\lambda}[(\mathrm{div}\,F)_n]\|_{L^p(D)}+\beta\Big\|\frac{\Phi^{(3)}_{n,\lambda}[(\mathrm{div}\,F)_n]}{|x|}\Big\|_{L^1(D)}\\ &\le|\lambda|\int_1^\infty|sI_{\mu_n}(\sqrt{\lambda}s)\widetilde{F}^{(3)}_n(s)|\bigg(\Big(\int_s^\infty|K_{\mu_n}(\sqrt{\lambda}r)|^pr\,\mathrm{d}r\Big)^{\frac1p}+\beta\int_s^\infty|K_{\mu_n}(\sqrt{\lambda}r)|\,\mathrm{d}r\bigg)\,\mathrm{d}s\,.\end{aligned}$$

By (4.154) and (4.156) for $k=0$ in Lemma 4.5.2 and (4.172) and (4.173) in Lemma 4.5.5, we have

$$\begin{aligned}&\|\Phi^{(3)}_{n,\lambda}[(\mathrm{div}\,F)_n]\|_{L^p(D)}+\beta\Big\|\frac{\Phi^{(3)}_{n,\lambda}[(\mathrm{div}\,F)_n]}{|x|}\Big\|_{L^1(D)}\\ &\le C\beta^{-1}|\lambda|^{\frac12}\int_1^{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}}|F_n(s)|s\,\mathrm{d}s+C|\lambda|^{\frac14}\int_{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}}^\infty s^{-\frac12-\gamma'}|s^{\gamma'}F_n(s)|s\,\mathrm{d}s\,,\end{aligned}$$

which implies (4.121) since the condition $\gamma' \in (\frac{1}{2}, 1)$ is assumed.
(II) Estimates of $\Phi^{(l)}_{n,\lambda}[(\operatorname{div} F)_n]$, $l \in \{4,5,6\}$: We give a proof only for $\Phi^{(6)}_{n,\lambda}[(\operatorname{div} F)_n]$ since the proofs for $\Phi^{(4)}_{n,\lambda}[(\operatorname{div} F)_n]$ and $\Phi^{(5)}_{n,\lambda}[(\operatorname{div} F)_n]$ are similar. After using the Minkowski inequality and the Fubini theorem, by (4.150), (4.152), and (4.155) for $k = 0$ in Lemma 4.5.2 and (4.174) and (4.175) in Lemma 4.5.5, we observe that

$$\begin{aligned}
&\|\Phi^{(6)}_{n,\lambda}[(\operatorname{div} F)_n]\|_{L^p(D)} + \Big\|\frac{\Phi^{(6)}_{n,\lambda}[(\operatorname{div} F)_n]}{|x|}\Big\|_{L^1(D)} \\
&\le |\lambda| \int_1^\infty \big|sK_{\mu_n}(\sqrt{\lambda}s)\,\widetilde{F}^{(6)}_n(s)\big| \bigg(\Big(\int_1^s |I_{\mu_n}(\sqrt{\lambda}r)|^p r\,\mathrm{d}r\Big)^{\frac{1}{p}} + \int_1^s |I_{\mu_n}(\sqrt{\lambda}r)|\,\mathrm{d}r\bigg)\,\mathrm{d}s \\
&\le C|\lambda|^{\frac{1}{2}} \int_1^{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}} |F_n(s)|s\,\mathrm{d}s + C|\lambda|^{\frac{1}{4}} \int_{\frac{1}{\mathrm{Re}(\sqrt{\lambda})}}^\infty s^{-\frac{1}{2}-\gamma'}|s^{\gamma'}F_n(s)|s\,\mathrm{d}s\,,
\end{aligned}$$

which leads to (4.121) by the condition $\gamma' \in (\frac{1}{2}, 1)$.
(III) Estimate of $\Phi^{(7)}_{n,\lambda}[(\operatorname{div} F)_n]$: The proof is straightforward using the results in Lemma 4.5.2 and thus we omit the details.
(iii) Estimate of $\big|\big\langle \omega^{\mathrm{ed}\,(1)}_{\operatorname{div} F,n}, |x|^{-1}(w^{\mathrm{ed}}_{f,r})_n\big\rangle_{L^2(D)}\big|$: We omit since the proof is parallel to that for (4.97) in Theorem 4.3.11 using (4.114) in Corollary 4.3.18. The proof is complete. □

4.3.3 Problem III: No external force and boundary data b

In this subsection we give the estimates for $(w, r) = (w^{\mathrm{ed}}_b, r^{\mathrm{ed}}_b)$ solving the next problem:

$$\begin{cases} \lambda w - \Delta w + \beta U^\perp \mathrm{rot}\, w + \nabla r \,=\, 0\,, \quad x \in D\,, \\ \operatorname{div} w \,=\, 0\,, \quad x \in D\,, \\ w|_{\partial D} \,=\, b\,. \end{cases} \tag{$\mathrm{RS}^{\mathrm{ed}}_b$}$$

Firstly we prove the representation formula to the problem ($\mathrm{RS}^{\mathrm{ed}}_b$).

Lemma 4.3.21 *Let $|n| = 1$ and $b \in L^\infty(\partial D)^2$, and let $\lambda \in \mathbb{C} \setminus (\overline{\mathbb{R}_-} \cup \mathcal{Z}(F_n))$. Suppose that w^{ed}_b is a solution to* ($\mathrm{RS}^{\mathrm{ed}}_b$)*. Then the n-Fourier modes $w^{\mathrm{ed}}_{b,n}$ and $\omega^{\mathrm{ed}}_{b,n} = (\mathrm{rot}\, w^{\mathrm{ed}}_{b,n})e^{-in\theta}$ satisfy the following representations*:

$$w^{\mathrm{ed}}_{b,n} \,=\, \frac{T_n(b)}{F_n(\sqrt{\lambda};\beta)} V_n[K_{\mu_n}(\sqrt{\lambda}\,\cdot)] + \frac{\mathcal{V}_n[b](\theta)}{r^2}\,, \tag{4.123}$$

$$\omega^{\mathrm{ed}}_{b,n}(r) \,=\, \frac{T_n(b)}{F_n(\sqrt{\lambda};\beta)}\, K_{\mu_n}(\sqrt{\lambda}r)\,, \tag{4.124}$$

where the operator $T_n(b)$ and the vector field $\mathcal{V}_n[b](\theta)$ are defines as

$$T_n(b) \,=\, \frac{b_{r,n}}{in} - b_{\theta,n}\,, \qquad \mathcal{V}_n[b](\theta) \,=\, b_{r,n}e^{in\theta}\mathbf{e}_r + \frac{b_{r,n}}{in}e^{in\theta}\mathbf{e}_\theta\,. \tag{4.125}$$

Here $\mathcal{Z}(F_n)$ is the set in (4.60) *and $V_n[\,\cdot\,]$ is the Biot-Savart law in* (4.27)*.*

Proof: It is easy to see that $u = \frac{T_n(b)}{F_n(\sqrt{\lambda};\beta)} V_n[K_{\mu_n}(\sqrt{\lambda}\,\cdot)]$ solves

$$\begin{cases} \lambda u - \Delta u + \beta U^\perp \mathrm{rot}\, u + \nabla p \,=\, 0\,, \quad x \in D\,, \\ \operatorname{div} u \,=\, 0\,, \quad x \in D\,, \\ u_r|_{\partial D} \,=\, 0\,, \quad u_\theta|_{\partial D} \,=\, -T_n(b)\,, \end{cases} \tag{4.126}$$

with some pressure $p \in W^{1,1}_{\mathrm{loc}}(\overline{\Omega})$. The vector field $\frac{\mathcal{V}_n[b](\theta)}{r^2}$ corrects the boundary condition in (4.126) so that $u + \frac{\mathcal{V}_n[b](\theta)}{r^2}$ solves ($\mathrm{RS}^{\mathrm{ed}}_b$) replacing b by b_n. The proof is complete. □

The estimates for $w^{\mathrm{ed}}_{b,n}$ and $\omega^{\mathrm{ed}}_{b,n}$ in Lemma 4.3.21 are the main results of this subsection. We recall that β_0 is the constant in Proposition 4.3.1.

Theorem 4.3.22 *Let $|n| = 1$, $p \in (1, \infty]$, and $q \in (1, \infty)$. Fix $\epsilon \in (0, \frac{\pi}{2})$. Then there is a positive constant $C = C(p, q, \epsilon)$ independent of β such that the following statement holds. Let $b \in L^\infty(\partial D)^2$, $f \in L^q(D)^2$, and $\beta \in (0, \beta_0)$. Then for $\lambda \in \Sigma_{\pi-\epsilon} \cap \mathcal{B}_{e^{-\frac{1}{6\beta}}}(0)$ we have*

$$\|w^{\mathrm{ed}}_{b,n}\|_{L^p(D)} \le \frac{C}{\beta}|\lambda|^{-\frac{1}{p}}\|b\|_{L^\infty(\partial D)}\,, \tag{4.127}$$

$$\|\frac{w^{\mathrm{ed}}_{b,n}}{|x|}\|_{L^2(D)} \le \frac{C}{\beta^2}\|b\|_{L^\infty(\partial D)}\,, \tag{4.128}$$

$$\|\omega^{\mathrm{ed}}_{b,n}\|_{L^2(D)} \le \frac{C}{\beta}\|b\|_{L^\infty(\partial D)}\,, \tag{4.129}$$

$$\big|\big\langle \omega^{\mathrm{ed}}_{b,n}, \frac{(w^{\mathrm{ed}}_{f,r})_n}{|x|}\big\rangle_{L^2(D)}\big| \le \frac{C}{\beta^4}|\lambda|^{-1+\frac{1}{q}}\|f\|_{L^q(D)}\|b\|_{L^\infty(\partial D)}\,. \tag{4.130}$$

Proof: The estimates (4.127) and (4.128) follow by Propositions 4.3.1 and 4.3.9, while (4.129) follows by Proposition 4.3.1 and (4.167) with $p = 2$ in Lemma 4.5.5 in Appendix 4.5.2. The proof for (4.130) is parallel to that for (4.97) in Theorem 4.3.11. The proof is complete. □

4.3.4 Resolvent estimate in region exponentially close to the origin

In this subsection we treat the problem (RS) when the resolvent parameter λ is exponentially close to the origin. We start with the a priori estimate of the term $\big\langle (\mathrm{rot}\, v)_n, \frac{v_{r,n}}{|x|}\big\rangle_{L^2(D)}$, $|n| = 1$, when $0 < |\lambda| < e^{-\frac{1}{6\beta}}$, which is needed in closing the energy computation. We recall that D denotes the exterior disk $\{x \in \mathbb{R}^2 \mid |x| > 1\}$, and that R, γ, and κ are defined in Assumption 4.1.1. Let β_0 be the constant in Proposition 4.3.1.

Proposition 4.3.23 *Let $|n| = 1$, $q \in (1, 2]$, and $f \in L^q(\Omega)^2$, and let $\lambda \in \Sigma_{\pi-\epsilon} \cap \mathcal{B}_{e^{-\frac{1}{6\beta}}}(0)$ for some $\epsilon \in (0, \frac{\pi}{2})$. Suppose that $v \in D(\mathbb{A}_V)$ is a solution to* (RS). *Then we have*

$$\big|\big\langle (\mathrm{rot}\, v)_n, \frac{v_{r,n}}{|x|}\big\rangle_{L^2(D)}\big| \le \frac{C}{\beta^5}|\lambda|^{-2+\frac{2}{q}}\|f\|^2_{L^q(\Omega)} + \frac{K}{\beta^5}(\beta^\kappa d + \beta d^{\frac{1}{2}})^2\|\nabla v\|^2_{L^2(\Omega)}\,, \tag{4.131}$$

as long as $\beta \in (0, \beta_0)$. The constant C is independent of β and depends on γ, q, and ϵ, while K is greater than 1 and independent of β and q, and depends on γ and ϵ.

Proof: In this proof we denote the function space $L^q(D)$ by L^q to simplify notation. Firstly we fix a positive number $\gamma' \in (\frac{1}{2}, \gamma)$, and set $F = -(R \otimes v + v \otimes R)|_D$ and $b = \mathcal{P}_n v|_{\partial D}$. It is easy to see that F belongs to the function space $X_{\gamma'}(D)$ defined in (4.99), and that $b \in L^\infty(\partial D)^2$. Moreover, a direct calculation and Assumption 4.1.1 imply that

$$\||x|^{\gamma'}F\|_{L^2} \le K_0\beta^\kappa d\|\nabla v\|_{L^2(\Omega)}\,, \qquad \|b\|_{L^\infty(\partial D)} \le K_0 d^{\frac{1}{2}}\|\nabla v\|_{L^2(\Omega)}\,. \tag{4.132}$$

Here K_0 denotes the constant which depends on γ and is independent of β and $q \in (1,2]$. In the following we use the notations in Subsections 4.3.1–4.3.3. Since $v|_D$ is a solution to the problem (RS$^{\rm ed}$), by the solution formula we have the decompositions for v_n, $|n|=1$:

$$v_n = w^{\rm ed}_{f,n} + w^{\rm ed}_{{\rm div} F,n} + w^{\rm ed}_{b,n} \quad \text{in } D\,, \tag{4.133}$$

$$({\rm rot}\, v)_n = \omega^{\rm ed}_{f,n} + \omega^{\rm ed}_{{\rm div} F,n} + \omega^{\rm ed}_{b,n} \quad \text{in } D\,. \tag{4.134}$$

Then, in view of (4.134), the assertion (4.131) follows from estimating the next three terms:

$$\big|\big\langle \omega^{\rm ed}_{f,n}, \frac{v_{r,n}}{|x|}\big\rangle_{L^2}\big|\,, \qquad \big|\big\langle \omega^{\rm ed}_{{\rm div}F,n}, \frac{v_{r,n}}{|x|}\big\rangle_{L^2}\big|\,, \qquad \big|\big\langle \omega^{\rm ed}_{b,n}, \frac{v_{r,n}}{|x|}\big\rangle_{L^2}\big|\,.$$

(i) Estimate of $|\langle \omega^{\rm ed}_{f,n}, \frac{v_{r,n}}{|x|}\rangle_{L^2}|$: We fix a number $p \in (\frac{2}{\gamma'}, \infty)$. Note that $p \in (q,\infty)$ holds since $\frac{2}{\gamma'} > 2$. Then setting $p' = \frac{p}{p-1} \in (1,q)$ and using the Hölder inequality, we see that

$$\big|\big\langle \omega^{\rm ed}_{f,n}, \frac{v_{r,n}}{|x|}\big\rangle_{L^2}\big| \le \big|\big\langle \omega^{{\rm ed}\,(1)}_{f,n}, \frac{v_{r,n}}{|x|}\big\rangle_{L^2}\big| + \big\|\frac{\omega^{{\rm ed}\,(2)}_{f,n}}{|x|}\big\|_{L^{p'}} \|v_n\|_{L^p}\,. \tag{4.135}$$

From (4.95) and (4.97) in Theorem 4.3.11, (4.107) in Theorem 4.3.13, and (4.128) in Theorem 4.3.22 we observe that

$$\begin{aligned}\big|\big\langle \omega^{{\rm ed}\,(1)}_{f,n}, \frac{v_{r,n}}{|x|}\big\rangle_{L^2}\big| &\le \big|\big\langle \omega^{{\rm ed}\,(1)}_{f,n}, \frac{(w^{\rm ed}_{f,r})_n}{|x|}\big\rangle_{L^2}\big| + \|\omega^{{\rm ed}\,(1)}_{f,n}\|_{L^2}\Big(\big\|\frac{w^{\rm ed}_{{\rm div}F,n}}{|x|}\big\|_{L^2} + \big\|\frac{w^{\rm ed}_{b,n}}{|x|}\big\|_{L^2}\Big)\\ &\le \frac{C}{\beta^5}|\lambda|^{-1+\frac1q}\|f\|_{L^q}\Big(|\lambda|^{-1+\frac1q}\|f\|_{L^q} + \big(\||x|^{\gamma'}F\|_{L^2} + \beta\|b\|_{L^\infty(\partial D)}\big)\Big)\,.\end{aligned}$$

Then by (4.132) we find

$$\big|\big\langle \omega^{{\rm ed}\,(1)}_{f,n}, \frac{v_{r,n}}{|x|}\big\rangle_{L^2}\big| \le \frac{C}{\beta^5}|\lambda|^{-1+\frac1q}\|f\|_{L^q}\big(|\lambda|^{-1+\frac1q}\|f\|_{L^q} + (\beta^\kappa d + \beta d^{\frac12})\|\nabla v\|_{L^2(\Omega)}\big)\,. \tag{4.136}$$

On the other hand, since $\frac1p + \frac1{p'} = 1$ holds, by using (4.96) replacing $\tilde q$ by p' in Theorem 4.3.11, (4.66) in Theorem 4.3.2, (4.106) in Theorem 4.3.13, and (4.127) in Theorem 4.3.22, we have

$$\begin{aligned}\big\|\frac{\omega^{{\rm ed}\,(2)}_{f,n}}{|x|}\big\|_{L^{p'}}\|v_n\|_{L^p} &\le \frac{C}{\beta^3}|\lambda|^{-1+\frac1q}\|f\|_{L^q}\Big(|\lambda|^{-1+\frac1q}\|f\|_{L^q} + \big(\||x|^{\gamma'}F\|_{L^2} + \beta\|b\|_{L^\infty(\partial D)}\big)\Big)\\ &\le \frac{C}{\beta^3}|\lambda|^{-1+\frac1q}\|f\|_{L^q}\big(|\lambda|^{-1+\frac1q}\|f\|_{L^q} + (\beta^\kappa d + \beta d^{\frac12})\|\nabla v\|_{L^2(\Omega)}\big)\,.\end{aligned} \tag{4.137}$$

Then inserting (4.136) and (4.137) into (4.135) we obtain

$$\big|\big\langle \omega^{\rm ed}_{f,n}, \frac{v_{r,n}}{|x|}\big\rangle_{L^2}\big| \le \frac{C}{\beta^5}|\lambda|^{-1+\frac1q}\|f\|_{L^q}\big(|\lambda|^{-1+\frac1q}\|f\|_{L^q} + (\beta^\kappa d + \beta d^{\frac12})\|\nabla v\|_{L^2(\Omega)}\big)\,. \tag{4.138}$$

(ii) Estimate of $|\langle \omega^{\rm ed}_{{\rm div}F,n}, \frac{v_{r,n}}{|x|}\rangle_{L^2}|$: By using the Hölder inequality we find

$$\begin{aligned}\big|\big\langle \omega^{\rm ed}_{{\rm div}F,n}, \frac{v_{r,n}}{|x|}\big\rangle_{L^2}\big| &\le \|\omega^{\rm ed}_{{\rm div}F,n}\|_{L^2}\Big(\big\|\frac{w^{\rm ed}_{{\rm div}F,n}}{|x|}\big\|_{L^2} + \big\|\frac{w^{\rm ed}_{b,n}}{|x|}\big\|_{L^2}\Big)\\ &\quad + \big|\big\langle \omega^{{\rm ed}\,(1)}_{{\rm div}F,n}, \frac{(w^{\rm ed}_{f,r})_n}{|x|}\big\rangle_{L^2}\big| + \big\|\frac{\omega^{{\rm ed}\,(2)}_{{\rm div}F,n}}{|x|}\big\|_{L^1}\big\|w^{\rm ed}_{f,n}\big\|_{L^\infty}\,.\end{aligned} \tag{4.139}$$

By Theorem 4.3.20, (4.107) in Theorem 4.3.13, and (4.128) in Theorem 4.3.22 we see that

$$\|\omega^{\rm ed}_{{\rm div}F,n}\|_{L^2}\Big(\big\|\frac{w^{\rm ed}_{{\rm div}F,n}}{|x|}\big\|_{L^2}+\big\|\frac{w^{\rm ed}_{b,n}}{|x|}\big\|_{L^2}\Big)\le\frac{K}{\beta^5}\||x|^{\gamma'}F\|_{L^2}\big(\||x|^{\gamma'}F\|_{L^2}+\beta\|b\|_{L^\infty(\partial D)}\big)$$
$$\le\frac{K}{\beta^5}(\beta^\kappa d+\beta d^{\frac12})^2\|\nabla v\|^2_{L^2(\Omega)}\,, \tag{4.140}$$

where we note that the constant K depends only on ϵ and γ, and is independent of β and, in particular, of $q\in(1,2]$. Theorem 4.3.20 and (4.66) with $p=\infty$ in Theorem 4.3.2 lead to

$$\big|\langle\omega^{{\rm ed}\,(1)}_{{\rm div}F,n},\frac{(w^{\rm ed}_{f,r})_n}{|x|}\rangle_{L^2}\big|+\|\frac{\omega^{{\rm ed}\,(2)}_{{\rm div}F,n}}{|x|}\|_{L^1}\|w^{\rm ed}_{f,n}\|_{L^\infty}$$
$$\le\frac{C}{\beta^5}|\lambda|^{-1+\frac1q}\|f\|_{L^q}\||x|^{\gamma'}F\|_{L^2}\le\frac{C}{\beta^5}\beta^\kappa d|\lambda|^{-1+\frac1q}\|f\|_{L^q}\|\nabla v\|_{L^2(\Omega)}\,. \tag{4.141}$$

Inserting (4.140) and (4.141) into (4.139) we have

$$\big|\langle\omega^{\rm ed}_{{\rm div}F,n},\frac{v_{r,n}}{|x|}\rangle_{L^2}\big|\le\frac{C}{\beta^5}\beta^\kappa d|\lambda|^{-1+\frac1q}\|f\|_{L^q(\Omega)}\|\nabla v\|_{L^2(\Omega)}+\frac{K}{\beta^5}(\beta^\kappa d+\beta d^{\frac12})^2\|\nabla v\|^2_{L^2(\Omega)}\,. \tag{4.142}$$

(iii) Estimate of $|\langle\omega^{\rm ed}_{b,n},\frac{v_{r,n}}{|x|}\rangle_{L^2}|$: Using the Schwartz inequality and Theorem 4.3.22 we find

$$\big|\langle\omega^{\rm ed}_{b,n},\frac{v_{r,n}}{|x|}\rangle_{L^2(D)}\big|\le\big|\langle\omega^{\rm ed}_{b,n},\frac{(w^{\rm ed}_{f,r})_n}{|x|}\rangle_{L^2}\big|+\|\omega^{\rm ed}_{b,n}\|_{L^2}\Big(\big\|\frac{w^{\rm ed}_{{\rm div}F,n}}{|x|}\big\|_{L^2}+\big\|\frac{w^{\rm ed}_{b,n}}{|x|}\big\|_{L^2}\Big)$$
$$\le\frac{1}{\beta^4}\Big(C|\lambda|^{-1+\frac1q}\|f\|_{L^q}\|b\|_{L^\infty(\partial D)}+K\|b\|_{L^\infty(\partial D)}\big(\||x|^{\gamma'}F\|_{L^2}+\beta\|b\|_{L^\infty(\partial D)}\big)\Big)$$
$$\le\frac{C}{\beta^5}\beta d^{\frac12}|\lambda|^{-1+\frac1q}\|f\|_{L^q(\Omega)}\|\nabla v\|_{L^2(\Omega)}+\frac{K}{\beta^5}(\beta^\kappa d+\beta d^{\frac12})^2\|\nabla v\|^2_{L^2(\Omega)}\,. \tag{4.143}$$

Finally we obtain the assertion (4.131) by collecting (4.138), (4.142), and (4.143), and using the Young inequality in the form

$$\frac{C}{\beta^5}(\beta^\kappa d+\beta d^{\frac12})|\lambda|^{-1+\frac1q}\|f\|_{L^q(\Omega)}\|\nabla v\|_{L^2(\Omega)}$$
$$\le\frac{C}{\beta^5}|\lambda|^{-2+\frac2q}\|f\|^2_{L^q(\Omega)}+\frac{(\beta^\kappa d+\beta d^{\frac12})^2}{\beta^5}\|\nabla v\|^2_{L^2(\Omega)}\,.$$

The proof is complete. □

Now we shall establish the resolvent estimate to (RS) when $0<|\lambda|<e^{-\frac{1}{6\beta}}$, by closing the energy computation starting from Proposition 4.2.1 in Subsection 4.2.3.

Proposition 4.3.24 *Let $\epsilon\in(0,\frac{\pi}{4})$, and let β_1, β_0, and K be the constants respectively in Propositions 4.2.1, 4.3.1, and 4.3.23. Then the following statements hold.*
(1) *Fix positive numbers $\beta_3\in(0,\min\{\beta_1,\beta_0\})$ and $\mu_*\in(0,(64K)^{-1})$. Then the set*

$$\Sigma_{\frac34\pi-\epsilon}\cap\mathcal{B}_{e^{-\frac{1}{6\beta}}}(0) \tag{4.144}$$

is included in the resolvent $\rho(-\mathbb{A}_V)$ for any $\beta\in(0,\beta_3)$ and $d\in(0,\mu_*\beta^2)$.

(2) *Let* $q \in (1.2]$ *and* $f \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$*. Then we have*

$$
\begin{aligned}
\|(\lambda + \mathbb{A}_V)^{-1} f\|_{L^2(\Omega)} &\le \frac{C}{\beta^2} |\lambda|^{-\frac{3}{2}+\frac{1}{q}} \|f\|_{L^q(\Omega)} , \quad \lambda \in \Sigma_{\frac{3}{4}\pi - \epsilon} \cap \mathcal{B}_{e^{-\frac{1}{6\beta}}}(0) , \\
\|\nabla(\lambda + \mathbb{A}_V)^{-1} f\|_{L^2(\Omega)} &\le \frac{C}{\beta^2} |\lambda|^{-1+\frac{1}{q}} \|f\|_{L^q(\Omega)} , \quad \lambda \in \Sigma_{\frac{3}{4}\pi - \epsilon} \cap \mathcal{B}_{e^{-\frac{1}{6\beta}}}(0) ,
\end{aligned}
\tag{4.145}
$$

as long as $\beta \in (0, \beta_3)$ *and* $d \in (0, \mu_* \beta^2)$*. The constant* C *is independent of* β*.*

Proof: (1) Let $\lambda \in \Sigma_{\frac{3}{4}\pi - \epsilon} \cap \mathcal{B}_{e^{-\frac{1}{6\beta}}}(0)$ and suppose that $v \in D(\mathbb{A}_V)$ is a solution to (RS). Since $d \in (0, \mu_* \beta^2)$ ensures $K(\beta^\kappa d + \beta d^{\frac{1}{2}})^2 \beta^{-4} \le \frac{1}{8}$, by inserting (4.131) in Proposition 4.3.23 into (4.29) and (4.30) in Proposition 4.2.1, and by combining them we find

$$
\begin{aligned}
&\big(|\mathrm{Im}(\lambda)| + \mathrm{Re}(\lambda)\big) \|v\|^2_{L^2(\Omega)} + \frac{1}{4} \|\nabla v\|^2_{L^2(\Omega)} \\
&\qquad \le \frac{C}{\beta^4} |\lambda|^{-2+\frac{2}{q}} \|f\|^2_{L^q(\Omega)} + C \|f\|^{\frac{2q}{3q-2}}_{L^q(\Omega)} \|v\|^{\frac{4(q-1)}{3q-2}}_{L^2(\Omega)} .
\end{aligned}
\tag{4.146}
$$

Then, since $\lambda \in \Sigma_{\frac{3}{4}\pi - \epsilon}$ implies that $|\mathrm{Im}(\lambda)| + \mathrm{Re}(\lambda) > c|\lambda|$ holds with some positive constant $c = c(\epsilon)$, the assertion $\Sigma_{\frac{3}{4}\pi - \epsilon} \cap \mathcal{B}_{e^{-\frac{1}{6\beta}}}(0) \subset \rho(-\mathbb{A}_V)$ follows.
(2) The estimate (4.145) can be easily checked by using (4.146). The proof is complete. □

4.4 Proof of Theorem 4.1.3

In this section we prove Theorem 4.1.3. The proof is an easy consequence of Propositions 4.2.2 and 4.3.24 respectively in Subsections 4.2.3 and 4.3.4.

Proof of Theorem 4.1.3: Let β_2 be the constant in Proposition 4.2.2. We note that $\mathcal{S}_{\beta_2} \cap \mathcal{B}_{e^{-\frac{1}{6\beta_2}}}(0) \ne \emptyset$ holds since $12 e^{\frac{1}{e}} \beta_2^2 < 1$ follows from the condition $\beta_2 \in (0, \frac{1}{12})$. Then there is a constant $\epsilon_0 \in (\frac{\pi}{4}, \frac{\pi}{2})$ such that the sector $\Sigma_{\pi - \epsilon_0}$ is included in the set $\mathcal{S}_\beta \cup \mathcal{B}_{e^{-\frac{1}{6\beta}}}(0)$ for any $\beta \in (0, \beta_2)$.
Let β_3 be the constant in Proposition 4.3.24. Fix a number $\beta_* \in (0, \min\{\beta_2, \beta_3\})$. Then by Propositions 4.2.2 and 4.3.24, there is a positive constant μ_* such that the sector $\Sigma_{\pi - \epsilon_0}$ is included in the resolvent $\rho(-\mathbb{A}_V)$ as long as $\beta \in (0, \beta_*)$ and $d \in (0, \mu_* \beta^2)$. Moreover, from the same propositions, for $q \in (1, 2]$ and $f \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$ we have

$$
\begin{aligned}
\|(\lambda + \mathbb{A}_V)^{-1} f\|_{L^2(\Omega)} &\le \frac{C}{\beta^2} |\lambda|^{-\frac{3}{2}+\frac{1}{q}} \|f\|_{L^q(\Omega)} , \quad \lambda \in \Sigma_{\pi - \epsilon_0} , \\
\|\nabla(\lambda + \mathbb{A}_V)^{-1} f\|_{L^2(\Omega)} &\le \frac{C}{\beta^2} |\lambda|^{-1+\frac{1}{q}} \|f\|_{L^q(\Omega)} , \quad \lambda \in \Sigma_{\pi - \epsilon_0} .
\end{aligned}
\tag{4.147}
$$

In particular, the first line in (4.147) implies the estimate (4.10) for $q = 2$. Next we consider the case $q \in (1, 2)$. Fix a number $\phi \in (\frac{\pi}{2}, \pi - \epsilon_0)$ and take a curve $\gamma(b) = \{z \in \mathbb{C} \mid |\arg z| = \phi\, , \, |z| \ge b\} \cup \{z \in \mathbb{C} \mid |\arg z| \le \phi\, , \, |z| = b\}$, $b \in (0, 1)$, oriented counterclockwise. Then the semigroup $e^{-t\mathbb{A}_V}$ admits a Dunford integral representation

$$
e^{-t\mathbb{A}_V} f = \frac{1}{2\pi i} \int_{\gamma(b)} e^{t\lambda} (\lambda + \mathbb{A}_V)^{-1} f \, \mathrm{d}\lambda , \qquad t > 0 ,
$$

for $f \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$. Then by taking the limit $b \to 0$ we observe from (4.147) that

$$\begin{aligned}\|e^{-t\mathbb{A}_V} f\|_{L^2(\Omega)} &\le \frac{C}{\beta^2}\|f\|_{L^q(\Omega)} \int_0^\infty s^{-\frac{3}{2}+\frac{1}{q}} e^{(\cos\phi)ts}\,\mathrm{d}s \\ &\le \frac{C}{\beta^2} t^{-\frac{1}{q}+\frac{1}{2}} \|f\|_{L^q(\Omega)}\,, \qquad t>0\,,\end{aligned}$$

which shows that (4.10) holds for $q \in (1,2)$. The estimate (4.11) can be obtained in a similar manner using the Dunford integral. This completes the proof of Theorem 4.1.3. □

4.5 Appendix

4.5.1 Asymptotics of the order $\mu_n(\beta)$ for small β

This appendix is devoted to the statement of the asymptotic behavior for $\mu_n(\beta) = (n^2 + in\beta)^{\frac{1}{2}}$, $\mathrm{Re}(\mu_n) > 0$, with $|n| = 1$ when the constant $\beta \in (0,1)$ in Assumption 4.1.1 reaches to zero. The following result is essentially proved in [44].

Lemma 4.5.1 ([44, Lemma B.1]) *Let $|n| = 1$. Then $\mu_n(\beta)$ satisfies the expansion*

$$\mathrm{Re}(\mu_n(\beta)) \;=\; 1 + \frac{\beta^2}{8} + O(\beta^4)\,, \qquad 0<\beta\ll 1\,, \tag{4.148}$$

$$\mathrm{Im}(\mu_n(\beta)) \;=\; \frac{\beta}{2} + O(\beta^3)\,, \qquad 0<\beta\ll 1\,. \tag{4.149}$$

4.5.2 Estimates of the Modified Bessel Function

In this appendix we collect the basic estimates for the modified Bessel functions $K_{\mu_n}(z)$ and $I_{\mu_n}(z)$ of the order $\mu_n = (n^2 + in\beta)^{\frac{1}{2}}$, $\mathrm{Re}(\mu_n) > 0$, with $|n| = 1$ and $\beta \in (0,1)$. We are especially interested in the β-dependence in each estimate, since our analysis in Section 4.3, where the results in this appendix are applied, requires the smallness of β. We denote by $\mathcal{B}_\rho(0)$ the disk in the complex plane $\mathbb{C}$ centered at the origin with radius $\rho > 0$.

Lemma 4.5.2 *Let $|n| = 1$, $k = 0, 1$, and $R \in [1, \infty)$. Fix $\epsilon \in (0, \frac{\pi}{2})$. Then there is a positive constant $C = C(R, \epsilon)$ independent of β such that the following statements hold.*
(1) *Let $z \in \Sigma_\epsilon \cap \mathcal{B}_R(0)$. Then $K_{\mu_n}(z)$ and $K_{\mu_n - 1}(z)$ satisfy the expansions*

$$K_{\mu_n}(z) \;=\; \frac{\Gamma(\mu_n)}{2}\Big(\frac{z}{2}\Big)^{-\mu_n} + R_n^{(1)}(z)\,, \tag{4.150}$$

$$K_{\mu_n-1}(z) \;=\; \frac{\pi}{2\sin((\mu_n-1)\pi)}\Big(\frac{1}{\Gamma(2-\mu_n)}\Big(\frac{z}{2}\Big)^{-\mu_n+1} - \frac{1}{\Gamma(\mu_n)}\Big(\frac{z}{2}\Big)^{\mu_n-1}\Big) + R_n^{(2)}(z)\,. \tag{4.151}$$

Here $\Gamma(z)$ denotes the Gamma function and the remainders $R_n^{(1)}(z)$ and $R_n^{(2)}(z)$ satisfy

$$|R_n^{(1)}(z)| \le C|z|^{2-\mathrm{Re}(\mu_n)}(1+|\log|z||)\,, \qquad z\in\Sigma_\epsilon\cap\mathcal{B}_R(0)\,, \tag{4.152}$$

$$|R_n^{(2)}(z)| \le C|z|^{3-\mathrm{Re}(\mu_n)}(1+|\log|z||)\,, \qquad z\in\Sigma_\epsilon\cap\mathcal{B}_R(0)\,. \tag{4.153}$$

(2) *The following estimates hold.*

$$|I_{\mu_n+k}(z)| \le C|z|^{\mathrm{Re}(\mu_n)+k}\,, \qquad z \in \Sigma_\epsilon \cap \mathcal{B}_R(0)\,, \tag{4.154}$$

$$|K_{\mu_n-k}(z)| \le C|z|^{-\frac{1}{2}} e^{-\mathrm{Re}(z)}\,, \qquad z \in \Sigma_\epsilon \cap \mathcal{B}_R(0)^{\mathrm{c}}\,, \tag{4.155}$$

$$|I_{\mu_n+k}(z)| \le C|z|^{-\frac{1}{2}} e^{\mathrm{Re}(z)}\,, \qquad z \in \Sigma_\epsilon \cap \mathcal{B}_R(0)^{\mathrm{c}}\,. \tag{4.156}$$

Proof: (1) The expansions (4.150) and (4.151) follow from the definition of $K_\mu(z)$ in Subsection 4.3.1 combined with the well-known Euler reflection formula for the Gamma function. The estimates of the remainder terms (4.152) and (4.153) are also consequences of the same definition, and we omit the calculations which are easily checked.
(2) The estimate (4.154) directly follows from the definition of $I_\mu(z)$ in Subsection 4.3.1. In order to prove (4.155) and (4.156), let us recall the integral formulas for $K_\mu(z)$ and $I_\mu(z)$:

$$K_\mu(z) \;=\; \frac{\pi^{\frac{1}{2}}}{\Gamma(\mu+\frac{1}{2})}\big(\frac{z}{2}\big)^\mu \int_0^\infty e^{-z\cosh t}(\sinh t)^{2\mu}\,\mathrm{d}t\,,$$

$$I_\mu(z) \;=\; \frac{1}{\pi^{\frac{1}{2}}\,\Gamma(\mu+\frac{1}{2})}\big(\frac{z}{2}\big)^\mu \int_0^\pi e^{z\cos\theta}(\sin\theta)^{2\mu}\,\mathrm{d}\theta\,,$$

which are valid if $\mathrm{Re}(\mu) > -\frac{1}{2}$ and $z \in \Sigma_{\frac{\pi}{2}}$ (see [1] page 376) . Then (4.155) and (4.156), especially the absence of the β-singularity in the right-hand sides, can be proved by using the identities $\cosh^2 t - \sinh^2 t = 1$ and $\cos^2\theta + \sin^2\theta = 1$. The proof is complete. □

In the following we present three lemmas without proofs, since they are straightforward adaptations of Lemma 4.5.2 and Lemma 4.5.1 in Appendix 4.5.1.

Lemma 4.5.3 *Let $|n| = 1$ and $k = 0, 1$, and let $\lambda \in \Sigma_{\pi-\epsilon} \cap \mathcal{B}_1(0)$ for some $\epsilon \in (0, \frac{\pi}{2})$. Then there is a constant $C > 0$ independent of β such that the following statements hold.*
(1) *If $1 \le \tau \le r \le \mathrm{Re}(\sqrt{\lambda})^{-1}$, then*

$$\Big|\int_\tau^r s^{2-k} K_{\mu_n-k}(\sqrt{\lambda}s)\,\mathrm{d}s\Big| \le \frac{C}{\beta^k}|\lambda|^{-\frac{\mathrm{Re}(\mu_n)}{2}+\frac{k}{2}} r^{-\mathrm{Re}(\mu_n)+3}\,. \tag{4.157}$$

(2) *If $1 \le \tau \le \mathrm{Re}(\sqrt{\lambda})^{-1} \le r$, then*

$$\Big|\int_\tau^r s^{2-k} K_{\mu_n-k}(\sqrt{\lambda}s)\,\mathrm{d}s\Big| \le \frac{C}{\beta^k}\,|\lambda|^{-\frac{3}{2}+\frac{k}{2}}\,. \tag{4.158}$$

(3) *If $\mathrm{Re}(\sqrt{\lambda})^{-1} \le \tau \le r$, then*

$$\int_\tau^r |s^{2-k} K_{\mu_n-k}(\sqrt{\lambda}s)|\,\mathrm{d}s \le C|\lambda|^{-\frac{3}{4}} \tau^{\frac{3}{2}-k} e^{-\mathrm{Re}(\sqrt{\lambda})\tau}\,. \tag{4.159}$$

(4) *If $1 \le \tau \le \mathrm{Re}(\sqrt{\lambda})^{-1}$, then*

$$\Big|\int_\tau^\infty s^{-k} K_{\mu_n-k}(\sqrt{\lambda}s)\,\mathrm{d}s\Big| \le \frac{C}{\beta^{1+k}}|\lambda|^{-\frac{\mathrm{Re}\,(\mu_n)}{2}+\frac{k}{2}} \tau^{-\mathrm{Re}\,(\mu_n)+1}\,. \tag{4.160}$$

(5) *If $\tau \ge \mathrm{Re}(\sqrt{\lambda})^{-1}$, then*

$$\int_\tau^\infty |s^{-k} K_{\mu_n-k}(\sqrt{\lambda}s)|\,\mathrm{d}s \le C|\lambda|^{-\frac{3}{4}} \tau^{-\frac{1}{2}-k} e^{-\mathrm{Re}(\sqrt{\lambda})\tau}\,. \tag{4.161}$$

Lemma 4.5.4 *Let* $|n| = 1$ *and* $k = 0, 1$, *and let* $\lambda \in \Sigma_{\pi-\epsilon} \cap \mathcal{B}_1(0)$ *for some* $\epsilon \in (0, \frac{\pi}{2})$. *Then there is a constant* $C > 0$ *independent of* β *such that the following statements hold.*
(1) *If* $1 \leq \tau \leq \mathrm{Re}(\sqrt{\lambda})^{-1}$, *then*

$$\int_1^\tau |s^{2-k} I_{\mu_n+k}(\sqrt{\lambda}s)| \, \mathrm{d}s \leq C|\lambda|^{\frac{\mathrm{Re}(\mu_n)}{2}+\frac{k}{2}} \tau^{\mathrm{Re}(\mu_n)+3} \,. \tag{4.162}$$

(2) *If* $\tau \geq \mathrm{Re}(\sqrt{\lambda})^{-1}$, *then*

$$\int_1^\tau |s^{2-k} I_{\mu_n+k}(\sqrt{\lambda}s)| \, \mathrm{d}s \leq C|\lambda|^{-\frac{3}{4}} \tau^{\frac{3}{2}-k} e^{\mathrm{Re}(\sqrt{\lambda})\tau} \,. \tag{4.163}$$

(3) *If* $1 \leq r \leq \tau \leq \mathrm{Re}(\sqrt{\lambda})^{-1}$, *then*

$$\int_r^\tau |s^{-k} I_{\mu_n+k}(\sqrt{\lambda}s)| \, \mathrm{d}s \leq C|\lambda|^{\frac{\mathrm{Re}(\mu_n)}{2}+\frac{k}{2}} \tau^{\mathrm{Re}(\mu_n)+1} \,. \tag{4.164}$$

(4) *If* $1 \leq r \leq \mathrm{Re}(\sqrt{\lambda})^{-1} \leq \tau$, *then*

$$\int_r^\tau |s^{-k} I_{\mu_n+k}(\sqrt{\lambda}s)| \, \mathrm{d}s \leq C|\lambda|^{-\frac{3}{4}} \tau^{-\frac{1}{2}-k} e^{\mathrm{Re}(\sqrt{\lambda})\tau} \,. \tag{4.165}$$

(5) *If* $\mathrm{Re}(\sqrt{\lambda})^{-1} \leq r \leq \tau$, *then*

$$\int_r^\tau |s^{-k} I_{\mu_n+k}(\sqrt{\lambda}s)| \, \mathrm{d}s \leq C|\lambda|^{-\frac{3}{4}} \tau^{-\frac{1}{2}-k} e^{\mathrm{Re}(\sqrt{\lambda})\tau} \,. \tag{4.166}$$

Lemma 4.5.5 *Let* $|n| = 1$ *and* $p \in (1, \infty)$, *and let* $\lambda \in \Sigma_{\pi-\epsilon} \cap \mathcal{B}_1(0)$ *for some* $\epsilon \in (0, \frac{\pi}{2})$. *Then there is a constant* $C > 0$ *independent of* β *such that the following statements hold.*
(1) *If additionally* $p \in [2, \infty)$, *then*

$$\|K_{\mu_n}(\sqrt{\lambda}\,\cdot)\|_{L^p((1,\infty);r\,\mathrm{d}r)} \leq \frac{C}{(p\mathrm{Re}(\mu_n) - 2)^{\frac{1}{p}}} |\lambda|^{-\frac{\mathrm{Re}\,(\mu_n)}{2}} \,. \tag{4.167}$$

(2) *If* $1 \leq r \leq \mathrm{Re}(\sqrt{\lambda})^{-1}$, *then*

$$\left(\int_r^\infty |s^{-1} K_{\mu_n}(\sqrt{\lambda}s)|^p s \, \mathrm{d}s \right)^{\frac{1}{p}} \leq C|\lambda|^{-\frac{\mathrm{Re}(\mu_n)}{2}} r^{-\mathrm{Re}(\mu_n)-1+\frac{2}{p}} \,. \tag{4.168}$$

(3) *If* $r \geq \mathrm{Re}(\sqrt{\lambda})^{-1}$, *then*

$$\left(\int_r^\infty |s^{-1} K_{\mu_n}(\sqrt{\lambda}s)|^p s \, \mathrm{d}s \right)^{\frac{1}{p}} \leq C|\lambda|^{-\frac{1}{4}-\frac{1}{2p}} r^{-\frac{3}{2}+\frac{1}{p}} e^{-\mathrm{Re}(\sqrt{\lambda})r} \,. \tag{4.169}$$

(4) *If* $1 \leq r \leq \mathrm{Re}(\sqrt{\lambda})^{-1}$, *then*

$$\left(\int_1^r |s^{-1} I_{\mu_n}(\sqrt{\lambda}s)|^p s \, \mathrm{d}s \right)^{\frac{1}{p}} \leq C|\lambda|^{\frac{\mathrm{Re}\,(\mu_n)}{2}} r^{\mathrm{Re}(\mu_n)-1+\frac{2}{p}} \,. \tag{4.170}$$

(5) *If* $r \geq \mathrm{Re}(\sqrt{\lambda})^{-1}$, *then*

$$\left(\int_1^r |s^{-1} I_{\mu_n}(\sqrt{\lambda}s)|^p s \, \mathrm{d}s \right)^{\frac{1}{p}} \leq C|\lambda|^{-\frac{1}{4}-\frac{1}{2p}} r^{-\frac{3}{2}+\frac{1}{p}} e^{\mathrm{Re}(\sqrt{\lambda})r} \,. \tag{4.171}$$

(6) *If additionally* $p \in [2, \infty)$ *and if* $1 \leq r \leq \mathrm{Re}(\sqrt{\lambda})^{-1}$, *then*

$$\begin{aligned}&\Big(\int_r^\infty |K_{\mu_n}(\sqrt{\lambda}s)|^p s \,\mathrm{d}s\Big)^{\frac{1}{p}} + \beta \int_r^\infty |K_{\mu_n}(\sqrt{\lambda}s)| \,\mathrm{d}s \\ &\leq \frac{C}{\beta}|\lambda|^{-\frac{\mathrm{Re}(\mu_n)}{2}} r^{-\mathrm{Re}(\mu_n)+1}\,.\end{aligned} \tag{4.172}$$

(7) *If additionally* $p \in [2, \infty)$ *and if* $r \geq \mathrm{Re}(\sqrt{\lambda})^{-1}$, *then*

$$\Big(\int_r^\infty |K_{\mu_n}(\sqrt{\lambda}s)|^p s \,\mathrm{d}s\Big)^{\frac{1}{p}} + \int_r^\infty |K_{\mu_n}(\sqrt{\lambda}s)| \,\mathrm{d}s \leq C|\lambda|^{-\frac{1}{2}} e^{-\mathrm{Re}(\sqrt{\lambda})r}\,. \tag{4.173}$$

(8) *If additionally* $p \in [2, \infty)$ *and if* $1 \leq r \leq \mathrm{Re}(\sqrt{\lambda})^{-1}$, *then*

$$\Big(\int_1^r |I_{\mu_n}(\sqrt{\lambda}s)|^p s \,\mathrm{d}s\Big)^{\frac{1}{p}} + \int_1^r |I_{\mu_n}(\sqrt{\lambda}s)| \,\mathrm{d}s \leq C|\lambda|^{\frac{\mathrm{Re}(\mu_n)}{2}} r^{\mathrm{Re}(\mu_n)+1}\,. \tag{4.174}$$

(9) *If additionally* $p \in [2, \infty)$ *and if* $r \geq \mathrm{Re}(\sqrt{\lambda})^{-1}$, *then*

$$\Big(\int_1^r |I_{\mu_n}(\sqrt{\lambda}s)|^p s \,\mathrm{d}s\Big)^{\frac{1}{p}} + \int_1^r |I_{\mu_n}(\sqrt{\lambda}s)| \,\mathrm{d}s \leq C|\lambda|^{-\frac{1}{2}} e^{\mathrm{Re}(\sqrt{\lambda})r}\,. \tag{4.175}$$

4.5.3 Proof of Proposition 4.3.1

Proposition 4.3.1 is a direct consequence of the next lemma. Let us recall that $\mathcal{B}_\rho(0)$ denotes the disk in the complex plane $\mathbb{C}$ centered at the origin with radius $\rho > 0$.

Lemma 4.5.6 *Let* $|n| = 1$. *Then for any* $\epsilon \in (0, \frac{\pi}{2})$ *there is a positive constant* $\beta_0 = \beta_0(\epsilon)$ *depending only on* ϵ *such that as long as* $\beta \in (0, \beta_0)$ *and* $\lambda \in \Sigma_{\pi-\epsilon} \cap \mathcal{B}_{\beta^4}(0)$ *we have*

$$|F_n(\sqrt{\lambda}; \beta)| \geq \frac{C}{\beta}|\lambda|^{-\frac{\mathrm{Re}(\mu_n)}{2}} \min\{1\,, -\beta^2 \log|\lambda|\}\,, \tag{4.176}$$

where $F_n(\sqrt{\lambda}; \beta)$ *is the function in* (4.59) *and the constant* C *depends only on* ϵ.

Proof: The proof is carried out with the similar spirit as in [44, Proposition 3.34], where the nonexistence of zeros of $F_n(\sqrt{\lambda}; \beta)$ in $\lambda \in \mathcal{B}_{\beta^4}(0)$ is proved. However, its proof is based on a contradiction argument, and quantitative estimates are not explicitly stated. Hence here we give the lower bound estimate of $|F_n(\sqrt{\lambda}; \beta)|$ for completeness.

Let $\lambda \in \Sigma_{\pi-\epsilon} \cap \mathcal{B}_{\frac{1}{2}}(0)$ and set $\zeta_n = \zeta_n(\beta) = \mu_n(\beta) - 1$. Then, by combining Lemmas 3.31–3.33 and Corollary A.8 in [44], we observe that the next expansion holds:

$$\zeta_n F_n(\sqrt{\lambda}; \beta) \;=\; \frac{\Gamma(1+\zeta_n)}{\sqrt{\lambda}}\Big(\frac{\sqrt{\lambda}}{2}\Big)^{-\zeta_n}\Big(1 - \big(e^{\gamma(\zeta_n)}\frac{\sqrt{\lambda}}{2}\big)^{\zeta_n} + R_n(\lambda)\Big)\,, \tag{4.177}$$

for sufficiently small β depending on $\epsilon \in (0, \frac{\pi}{2})$. Here the function $\gamma(\zeta_n)$ have the expansion

$$\gamma(\zeta_n) \;=\; \gamma + O(|\zeta_n|) \quad \text{as} \quad |\zeta_n| \to 0\,, \tag{4.178}$$

where γ denotes the Euler constant $\gamma = 0.5772\cdots$. The remainder R_n in (4.177) satisfies

$$|R_n(\lambda)| \le C_1|\lambda|^{\frac{\mathrm{Re}(\mu_n)}{2}}, \qquad \lambda \in \Sigma_{\pi-\epsilon} \cap \mathcal{B}_{\frac{1}{2}}(0), \tag{4.179}$$

with a constant $C_1 = C_1(\epsilon)$ independent of small β. To simplify notation we set

$$z = \sqrt{\lambda}, \qquad \tilde{z} = e^{\gamma(\zeta_n)}\frac{\sqrt{\lambda}}{2}, \qquad \theta(\tilde{z}) = \arg\tilde{z}. \tag{4.180}$$

If β is sufficiently small, then we see from (4.178) and (4.180) that

$$\frac{1}{2} \le \Big|\frac{\tilde{z}}{z}\Big| \le 1, \qquad |\theta(\tilde{z})| \le \frac{\pi}{2} - \frac{\epsilon}{4}, \qquad \lambda \in \Sigma_{\pi-\epsilon} \cap \mathcal{B}_{\frac{1}{2}}(0). \tag{4.181}$$

Now we set

$$h(\tilde{z},\zeta_n) = \mathrm{Re}(\zeta_n)\log|\tilde{z}| - \mathrm{Im}(\zeta_n)\theta(\tilde{z}), \tag{4.182}$$

$$\begin{aligned}\Omega(\tilde{z},\zeta_n) &= \mathrm{Re}(\zeta_n)\theta(\tilde{z}) + \mathrm{Im}(\zeta_n)\log|\tilde{z}| \\ &= \big(\mathrm{Re}(\zeta_n) + \frac{\mathrm{Im}(\zeta_n)^2}{\mathrm{Re}(\zeta_n)}\big)\theta(\tilde{z}) + \frac{\mathrm{Im}(\zeta_n)}{\mathrm{Re}(\zeta_n)}h(\tilde{z},\zeta_n).\end{aligned} \tag{4.183}$$

Then it is easy to see that

$$1 - \tilde{z}^{\zeta_n} = 1 - e^{h(\tilde{z},\zeta_n)}e^{i\Omega(\tilde{z},\zeta_n)}. \tag{4.184}$$

In the following we show the lower bound estimate of $|1 - \tilde{z}^{\zeta_n}|$. Firstly let us take a small positive constant $\kappa = \kappa(\epsilon) \ll 1$ so that

$$\big(\mathrm{Re}(\zeta_n) + (1+\kappa)\frac{\mathrm{Im}(\zeta_n)^2}{\mathrm{Re}(\zeta_n)}\big)\big(\frac{\pi}{2} - \frac{\epsilon}{4}\big) < \pi \tag{4.185}$$

holds. The existence of such κ is verified by using Lemma 4.5.1 in Appendix 4.5.1 if β is sufficiently small depending on ϵ. Note that the smallness of κ depends only on ϵ.
(i) Case $|h(\tilde{z},\zeta_n)| \le \kappa|\mathrm{Im}(\zeta_n)||\theta(\tilde{z})|$: In this case, (4.181), (4.183), and (4.185) ensure that

$$|\Omega(\tilde{z},\zeta_n)| < \pi, \tag{4.186}$$

and thus that $e^{i\Omega(\tilde{z},\zeta_n)}$ is close to 1 if and only if $\Omega(\tilde{z},\zeta_n)$ is close to 0. From (4.182) we have

$$-\mathrm{Re}(\zeta_n)\log|\tilde{z}| \le (1+\kappa)|\mathrm{Im}(\zeta_n)||\theta(\tilde{z})|,$$

which leads to, for sufficiently small β,

$$|\theta(\tilde{z})| \ge -\frac{1}{1+\kappa}\frac{\mathrm{Re}(\zeta_n)}{|\mathrm{Im}(\zeta_n)|}\log|\tilde{z}| \ge -\frac{\beta}{2}\log|\tilde{z}|,$$

where $\frac{\mathrm{Re}(\zeta_n)}{|\mathrm{Im}(\zeta_n)|} = \frac{\beta}{4} + O(\beta^3)$ is applied in Lemma 4.5.1. Then from (4.183) we have

$$|\Omega(\tilde{z},\zeta_n)| \ge \big(\mathrm{Re}(\zeta_n) + (1-\kappa)\frac{\mathrm{Im}(\zeta_n)^2}{\mathrm{Re}(\zeta_n)}\big)|\theta(\tilde{z})| \ge -\beta\log|\tilde{z}|,$$

if β is small enough. On the other hand, it is straightforward to see that

$$|1-\tilde{z}^{\zeta_n}| \geq \max\{|1-e^{h(\tilde{z},\zeta_n)}\cos\Omega(\tilde{z},\zeta_n)|\,,\ e^{h(\tilde{z},\zeta_n)}|\sin\Omega(\tilde{z},\zeta_n)|\}\,.$$

Since $e^{h(\tilde{z},\zeta_n)} \in [\frac{1}{2},\frac{3}{2}]$, $|\sin x| \geq \frac{2|x|}{\pi}$ on $|x| \in [0,\frac{\pi}{2}]$, and $1 > \frac{|\Omega(\tilde{z},\zeta_n)|}{\pi}$ by (4.186), we have

$$|1-\tilde{z}^{\zeta_n}| \geq \min\{1\,,\ \frac{|\Omega(\tilde{z},\zeta_n)|}{\pi}\} \geq -\frac{\beta}{\pi}\log|\tilde{z}|\,. \tag{4.187}$$

(ii) Case $|h(\tilde{z},\zeta_n)| > \kappa|\mathrm{Im}(\zeta_n)||\theta(\tilde{z})|$: When $|\theta(\tilde{z})| > -\frac{1}{2}\frac{\mathrm{Re}(\zeta_n)}{|\mathrm{Im}(\zeta_n)|}\log|\tilde{z}|$, we have

$$|h(\tilde{z},\zeta_n)| \geq -\frac{\kappa\beta^2}{2}\log|\tilde{z}|\,.$$

On the other hand, when $|\theta(\tilde{z})| \leq -\frac{1}{2}\frac{\mathrm{Re}(\zeta_n)}{|\mathrm{Im}(\zeta_n)|}\log|\tilde{z}|$, (4.182) implies that

$$|h(\tilde{z},\zeta_n)| \geq -\frac{1}{2}\mathrm{Re}(\zeta_n)\log|\tilde{z}| \geq -\frac{\beta^2}{2}\log|\tilde{z}|\,.$$

Thus in the case (ii), since $|1-\tilde{z}^{\zeta_n}| \geq \big|1-|\tilde{z}^{\zeta_n}|\big| = |1-e^{h(\tilde{z},\zeta_n)}|$, we observe that

$$|1-\tilde{z}^{\zeta_n}| \geq \min\{1\,,\ |h(\tilde{z},\zeta_n)|\} \geq \min\{1\,,-\frac{\kappa\beta^2}{2}\log|\tilde{z}|\}\,. \tag{4.188}$$

Hence, by collecting (4.181), (4.187), and (4.188), we have the next lower estimate of $|1-\tilde{z}^{\zeta_n}|$:

$$|1-\tilde{z}^{\zeta_n}| \geq \frac{\kappa}{4}\min\{1\,,-\beta^2\log|z|\}\,. \tag{4.189}$$

Finally by inserting (4.179) and (4.189) into (4.177) we obtain

$$|\zeta_n F_n(\sqrt{\lambda};\beta)| \geq C|\lambda|^{-\frac{\mathrm{Re}(\mu_n)}{2}}\big(\kappa\min\{1\,,-\beta^2\log|z|\} - C_1|\lambda|^{\frac{\mathrm{Re}(\mu_n)}{2}}\big)\,,$$

which implies the assertion (4.176) if $\lambda \in \Sigma_{\pi-\epsilon}\cap\mathcal{B}_{\beta^4}(0)$ and β is sufficiently small depending on ϵ. The proof is complete. □

4.6 Future work: Large-time estimates of the semigroup and its application

This section is devoted to a brief presentation of future work related to the analysis in the previous sections. In Theorem 4.1.3 we obtained the L^p-L^q estimates for the semigroup $e^{-t\mathbb{A}_V}$, however, they are singular in the small parameter β. Especially, these singularities lead to the restriction on the size of the initial data in Theorem 4.1.4. Our aim in this section is to derive the semigroup estimates without the β-singularity by allowing *slower decays in time*. To make the problem simple, let us consider the case formally $d=0$ in Assumption 4.1.1. Then we obtain the rotating flow $\alpha U = \alpha\frac{x^\perp}{|x|^2}$, $\alpha \in (0,1)$, on the exterior disk $D = \{x \in \mathbb{R}^2 \mid |x| > 1\}$, and the perturbed Stokes equations (PS) are now written as

$$\begin{cases} \partial_t v - \Delta v + \alpha(U\cdot\nabla v + v\cdot\nabla U) + \nabla q = 0\,, & t>0\,,\ x\in D\,,\\ \mathrm{div}\, v = 0\,, & t\geq 0\,,\ x\in D\,,\\ v|_{\partial D} = 0\,, & t>0\,,\\ v|_{t=0} = v_0\,, & x\in D\,. \end{cases} \tag{PS$_\alpha$}$$

To state the main result let us recall some notations related to the problem (PS_α). We denote by $L^2_\sigma(D)$ the L^2-closure of $C^\infty_{0,\sigma}(D)$ and by $\mathbb{P}$ the Helmholtz projection $\mathbb{P} : L^2(D)^2 \to L^2_\sigma(D)$. The Stokes operator $\mathbb{A}$ with the domain $D_{L^2}(\mathbb{A}) = L^2_\sigma(D) \cap W^{1,2}_0(D)^2 \cap W^{2,2}(D)^2$ is defined as $\mathbb{A} = -\mathbb{P}\Delta$. Then we define the perturbed Stokes operator $\mathbb{A}_\alpha$ as

$$\begin{aligned} D_{L^2}(\mathbb{A}_\alpha) \, &= \, D_{L^2}(\mathbb{A}) \,, \\ \mathbb{A}_\alpha v \, &= \, \mathbb{A} v + \alpha \mathbb{P}(U \cdot \nabla v + v \cdot \nabla U) \,. \end{aligned}$$

Again, the perturbation theory for sectorial operators leads to the generation of an analytic semigroup by $-\mathbb{A}_\alpha$ in $L^2_\sigma(D)$, which we denote as $e^{-t\mathbb{A}_\alpha}$. Our main result in this section is the following exponentially large time estimates for the semigroup $e^{-t\mathbb{A}_\alpha}$:

Theorem 4.6.1 *There is a positive constant α_* such that if $\alpha \in (0, \alpha_*)$ then the following statement holds. Let $q \in (1, 2]$. Then we have*

$$\|e^{-t\mathbb{A}_\alpha} f\|_{L^2(D)} \le \begin{cases} C t^{-\frac{1}{q}+\frac{1}{2}} \|f\|_{L^q(D)} \,, & t \in (0, e^{\frac{1}{6\alpha}}] \,, \\ C\alpha (\log t)^3 t^{-\frac{1}{q}+\frac{1}{2}} \|f\|_{L^q(D)} \,, & t \in (e^{\frac{1}{6\alpha}}, \infty) \,, \end{cases} \tag{4.190}$$

$$\|\nabla e^{-t\mathbb{A}_\alpha} f\|_{L^2(D)} \le \begin{cases} C t^{-\frac{1}{q}} \|f\|_{L^q(D)} \,, & t \in (0, e^{\frac{1}{6\alpha}}] \,, \\ C\alpha^2 (\log t)^{\frac{11}{2}} t^{-\frac{1}{q}} \|f\|_{L^q(D)} \,, & t \in (e^{\frac{1}{6\alpha}}, \infty) \,, \end{cases} \tag{4.191}$$

for $f \in L^2_\sigma(D) \cap L^q(D)^2$. Here the constant C is independent of α and depends on q.

Remark 4.6.2 (1) Compared with the L^p-L^q estimate in Theorem 4.1.3, the new estimate in (4.190) or (4.191) is uniformly bounded in sufficiently small $\alpha \in (0, 1)$ for each fixed $t \in (0, \infty)$, while the bound in the right-hand side decays slower or even grows in time.
(2) The logarithmic factors $\log t$ in (4.190) and (4.191) are due to the factor $|\log|\lambda||$ in the resolvent estimates in Theorem 4.6.9 (2) of Subsection 4.6.1. The appearance of the logarithm $|\log|\lambda||$ is a consequence of the resolution of the singularity $\frac{1}{\alpha}$ in the resolvent estimates; see the proof of Theorem 4.6.7 in Subsection 4.6.1 for details.

We can prove the nonlinear stability of αU by applying Theorem 4.6.1. The integral form of Navier-Stokes equations is given by

$$v(t) \, = \, e^{-t\mathbb{A}_\alpha} v_0 - \int_0^t e^{-(t-s)\mathbb{A}_\alpha} \mathbb{P}(v \cdot \nabla v)(s) \, \mathrm{d}s \,, \qquad t > 0 \,. \tag{INS_α}$$

The following theorem is proved by the same argument as Theorem 4.1.4, and hence we omit the details. Its novelty is a relaxed smallness condition for fast decaying initial data.

Theorem 4.6.3 *Let $q \in (1, 2)$ and let α_* be the constant in Theorem 4.6.1. Then there is a positive constant ξ_* depending only on q such that if $\alpha \in (0, \alpha_*)$ and $\|v_0\|_{L^q(D)} \in (0, \xi_*)$ then there exists a unique solution $v \in C([0, \infty); L^2_\sigma(D)) \cap C((0, \infty); W^{1,2}_0(D)^2)$ to* (INS_α) *satisfying*

$$\lim_{t\to\infty} (\log t)^{-3-\frac{5k}{2}} t^{\frac{1}{q}-\frac{1}{2}+\frac{k}{2}} \|\nabla^k v\|_{L^2(D)} \, = \, 0 \,, \qquad k \, = \, 0, 1 \,. \tag{4.192}$$

The proof of Theorem 4.6.1 is carried out in a similar manner as in the main sections. We consider the resolvent problem associated with (PS$_\alpha$), which is now written as

$$\begin{cases} \lambda v - \Delta v + \alpha U^{\perp}\mathrm{rot}\, v + \nabla q \, = \, f \,, \quad x \in D \,, \\ \qquad\qquad\qquad\qquad\quad \mathrm{div}\, v \, = \, 0 \,, \quad x \in D \,, \\ \qquad\qquad\qquad\qquad\quad v|_{\partial D} \, = \, 0 \,. \end{cases} \tag{RS$_\alpha$}$$

Although the problem (RS$_\alpha$) is already introduced as (RS$^{\mathrm{ed}}_f$) and analyzed in Subsection 4.3.1, we revisit this problem to derive the resolvent estimates which lead to Theorem 4.6.1.

This section is organized as follows. In Subsection 4.6.1 we prove the resolvent estimates for (RS$_\alpha$) uniformly bounded in $\alpha \in (0,1)$. The most important part is Theorem 4.6.7 where the correspondence between the singularities $\frac{1}{\alpha}$ and $|\log|\lambda||$ is observed. In Subsection 4.6.2 we prove Theorem 4.6.1 by using the Dunford integral formula.

4.6.1 Resolvent analysis to (RS$_\alpha$)

In this subsection we prove the resolvent estimates to (RS$_\alpha$). The procedure is simpler than that adopted in Sections 4.2 and 4.3 thanks to the symmetry of the domain D. We firstly derive an a priori estimate to (RS$_\alpha$) which can be obtained by the energy method.

Proposition 4.6.4 *Let $q \in (1,2]$, $f \in L^q(D)^2$, and $\lambda \in \mathbb{C}$. Suppose that $v \in D(\mathbb{A}_\alpha)$ is a solution to (RS$_\alpha$). Then the following statements hold.*
(1) *Let $|n| = 1$. Then we have*

$$\begin{aligned} &\mathrm{Re}(\lambda)\|\mathcal{P}_n v\|^2_{L^2(D)} + \frac{3}{4}\|\nabla \mathcal{P}_n v\|^2_{L^2(D)} \\ &\qquad \le \alpha|\langle U^{\perp}\mathrm{rot}\,\mathcal{P}_n v, \mathcal{P}_n v\rangle_{L^2(D)}| + C\|f\|^{\frac{2q}{3q-2}}_{L^q(D)}\|\mathcal{P}_n v\|^{\frac{4(q-1)}{3q-2}}_{L^2(D)} \,, \end{aligned} \tag{4.193}$$

$$\begin{aligned} |\mathrm{Im}(\lambda)|\|\mathcal{P}_n v\|^2_{L^2(D)} &\le \frac{1}{4}\|\nabla \mathcal{P}_n v\|^2_{L^2(D)} + \alpha|\langle U^{\perp}\mathrm{rot}\,\mathcal{P}_n v, \mathcal{P}_n v\rangle_{L^2(D)}| \\ &\qquad + C\|f\|^{\frac{2q}{3q-2}}_{L^q(D)}\|\mathcal{P}_n v\|^{\frac{4(q-1)}{3q-2}}_{L^2(D)} \,, \end{aligned} \tag{4.194}$$

where the constant C is independent of α.
(2) *There is a constant $\alpha_1 \in (0,1)$ such that we have as long as $\alpha \in (0,\alpha_1)$,*

$$\mathrm{Re}(\lambda)\|\mathcal{Q}_0 v\|^2_{L^2(D)} + \frac{3}{4}\|\nabla \mathcal{Q}_0 v\|^2_{L^2(D)} \le C\|f\|^{\frac{2q}{3q-2}}_{L^q(D)}\|\mathcal{Q}_0 v\|^{\frac{4(q-1)}{3q-2}}_{L^2(D)} \,, \tag{4.195}$$

$$|\mathrm{Im}(\lambda)|\|\mathcal{Q}_0 v\|^2_{L^2(D)} \le \frac{1}{4}\|\nabla \mathcal{Q}_0 v\|^2_{L^2(D)} + C\|f\|^{\frac{2q}{3q-2}}_{L^q(D)}\|\mathcal{Q}_0 v\|^{\frac{4(q-1)}{3q-2}}_{L^2(D)} \,, \tag{4.196}$$

where the constant C is independent of α.

Remark 4.6.5 We immediately obtain from (4.195) and (4.196),

$$\big(|\mathrm{Im}(\lambda)| + \mathrm{Re}(\lambda)\big)\|\mathcal{Q}_0 v\|^2_{L^2(D)} + \frac{1}{2}\|\nabla \mathcal{Q}_0 v\|^2_{L^2(D)} \le 2C\|f\|^{\frac{2q}{3q-2}}_{L^q(D)}\|\mathcal{Q}_0 v\|^{\frac{4(q-1)}{3q-2}}_{L^2(D)} \,, \tag{4.197}$$

which, combined with (4.196), particularly implies that we have as long as $\alpha \in (0,\alpha_1)$,

$$\begin{aligned} \|\mathcal{Q}_0(\lambda + \mathbb{A}_\alpha)^{-1} f\|_{L^2(D)} &\le C|\lambda|^{-\frac{3}{2}+\frac{1}{q}}\|f\|_{L^q(D)} \,, \qquad \lambda \in \rho(-\mathbb{A}_\alpha) \cap \Sigma_{\frac{3}{4}\pi} \,, \\ \|\nabla \mathcal{Q}_0(\lambda + \mathbb{A}_\alpha)^{-1} f\|_{L^2(D)} &\le C|\lambda|^{-1+\frac{1}{q}}\|f\|_{L^q(D)} \,, \qquad \lambda \in \rho(-\mathbb{A}_\alpha) \cap \Sigma_{\frac{3}{4}\pi} \,, \end{aligned} \tag{4.198}$$

for $f \in L^2_\sigma(D) \cap L^q(D)^2$. Here the constant C is independent of α.

Proof: Note that the Fourier series expansion can be used on D thanks to the symmetry. Then the assertions of (1) and (2) follow from taking inner product respectively with $\mathcal{P}_n v$, $|n| = 1$, and $\mathcal{Q}_0 v$ to the first equation of (RS$_\alpha$), and making similar computations as ones performed in the proof of Proposition 4.2.1 in Section 4.2. This completes the proof. □

Resolvent estimates away from the origin

We shall prove the resolvent estimates to (RS$_\alpha$) for the case when the resolvent parameter $\lambda \in \mathbb{C}$ is outside of an exponentially small disk centered at the origin.

Proposition 4.6.6 *Let α_1 be the constant in Proposition 4.6.4. Then the following statements hold.*
(1) *Fix a positive number $\alpha_2 \in (0, \min\{\frac{1}{12}, \alpha_1\})$. Then the set*

$$\mathcal{S}_\alpha = \big\{\lambda \in \mathbb{C} \mid |\mathrm{Im}(\lambda)| > -\mathrm{Re}(\lambda) + 12 e^{\frac{1}{e}} \alpha^2 e^{-\frac{1}{6\alpha}}\big\} \tag{4.199}$$

is included in the resolvent $\rho(-\mathbb{A}_\alpha)$ for any $\alpha \in (0, \alpha_2)$.
(2) *Let $q \in (1, 2]$ and $f \in L^2_\sigma(D) \cap L^q(D)^2$. Then we have*

$$\begin{aligned} \|(\lambda + \mathbb{A}_\alpha)^{-1} f\|_{L^2(D)} &\le C|\lambda|^{-\frac{3}{2}+\frac{1}{q}} \|f\|_{L^q(D)}, \qquad \lambda \in \mathcal{S}_\alpha \cap \mathcal{B}_{\frac{1}{2}e^{-\frac{1}{6\alpha}}}(0)^{\mathrm{c}}, \\ \|\nabla(\lambda + \mathbb{A}_\alpha)^{-1} f\|_{L^2(D)} &\le C|\lambda|^{-1+\frac{1}{q}} \|f\|_{L^q(D)}, \qquad \lambda \in \mathcal{S}_\alpha \cap \mathcal{B}_{\frac{1}{2}e^{-\frac{1}{6\alpha}}}(0)^{\mathrm{c}}, \end{aligned} \tag{4.200}$$

as long as $\alpha \in (0, \alpha_2)$. The constant C is independent of α.

Proof: (1) Let $|n| = 1$. Then by applying the same argument as in the proof of Proposition 4.2.2 in Section 4.2, we see from (4.193) and (4.194) in Proposition 4.6.4 that

$$\begin{aligned} &\big(|\mathrm{Im}(\lambda)| + \mathrm{Re}(\lambda) - 12 e^{\frac{1}{e}} \alpha^2 e^{-\frac{1}{6\alpha}}\big) \|\mathcal{P}_n v\|^2_{L^2(D)} + \frac{1}{4} \|\nabla \mathcal{P}_n v\|^2_{L^2(D)} \\ &\le C \|f\|^{\frac{2q}{3q-2}}_{L^q(D)} \|\mathcal{P}_n v\|^{\frac{4(q-1)}{3q-2}}_{L^2(D)} \end{aligned} \tag{4.201}$$

holds as long as $\alpha \in (0, \frac{1}{12})$. The assertion follows by (4.197) in Remark 4.6.5 and (4.201).
(2) The argument in the proof of Proposition 4.2.2 implies that we have

$$\begin{aligned} \|\mathcal{P}_n(\lambda + \mathbb{A}_\alpha)^{-1} f\|_{L^2(D)} &\le C|\lambda|^{-\frac{3}{2}+\frac{1}{q}} \|f\|_{L^q(D)}, \qquad \lambda \in \mathcal{S}_\alpha \cap \mathcal{B}_{\frac{1}{2}e^{-\frac{1}{6\alpha}}}(0)^{\mathrm{c}}, \\ \|\nabla \mathcal{P}_n(\lambda + \mathbb{A}_\alpha)^{-1} f\|_{L^2(D)} &\le C|\lambda|^{-1+\frac{1}{q}} \|f\|_{L^q(D)}, \qquad \lambda \in \mathcal{S}_\alpha \cap \mathcal{B}_{\frac{1}{2}e^{-\frac{1}{6\alpha}}}(0)^{\mathrm{c}}, \end{aligned} \tag{4.202}$$

from (4.193) and (4.194) in Proposition 4.6.4. Then the estimates in (4.200) result from (4.198) in Remark 4.6.5 and (4.202) since $\mathcal{S}_\alpha \subset \Sigma_{\frac{3}{4}\pi}$ holds. The proof is complete. □

Resolvent estimates near the origin

Next we consider the estimates to (RS_α) for the case when the resolvent parameter $\lambda \in \mathbb{C}$ is in a neighbourhood of the origin. Concerning the part $\mathcal{Q}_0 v$ of v, we have proved a priori estimates in Remark 4.6.5 leading to the resolvent estimates of $\mathcal{Q}_0 v$ for all $\lambda \in \Sigma_{\frac{3}{4}\pi}$. Hence our task is to derive the estimates for the part $\mathcal{P}_n v$ of v with $|n| = 1$. We note that, contrary to the procedure in the main sections, there is no need to employ the energy method because the Fourier series expansion is available on the fluid domain D thanks to the symmetry.

Theorem 4.6.7 *Let $|n| = 1$ and $\epsilon \in (0, \frac{\pi}{2})$. Then there is a positive constant $\alpha_3 = \alpha_3(\epsilon)$ such that if $\alpha \in (0, \alpha_3)$ then the following statement holds. Let $1 \le q < p \le \infty$ or $1 < q \le p < \infty$, and let $f \in C_0^\infty(D)$. Then we have for $\lambda \in \Sigma_{\pi-\epsilon} \cap \mathcal{B}_{e^{-\frac{1}{6\alpha}}}(0)$,*

$$\|\mathcal{P}_n v\|_{L^p(D)} \le C\alpha |\log|\lambda||^3 |\lambda|^{-1+\frac{1}{q}-\frac{1}{p}} \|f\|_{L^q(D)}\,, \tag{4.203}$$

$$\|\mathrm{rot}\,\mathcal{P}_n v\|_{L^2(D)} \le C\alpha |\log|\lambda||^{\frac{5}{2}} |\lambda|^{-\frac{1}{2}} \|f\|_{L^2(D)}\,, \tag{4.204}$$

where the constant $C = C(q, p, \epsilon)$ is independent of α. Moreover, (4.203) *holds all for $f \in L^q(D)^2$, and* (4.204) *holds all for $f \in L^2(D)^2$.*

Proof: (i) Estimate (4.203): In Subsection 4.3.1, where $\mathcal{P}_n v$ is denoted by $w^{\mathrm{ed}}_{f,n}$ and α is replaced with β, we have already obtained the L^p-L^q resolvent estimates for $\mathcal{P}_n v$ without the logarithmic factor $|\log|\lambda||^3$ but with $\frac{1}{\beta^2}$ singularity in the right-hand side of (4.203); see the estimate (4.66) in Theorem 4.3.2. The proof is based on the representation formula (4.61) for $w^{\mathrm{ed}}_{f,n}$ in Subsection 4.3.1. By following the computations in Subsection 4.3.1 again, we can check that the factor $\frac{1}{\beta^2}$ in (4.66) arises from the estimates in Appendix 4.5.2, more precisely, (4.151) in Lemma 4.5.2, and (4.157), (4.158), and (4.160) in Lemma 4.5.3. On the other hand, the β-singularity in these bounds can be replaced with the logarithmic factor $|\log|\lambda||$. Indeed, by a direct calculation we have for $\lambda \in \Sigma_{\pi-\epsilon} \cap \mathcal{B}_{\frac{1}{2}}(0)$,

$$|K_{\mu_n-1}(\sqrt{\lambda} r)| \le C|\log|\lambda||\,|\lambda|^{-\frac{\mathrm{Re}(\mu_n)}{2}+\frac{1}{2}} r^{-\mathrm{Re}(\mu_n)+1}\,, \quad 1 \le r \le \mathrm{Re}(\sqrt{\lambda})^{-1}\,,$$

$$\left| \int_\tau^r K_{\mu_n}(\sqrt{\lambda} s)\,\mathrm{d}s \right| \le C|\log|\lambda||\,|\lambda|^{-\frac{\mathrm{Re}(\mu_n)}{2}} \tau^{-\mathrm{Re}(\mu_n)+1}\,, \quad 1 \le \tau \le r \le \mathrm{Re}(\sqrt{\lambda})^{-1}\,,$$

and the constant C can be taken to be uniform in small β. Then we see that the factor $\frac{1}{\beta}$ in Lemmas 4.5.2 and 4.5.3 can be replaced with $|\log|\lambda||$. This concludes the replacement of $\frac{1}{\beta^2} = \beta\frac{1}{\beta^3}$ by $\beta|\log|\lambda||^3$ in the estimate (4.66), which implies that (4.203) holds.
(ii) Estimate (4.204): The proof is done by reproducing similar computations in [44, Subsection 3.3.2] based on the representation formula (4.64) in Subsection 4.3.1, and using the $\frac{1}{\beta}$-removed estimates in the above proof of (4.203). The details are omitted. We note that the factor $|\log|\lambda||^{\frac{1}{2}}$ in (4.204) originates from the following estimate with $p = 2$:

$$\|K_{\mu_n}(\sqrt{\lambda}\,\cdot\,)\|_{L^p((1,\infty);r\,\mathrm{d}r)} \le C_p |\log|\lambda||^{\frac{1}{p}} |\lambda|^{-\frac{\mathrm{Re}\,(\mu_n)}{2}}\,, \quad p \in [2,\infty)\,,$$

which corresponds to the estimate (4.167) in Lemma 4.5.5 with $\frac{1}{\beta}$ replaced with $|\log|\lambda||$ in Appendix 4.5.2. This completes the proof of Theorem 4.6.7. □

The following proposition follows from the estimates in Remark 4.6.5 and Theorem 4.6.7.

Proposition 4.6.8 *Let α_1 and α_3 be the constants respectively in Proposition 4.6.4 and Theorem 4.6.7. Then the following statements hold.*
(1) *Fix a positive number $\alpha_4 \in (0, \min\{\alpha_1, \alpha_3\})$. Then the set*

$$\mathcal{T}_\alpha = \Sigma_{\frac{3}{4}\pi} \cap \mathcal{B}_{e^{-\frac{1}{6\alpha}}}(0) \tag{4.205}$$

is included in the resolvent $\rho(-\mathbb{A}_\alpha)$ for any $\alpha \in (0, \alpha_4)$.
(2) *Let $q \in (1, 2]$ and $f \in L^2_\sigma(D) \cap L^q(D)^2$. Then we have*

$$\begin{aligned} \|(\lambda + \mathbb{A}_\alpha)^{-1} f\|_{L^2(D)} &\le C\alpha|\log|\lambda||^3 |\lambda|^{-\frac{3}{2}+\frac{1}{q}} \|f\|_{L^q(D)}\,, \qquad \lambda \in \mathcal{T}_\alpha\,, \\ \|\nabla(\lambda + \mathbb{A}_\alpha)^{-1} f\|_{L^2(D)} &\le C\alpha|\log|\lambda||^{\frac{5}{2}} |\lambda|^{-\frac{1}{2}} \|f\|_{L^2(D)}\,, \qquad \lambda \in \mathcal{T}_\alpha\,, \end{aligned} \tag{4.206}$$

as long as $\alpha \in (0, \alpha_4)$. The constant C is independent of α.

Proof: (1) Let $\epsilon = \frac{\pi}{4}$ in Theorem 4.6.7. Then the assertion is an immediate consequence from the estimates (4.197) in Remark 4.6.5 and (4.203) in Theorem 4.6.7.
(2) Let $\epsilon = \frac{\pi}{4}$ in Theorem 4.6.7 again. Then the first line in (4.206) results from (4.198) in Remark 4.6.5 and (4.203) in Theorem 4.6.7 with $p = 2$, while the second line in (4.206) follows from (4.198) and (4.204) in Theorem 4.6.7 combined with the following inequality

$$\|\nabla h\|_{L^2(D)} \le C\|\mathrm{rot}\, h\|_{L^2(D)}\,,$$

which is valid for $h \in W^{1,2}_0(D) \cap L^2_\sigma(D)$. This completes the proof. □

We obtain the following theorem by combining the results in Propositions 4.6.6 and 4.6.8.

Theorem 4.6.9 *Let α_2 and α_4 be the constants respectively in Propositions 4.6.6 and 4.6.8. Then the following statements hold.*
(1) *Fix a positive number $\alpha_* \in (0, \min\{\alpha_2, \alpha_4\})$. Then there is a constant $\epsilon_0 \in (\frac{\pi}{4}, \frac{\pi}{2})$ such that the sector $\Sigma = \Sigma_{\pi-\epsilon_0}$ is included in the resolvent $\rho(-\mathbb{A}_\alpha)$ for any $\alpha \in (0, \alpha_*)$.*
(2) *Let $q \in (1, 2]$ and $f \in L^2_\sigma(D) \cap L^q(D)^2$. Then we have*

$$\begin{aligned} &\|(\lambda + \mathbb{A}_\alpha)^{-1} f\|_{L^2(D)} \\ &\le \begin{cases} C|\lambda|^{-\frac{3}{2}+\frac{1}{q}} \|f\|_{L^q(D)}\,, & \lambda \in \Sigma \cap \mathcal{B}_{e^{-\frac{1}{6\alpha}}}(0)^{\mathrm{c}}\,, \\ C\alpha|\log|\lambda||^3 |\lambda|^{-\frac{3}{2}+\frac{1}{q}} \|f\|_{L^q(D)}\,, & \lambda \in \Sigma \cap \mathcal{B}_{e^{-\frac{1}{6\alpha}}}(0)\,, \end{cases} \end{aligned} \tag{4.207}$$

$$\begin{aligned} &\|\nabla(\lambda + \mathbb{A}_\alpha)^{-1} f\|_{L^2(D)} \\ &\le \begin{cases} C|\lambda|^{-\frac{1}{2}} \|f\|_{L^2(D)}\,, & \lambda \in \Sigma \cap \mathcal{B}_{e^{-\frac{1}{6\alpha}}}(0)^{\mathrm{c}}\,, \\ C\alpha|\log|\lambda||^{\frac{5}{2}} |\lambda|^{-\frac{1}{2}} \|f\|_{L^2(D)}\,, & \lambda \in \Sigma \cap \mathcal{B}_{e^{-\frac{1}{6\alpha}}}(0)\,, \end{cases} \end{aligned} \tag{4.208}$$

as long as $\alpha \in (0, \alpha_*)$. The constant C is independent of α.

Proof: (1) Firstly we note that $\mathcal{S}_{\alpha_*} \cap \mathcal{T}_{\alpha_*} = \mathcal{S}_{\alpha_*} \cap \mathcal{B}_{e^{-\frac{1}{6\alpha_*}}}(0) \neq \emptyset$ holds since $12e^{\frac{1}{e}}\alpha_*^2 < 1$ from the condition $\alpha_* \in (0, \frac{1}{12})$. Hence there is a constant $\epsilon_0 \in (\frac{\pi}{4}, \frac{\pi}{2})$ such that the sector $\Sigma = \Sigma_{\pi-\epsilon_0}$ is included in the set $\mathcal{S}_\alpha \cup \mathcal{B}_{e^{-\frac{1}{6\alpha}}}(0)$ for any $\alpha \in (0, \alpha_*)$. Then by Propositions 4.6.6 and 4.6.8, the sector Σ is included in the resolvent $\rho(-\mathbb{A}_\alpha)$ as long as $\alpha \in (0, \alpha_*)$.
(2) The estimates in (4.207) and (4.208) follow from the ones in Propositions 4.6.6 and 4.6.8. The proof is complete. □

4.6.2 Proof of Theorem 4.6.1

In this subsection we prove Theorem 4.6.1. The proof is a consequence of Theorem 4.6.9.

Proof of Theorem 4.6.1: We denote the function space $L^q(D)$ by L^q in this proof for simplicity. Let $t \in (0,\infty)$, and let $\alpha \in (0,\alpha_*)$ and $\epsilon_0 \in (\frac{\pi}{4},\frac{\pi}{2})$ be the constants in Theorem 4.6.9. We fix a number $\phi \in (\frac{\pi}{2},\pi-\epsilon_0)$ and take a curve $\gamma(t) = \{z \in \mathbb{C} \mid |\arg z| = \phi\,,\ |z| \geq \frac{1}{t}\} \cup \{z \in \mathbb{C} \mid |\arg z| \leq \phi\,,\ |z| = \frac{1}{t}\}$ oriented counterclockwise. We note that $\gamma(t) \in \rho(-\mathbb{A}_\alpha)$ for any $t \in (0,\infty)$ since $\Sigma_{\pi-\epsilon_0} \subset \rho(-\mathbb{A}_\alpha)$. Then, for $f \in L^2_\sigma(D) \cap L^q(D)^2$, the semigroup $e^{-t\mathbb{A}_\alpha}$ admits a Dunford integral representation as

$$e^{-t\mathbb{A}_\alpha} f \,=\, \frac{1}{2\pi i}\int_{\gamma(t)} e^{t\lambda}(\lambda+\mathbb{A}_\alpha)^{-1} f \,\mathrm{d}\lambda\,, \quad t>0\,.$$

We give the estimate of $\|\nabla^k e^{-t\mathbb{A}_\alpha} f\|_{L^2}$ for $k \in \{0,1\}$. By a direct calculation we have

$$\begin{aligned}\|\nabla^k e^{-t\mathbb{A}_\alpha} f\|_{L^2} \leq C\int_{\frac{1}{t}}^{\infty} e^{(\cos\phi)ts}\|\nabla^k(se^{i\phi}+\mathbb{A}_\alpha)^{-1} f\|_{L^2}\,\mathrm{d}s \\ +\frac{C}{t}\int_{-\phi}^{\phi}\|\nabla^k(\frac{1}{t}e^{i\theta}+\mathbb{A}_\alpha)^{-1} f\|_{L^2}\,\mathrm{d}\theta\,.\end{aligned}$$

(i) Case $t \in (0, e^{\frac{1}{6\alpha}}]$: Since $\frac{1}{t} \in [e^{-\frac{1}{6\alpha}},\infty)$ holds, we have from Theorem 4.6.9 (2),

$$\begin{aligned}\|e^{-t\mathbb{A}_\alpha} f\|_{L^2} &\leq C\|f\|_{L^q}\bigg(\int_{\frac{1}{t}}^{\infty} e^{(\cos\phi)ts}s^{-\frac{3}{2}+\frac{1}{q}}\,\mathrm{d}s + \int_{-\phi}^{\phi} t^{\frac{1}{2}-\frac{1}{q}}\,\mathrm{d}\theta\bigg) \\ &\leq Ct^{-\frac{1}{q}+\frac{1}{2}}\|f\|_{L^q}\,, \qquad (4.209)\\ \|\nabla e^{-t\mathbb{A}_\alpha} f\|_{L^2} &\leq C\|f\|_{L^2}\bigg(\int_{\frac{1}{t}}^{\infty} e^{(\cos\phi)ts}s^{-\frac{1}{2}}\,\mathrm{d}s + \int_{-\phi}^{\phi} t^{-\frac{1}{2}}\,\mathrm{d}\theta\bigg) \\ &\leq Ct^{-\frac{1}{2}}\|f\|_{L^2}\,. \qquad (4.210)\end{aligned}$$

Thus we have (4.190) for $t \in (0, e^{\frac{1}{6\alpha}}]$ from (4.209). The estimate (4.191) for $t \in (0, e^{\frac{1}{6\alpha}}]$ follows from (4.209) and (4.210) using the semigroup property of $e^{-t\mathbb{A}_\alpha}$.
(ii) Case $t \in (e^{\frac{1}{6\alpha}},\infty)$: We have $\frac{1}{t} \in (0, e^{-\frac{1}{6\alpha}})$ in this case. Then Theorem 4.6.9 (2) yields

$$\begin{aligned}\|e^{-t\mathbb{A}_\alpha} f\|_{L^2} &\leq C\|f\|_{L^q}\bigg(\alpha\int_{\frac{1}{t}}^{e^{-\frac{1}{6\alpha}}} e^{(\cos\phi)ts}|\log|s||^3 s^{-\frac{3}{2}+\frac{1}{q}}\,\mathrm{d}s + \int_{e^{-\frac{1}{6\alpha}}}^{\infty} e^{(\cos\phi)ts}s^{-\frac{3}{2}+\frac{1}{q}}\,\mathrm{d}s\bigg) \\ &\quad + C\alpha\|f\|_{L^q}\int_{-\phi}^{\phi}|\log t|^3 t^{\frac{1}{2}-\frac{1}{q}}\,\mathrm{d}\theta \\ &\leq C\alpha(\log t)^3 t^{-\frac{1}{q}+\frac{1}{2}}\|f\|_{L^q}\,, \qquad (4.211)\end{aligned}$$

and in the same manner we also have

$$\|\nabla e^{-t\mathbb{A}_\alpha} f\|_{L^2} \leq C\alpha(\log t)^{\frac{5}{2}} t^{-\frac{1}{2}}\|f\|_{L^2}\,. \qquad (4.212)$$

Hence we obtain (4.190) for the case $t \in (e^{\frac{1}{6\alpha}},\infty)$ from (4.211), and (4.191) for $t \in (e^{\frac{1}{6\alpha}},\infty)$ from (4.211), (4.212), and the semigroup property. This completes the proof. □

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