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Capelli identities with zero entries

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Abstract

In the Capelli identities and several variants of them, the entries of matrices in the identities are usually nonzero except a few cases of alternating matrices. In this paper we introduce Capelli identities in which there are zero entries, and, as an application, we compute b-functions of prehomogeneous vector spaces.

1 Introduction

Let t_{ij} be (independent) variables, and set

$$T = (t_{ij})_{1\leq i,j\leq n}, \qquad \frac{\partial}{\partial T} = \left(\frac{\partial}{\partial t_{ij}}\right)_{1\leq i,j\leq n}.$$

Then the original Capelli identity is the following equation in the ring of the differential operators with polynomial coefficients [1]:

$$\det({}^tT)\det\left(\frac{\partial}{\partial T}\right) = \det\left({}^tT\frac{\partial}{\partial T} + \begin{pmatrix} n-1 & & \\ & n-2 & \\ & & \ddots & \\ & & & 0\end{pmatrix}\right), \tag{1}$$

where the determinant is defined as $\det(X) = \sum_\sigma \operatorname{sgn}(\sigma)X_{\sigma(1)1}X_{\sigma(2)2}\cdots X_{\sigma(n)n}$, which is called the column determinant.

Define a polynomial f and a differential operator $f^*(\partial)$ with constant coefficients by

$$f = \det({}^tT), \qquad f^*(\partial) = \det\left(\frac{\partial}{\partial T}\right).$$

Then the differentiation by $f^*(\partial)$ on f^{s+1} gives a scalar multiple of f^s:

$$f^*(\partial).f^{s+1} = b_f(s)f^s,$$

and $b_f(s) \in \mathbb{C}[s]$ is called the b-function of f. In this case it is known that $b_f(s) = (s+1)(s+2)\cdots(s+n)$, and the Capelli identity enables us to compute this b-function.

Next we recall a variant of the Capelli identity, where t_{ij} are variables satisfying $t_{ij} = t_{ji}$. There is an analogous identity in this setting. Set

$$T = (t_{ij})_{1\le i,j\le n}, \qquad \frac{\overline{\partial}}{\partial T} = \left(\frac{\overline{\partial}}{\partial t_{ij}}\right)_{1\le i,j\le n},$$

where

$$\frac{\overline{\partial}}{\partial t_{ij}} = \begin{cases} \dfrac{\partial}{\partial t_{ii}} & (i=j) \\ \dfrac{1}{2}\dfrac{\partial}{\partial t_{ij}} & (i\neq j) \end{cases}$$

Then the Capelli identity in this case is as follows [3]:

$$\det({}^tT)\det\left(\frac{\overline{\partial}}{\partial T}\right) = \det\left({}^tT\frac{\overline{\partial}}{\partial T} + \begin{pmatrix} (n-1)/2 & & \\ & (n-2)/2 & \\ & & \ddots & \\ & & & 0 \end{pmatrix}\right). \tag{2}$$

Define a polynomial f and a differential operator $f^*(\partial)$ with constant coefficients by

$$f = \det({}^tT), \qquad f^*(\partial) = \det\left(\frac{\overline{\partial}}{\partial T}\right). \tag{3}$$

Then the b-function is given by

$$f^*(\partial).f^{s+1} = b_f(s)f^s, \qquad b(s) = (s+1)(s+\frac{3}{2})(s+2)\cdots(s+\frac{n+1}{2}). \tag{4}$$

The Capelli identity again enables us to compute the b-function also in this case.

In the above two cases the matrix T has nonzero entries only. In this paper we consider the cases where T has zero entries, and prove the Capelli identities (Theorem 1). We hope the b-functions of $\det(T)$ are computed by using our Capelli identities, but we can not use the Capelli identities to compute all the b-functions at present. We give the b-functions computed by using our Capelli identity or in different ways (Propositions 5, 6, 7).

2 Capelli identities with zero entries

When some entries of T are zero, the Capelli identities (1) and (2) can hold.

Theorem 1. (1) Let the entries of T be (independent) variables or zero, and suppose that T satisfies the following conditions:

(A) In each row of T zero entries are at the end of the row.

(B) The number of the zero entries of a row is greater than or equal to that of the previous row.

In other words nonzero entries are placed just as a Young diagram. Then the identity (1) in Introduction holds.

(2) Let the entries of T be symmetric variables ($t_{ij} = t_{ji}$) or zero, that is, T is a symmetric matrix containing zero entries. Suppose also that T is of the following form:

$$T = \begin{pmatrix} T_1 & T_2 \\ {}^tT_2 & 0 \end{pmatrix}, \quad (T_1 \text{ is } p \times p,\ T_2 \text{ is } p \times q, \text{ and } p + q = n),$$

where T_1 and T_2 have no zero entries. Then the identity (2) in Introduction holds.

2.1 Proof of Theorem 1 (1)

We denote $\partial/\partial t_{ij}$ by ∂_{ij} for short.

Let λ_i be the number of nonzero entries of the ith row of T, and therefore $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Note that the partition $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ corresponds to the Young diagram mentioned in the theorem. We interpret t_{ij} and ∂_{ij} are zero when $j > \lambda_i$. We define the 'characteristic function' corresponding to the nonzero entries of T:

$$\epsilon_{(i,j)} = \begin{cases} 1 & (j \leq \lambda_i) \\ 0 & (j > \lambda_i) \end{cases}$$

We use the exterior calculus for the proof. Let $e_1, e_2, \ldots, e_n$ be the standard basis of $\mathbb{C}^n$, and consider the algebra $A := \bigwedge \mathbb{C}^n \otimes_{\mathbb{C}} W$, which is the tensor product of the exterior algebra $\bigwedge \mathbb{C}^n$ and the Weyl algebra W generated by t_{ij} and ∂_{ij}. In denoting elements of A we write such as $e_1 e_2 t_{12} \partial_{23}$ instead of $e_1 \wedge e_2 \otimes t_{12}\partial_{23}$ for short.

Define some elements of A. Set

$$\eta_k = \sum_{i=1}^{n} e_i t_{ki} \quad (1 \leq k \leq n), \qquad \zeta_j = \sum_{i=1}^{n} e_i \left({}^tT \frac{\partial}{\partial T} \right)_{ij} \quad (1 \leq j \leq n),$$

where $({}^tT \cdot \partial/\partial T)_{ij}$ means the (i,j)-entry of the matrix. We can write ζ_j in other forms as

$$\zeta_j = \sum_{i,k=1}^{n} e_i t_{ki} \partial_{kj} = \sum_{k=1}^{n} \eta_k \partial_{kj}.$$

For a complex number u define $\zeta_j(u) = \zeta_j + u e_j$ $(1 \leq j \leq n)$, and we can write $\zeta_j(u)$ in another form as

$$\zeta_j(u) = \zeta_j + u e_j = \sum_{i=1}^{n} e_i \left({}^tT \frac{\partial}{\partial T} + u 1_n \right)_{ij},$$

where 1_n denotes the identity matrix of size n.

Lemma 2. For $l, j, k \in \{1, 2, \ldots, n\}$ and $u \in \mathbb{C}$ we have the following, where δ_{lk} denotes the Kronecker delta.

(1) $\partial_{lj}\eta_k = \eta_k \partial_{lj} + \delta_{lk}\epsilon_{(l,j)} e_j$

(2) $\zeta_j(u)\eta_k = -\eta_k(\zeta_j(u) - \epsilon_{(k,j)} e_j)$

Proof. (1)

$$\begin{aligned}\partial_{lj}\eta_k &= \partial_{lj}\sum_{i=1}^{n} e_i t_{ki} \\ &= \sum_{i=1}^{n} e_i \epsilon_{(l,j)}\epsilon_{(k,i)}(t_{ki}\partial_{lj} + \delta_{lk}\delta_{ji}) \\ &= \eta_k\partial_{lj} + \delta_{lk} e_j \epsilon_{(l,j)}\epsilon_{(k,j)} \\ &= \eta_k\partial_{lj} + \delta_{lk}\epsilon_{(l,j)} e_j.\end{aligned}$$

(2) We have

$$\begin{aligned}\zeta_j\eta_k &= \sum_{l=1}^{n} \eta_l\partial_{lj}\eta_k \\ &\overset{(1)}{=} \sum_{l=1}^{n} \eta_l(\eta_k\partial_{lj} + \delta_{lk}\epsilon_{(l,j)} e_j) \\ &= -\eta_k\zeta_j + \eta_k\epsilon_{(k,j)} e_j \\ &= -\eta_k(\zeta_j - \epsilon_{(k,j)} e_j).\end{aligned}$$

Then the desired equation is obtained by adding $ue_j\eta_k = -\eta_k ue_j$ to both sides. □

We start the proof of Theorem 1 (1), that is, we prove

$$\det({}^tT)\det\left(\frac{\partial}{\partial T}\right) = \det\left({}^tT\frac{\partial}{\partial T} + \begin{pmatrix} n-1 & & \\ & n-2 & \\ & & \ddots \\ & & & 0\end{pmatrix}\right),$$

where the (i,j)-entry t_{ij} of T and the (i,j)-entry ∂_{ij} of $\partial/\partial T$ are zero if and only if $j > \lambda_i$.

It is clear that

$$\zeta_1(n-1)\zeta_2(n-2)\cdots\zeta_n(0) = e_1e_2\cdots e_n\det\left({}^tT\frac{\partial}{\partial T} + \begin{pmatrix} n-1 & & \\ & n-2 & \\ & & \ddots \\ & & & 0\end{pmatrix}\right)$$

from the definition of (column) determinant. Next we compute the left-hand side of the above equation in another way. By using Lemma 2 (2) we have

$$\begin{aligned}&\zeta_1(n-1)\zeta_2(n-2)\cdots\zeta_n(0) \\ &= \zeta_1(n-1)\zeta_2(n-2)\cdots\zeta_{n-1}(1)\cdot\sum_{l_n=1}^{n}\eta_{l_n}\partial_{l_n,n} \\ &= (-1)^{n-1}\sum_{l_n=1}^{n}\eta_{l_n}\cdot(\zeta_1(n-1) - \epsilon_{(l_n,1)}e_1)\cdots(\zeta_{n-1}(1) - \epsilon_{(l_n,n-1)}e_{n-1})\cdot\partial_{l_n,n}.\end{aligned} \tag{5}$$

Suppose that $\partial_{l_n,n} \neq 0$ in the above expression. Then $\epsilon_{(l_n,n)} = 1$, and therefore every $\epsilon_{(l_n,j)}$ $(j \leq n)$ is equal to one by the definition of $\epsilon_{(i,j)}$ (recall 'Young diagram'). Thus we may assume that every $\epsilon_{(l_n,j)}$ in the expression is equal to one, and we have

$$
\begin{aligned}
&(\text{RHS of (5)}) \\
&= (-1)^{n-1}\sum_{l_n=1}^{n} \eta_{l_n}\cdot\zeta_1(n-2)\cdots\zeta_{n-1}(0)\cdot\partial_{l_n,n} \\
&= (-1)^{n-1}\sum_{l_n=1}^{n} \eta_{l_n}\cdot\zeta_1(n-2)\cdots\zeta_{n-2}(1)\cdot\sum_{l_{n-1}=1}^{n}\eta_{l_{n-1}}\partial_{l_{n-1},n-1}\cdot\partial_{l_n,n} \qquad (6)
\end{aligned}
$$

We can move $\eta_{l_{n-1}}$ to the left in this expression with parameters of ζ_j $(1 \le j \le n-2)$ decreasing by one as η_{l_n} moved. Similarly we repeat this operation, and obtain

$$
\begin{aligned}
(\text{RHS of (6)}) &= (-1)^{(n-1)n}\sum_{l_1,\dots,l_n=1}^{n}\eta_{l_1}\eta_{l_2}\cdots\eta_{l_n}\cdot\partial_{l_1,1}\partial_{l_2,2}\cdots\partial_{l_n,n} \\
&= \sum_{\sigma\in S_n}\eta_{\sigma(1)}\eta_{\sigma(2)}\cdots\eta_{\sigma(n)}\cdot\partial_{\sigma(1),1}\partial_{\sigma(2),2}\cdots\partial_{\sigma(n),n} \\
&= \sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\eta_1\eta_2\cdots\eta_n\cdot\partial_{\sigma(1),1}\partial_{\sigma(2),2}\cdots\partial_{\sigma(n),n} \\
&= e_1e_2\cdots e_n\det({}^tT)\det\left(\frac{\partial}{\partial T}\right).
\end{aligned}
$$

Thus we have proved the assertion.

2.2 Proof of Theorem 1 (2)

We denote $\partial/\partial t_{ij}$ by ∂_{ij}, and $\overline{\partial}/\partial t_{ij}$ by $\overline{\partial}_{ij}$ for short.

We define the 'characteristic function' corresponding to the nonzero entries of T:

$$
\epsilon_{(i,j)} = \begin{cases} 1 & (i \le p \text{ or } j \le p) \\ 0 & (i > p \text{ and } j > p) \end{cases}
$$

We interpret t_{ij} and ∂_{ij} (and $\overline{\partial}_{ij}$) to be zero when $\epsilon_{(i,j)} = 0$.

We use the exterior calculus again. We set $A = \bigwedge \mathbb{C}^n \otimes_{\mathbb{C}} W$ as in the proof of Theorem 1 (1). Note that $n = p + q$.

Define some elements of A. Set

$$
\eta_k = \sum_{i=1}^{n} e_i t_{ki} \quad (1 \le k \le n), \qquad \zeta_j = \sum_{i=1}^{n} e_i\left({}^tT\frac{\overline{\partial}}{\partial T}\right)_{ij} \quad (1 \le j \le n).
$$

We can write ζ_j in another form as

$$
\zeta_j = \sum_{k=1}^{n}\eta_k\overline{\partial}_{kj}.
$$

For a complex number u define $\zeta_j(u) = \zeta_j + ue_j$ $(1 \leq j \leq n)$, and we can write $\zeta_j(u)$ in another form as

$$\zeta_j(u) = \zeta_j + ue_j = \sum_{i=1}^{n} e_i \left({}^tT\frac{\overline{\partial}}{\partial T} + u1_n \right)_{ij}.$$

Lemma 3. For $k, j, l \in \{1, 2, \ldots, n\}$ and $u \in \mathbb{C}$ we have the following.
(1) $\overline{\partial}_{kj}\eta_l = \eta_l\overline{\partial}_{kj} + \epsilon_{(k,j)}(\delta_{kl}e_j + \delta_{jl}e_k)$
(2) $\zeta_j(u)\eta_l = -\eta_l(\zeta_j(u) - \epsilon_{(l,j)}e_j) + \delta_{lj}\sum_{k=1}^{n}\epsilon_{(k,j)}\eta_k e_k$

Proof. (1)

$$\begin{aligned}
\overline{\partial}_{kj}\eta_l &= \overline{\partial}_{kj}\sum_{i=1}^{n} e_i t_{li} \\
&= \sum_{i=1}^{n} \epsilon_{(k,j)}\epsilon_{(l,i)} e_i (t_{li}\overline{\partial}_{kj} + \delta_{kl}\delta_{ji} + \delta_{ki}\delta_{jl}) \\
&= \eta_k\overline{\partial}_{kj} + \epsilon_{(k,j)}\epsilon_{(l,j)}e_j\delta_{kl} + \epsilon_{(k,j)}\epsilon_{(l,k)}e_k\delta_{jl} \\
&= \eta_l\overline{\partial}_{kj} + \epsilon_{(k,j)}(\delta_{kl}e_j + \delta_{jl}e_k).
\end{aligned}$$

(2) We have

$$\begin{aligned}
\zeta_j\eta_l &= \sum_{k=1}^{n} \eta_k\overline{\partial}_{kj}\eta_l \\
&\overset{(1)}{=} \sum_{k=1}^{n} \eta_k(\eta_l\overline{\partial}_{kj} + \epsilon_{(k,j)}(\delta_{kl}e_j + \delta_{jl}e_k)) \\
&= -\eta_l\zeta_j + \eta_l\epsilon_{(l,j)}e_j + \sum_{k=1}^{n}\eta_k\epsilon_{(k,j)}\delta_{jl}e_k.
\end{aligned}$$

Then the desired equation is obtained by adding $ue_j\eta_l = -\eta_l ue_j$ to both sides. □

The next lemma is easy to show, and we omit the proof.

Lemma 4. We have

$$\sum_{k=1}^{n} \eta_k e_k = 0.$$

We start the proof of Theorem 1 (2), that is, we prove

$$\det({}^tT)\det\left(\frac{\overline{\partial}}{\partial T}\right) = \det\left({}^tT\frac{\overline{\partial}}{\partial T} + \begin{pmatrix} (n-1)/2 & & \\ & (n-2)/2 & \\ & & \ddots \\ & & & 0 \end{pmatrix}\right).$$

It is clear that

$$\zeta_1(n-1)\zeta_2(n-2)\cdots\zeta_n(0) = e_1e_2\cdots e_n\det\left({}^tT\frac{\overline{\partial}}{\partial T} + \begin{pmatrix} (n-1)/2 & & \\ & (n-2)/2 & \\ & & \ddots \\ & & & 0 \end{pmatrix}\right).$$

Next we compute the left-hand side of the above equation in another way. We have

$$\begin{aligned}&\zeta_1(n-1)\zeta_2(n-2)\cdots\zeta_n(0)\\&=\zeta_1(n-1)\zeta_2(n-2)\cdots\zeta_{n-1}(1)\cdot\sum_{l_n=1}^{n}\eta_{l_n}\overline{\partial}_{l_n,n}.\end{aligned}\tag{7}$$

Here we need some preparation. For $s > j$ it follows from Lemma 3 (2) that

$$\begin{aligned}&\zeta_j(u)\sum_{l=1}^{n}\eta_l\cdot(\text{some factors})\cdot\overline{\partial}_{ls}\\&=\sum_{l=1}^{n}\Big(-\eta_l(\zeta_j(u)-\epsilon_{(l,j)}e_j)+\delta_{lj}\sum_{k=1}^{n}\epsilon_{(k,j)}\eta_k e_k\Big)\cdot(\text{some factors})\cdot\overline{\partial}_{ls}.\end{aligned}$$

Suppose that $\overline{\partial}_{l,s}\neq 0$ in the above expression. Then $\epsilon_{(l,j)}=1$ by $j<s$, and therefore $\zeta_j(u)-\epsilon_{(l,j)}e_j$ becomes $\zeta_j(u-1)$. For the part of $\delta_{lj}\sum_{k=1}^{n}\epsilon_{(k,j)}\eta_k e_k$ we have only to consider the case where $\overline{\partial}_{js}\neq 0$ thanks to the factor δ_{lj}. Then at least one of j and s is less than or equal to p, and it turns out that $j\le p$ by $j<s$. When $j\le p$, every $\epsilon_{(k,j)}$ $(k=1,2,\ldots,n)$ is equal to one, and it follows from Lemma 4 that this part is zero. To summarize we have

$$\zeta_j(u)\sum_{l=1}^{n}\eta_l\cdot(\text{some factors})\cdot\overline{\partial}_{ls}=-\sum_{l=1}^{n}\eta_l\zeta_j(u-1)\cdot(\text{some factors})\cdot\overline{\partial}_{ls}.$$

Thanks to the preparation in the previous paragraph the computation goes similarly to the proof of Theorem 1 (1), and finally we have

$$(\text{RHS of (7)})=e_1e_2\cdots e_n\det({}^tT)\det\left(\frac{\overline{\partial}}{\partial T}\right).$$

Thus we have proved the assertion.

3 b-Functions

We can compute the b-functions of the prehomogeneous vector spaces corresponding to our Capelli identities.

We first consider the following prehomogeneous vector space, which corresponds to the Capelli identity of Theorem 1 (1). Define $n_1,n_2,\ldots,n_m$ as the multiplicities of the partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n)$. In other words the numbers of nonzero entries in the first n_1 rows of T are equal, those in the next n_2 rows are equal, and so on. Similarly define $n'_1,n'_2,\ldots,n'_m$ as the multiplicities of the conjugate of the partition λ. In other words the numbers of nonzero entries in the first n_1 columns of T are equal, those in the next n_2 columns are equal, and so on.

Define complex Lie groups P, P', G, and a vector space V by

$$P = \left\{ \begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1m} \\ 0 & P_{22} & \cdots & P_{2m} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & P_{mm} \end{pmatrix} \in GL_n(\mathbb{C}) \;\middle|\; P_{ii} \in GL_{n_i}(\mathbb{C})\ (i = 1, 2, \ldots, m) \right\},$$

$$P' = \left\{ \begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1m} \\ 0 & P_{22} & \cdots & P_{2m} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & P_{mm} \end{pmatrix} \in GL_n(\mathbb{C}) \;\middle|\; P_{ii} \in GL_{n'_i}(\mathbb{C})\ (i = 1, 2, \ldots, m) \right\},$$

$$G = P \times P',$$

$$V = \left\{ \begin{pmatrix} V_{11} & \cdots & V_{1,m-1} & V_{1m} \\ V_{21} & \cdots & V_{2,m-1} & 0 \\ \vdots & \iddots & \iddots & \vdots \\ V_{m1} & 0 & \cdots & 0 \end{pmatrix} \in \mathrm{Mat}_n(\mathbb{C}) \;\middle|\; V_{ij} \in \mathrm{Mat}(n_i, n'_j; \mathbb{C}) \right\}.$$

Namely, t_{ij} in Theorem 1 (1) is the linear coordinate system on a vector space of this form. Then G acts on V by $(g, h).A = gA^th$ $((g, h) \in G$ and $A \in V)$, and (G, V) is a prehomogeneous vector space. $f = \det(T)$ is a relative invariant (if f is a nonzero polynomial) corresponding to the character $\det g \cdot \det h$. We can compute the b-function of f only in a limited case where $m = 2$ and $n_2 = n'_2 = 1$.

Proposition 5. If $m = 2$ and $n_2 = n'_2 = 1$ in the above setting, then the b-function $b_f(s)$ of $f = \det(T)$ is given by

$$b_f(s) = (s+1)(s+2)\cdots(s+n_1-1)\cdot(s+n_1)^2.$$

Proof. We can compute the b-function by direct computation using our Capelli identity. □

We next consider the following prehomogeneous vector space, which corresponds to the Capelli identity of Theorem 1 (2). Let $p \geq q$ be positive integers. Define a Lie group G and a vector space V as

$$\begin{aligned} G &= GL_p(\mathbb{C}) \times GL_q(\mathbb{C}), \\ V &= \left\{ \begin{pmatrix} V_{11} & V_{12} \\ {}^tV_{12} & 0 \end{pmatrix} \in \mathrm{Sym}_{p+q}(\mathbb{C}) \;\middle|\; V_{11} \in \mathrm{Sym}_p(\mathbb{C}), V_{12} \in \mathrm{Mat}(p, q; \mathbb{C}) \right\} \\ &\simeq \mathrm{Sym}_p(\mathbb{C}) \oplus \mathrm{Mat}(p, q; \mathbb{C}), \end{aligned} \tag{8}$$

where $\mathrm{Sym}_p(\mathbb{C})$ denotes the set of symmetric matrices of size $p \times p$. Namely, t_{ij} in Theorem 1 (2) is the linear coordinate system on a vector space of this form. Then G acts on V by

$$(g, h).A = \begin{pmatrix} g & \\ & h \end{pmatrix} A \, {}^t\!\begin{pmatrix} g & \\ & h \end{pmatrix} \qquad ((g, h) \in G,\ A \in V),$$

and (G, V) is a prehomogeneous vector space.

There are two basic invariants for this prehomogeneous vector space:

$$\begin{aligned} f_1 &= \det(T') \qquad (T' = (t_{ij})_{1 \le i,j \le p}), \\ f_2 &= \det(T). \end{aligned} \tag{9}$$

The basic invariants f_1 and f_2 correspond to the character $\det g^2$ and $\det g^2 \cdot \det h^2$, respectively. The b-function of f_1 is equal to $(s+1)(s+3/2)\cdots(s+(p+1)/2)$ as seen in (4). We want to compute the b-function of f_2 by using our Capelli identity, but we have not succeeded at this point. Sato-Sugiyama [2] have computed the b-function as

$$b_{f_2}(s) = (s + \frac{p+1}{2})^{((p))}(s + \frac{p}{2})^{((q))}, \tag{10}$$

where $x^{((q))} = x(x-1/2)\cdots(x-(q-1)/2)$.

4 b-Function of several variables

In this section we focus on the prehomogeneous vector space (G, V) defined by (8), which is corresponding to Theorem 1 (2). We retain the notation there.

For a prehomogeneous vector space with more than one basic invariant, we can consider b-functions of several variables. In the case we are focusing the b-function $b_{d_1,d_2}(s_1, s_2)$ of two variables is defined as

$$f_1^*(\partial)^{d_1} f_2^*(\partial)^{d_2} . f_1^{s_1+d_1} f_2^{s_2+d_2} = b_{d_1,d_2}(s_1, s_2) f_1^{s_1} f_2^{s_2},$$

where $f_1^*(\partial)$ and $f_2^*(\partial)$ are defined similarly in the case of (3). It is easy to see that $b_{1,0}(s_1, s_2)$ and $b_{0,1}(s_1, s_2)$ determines all $b_{d_1,d_2}(s_1, s_2)$, and therefore our goal is to compute $b_{1,0}(s_1, s_2)$ and $b_{0,1}(s_1, s_2)$, which are achieved in Proposition 6 and Proposition 7, respectively. The definition of $b_{0,1}(0, s)$ reads as $f_2^*(\partial).f_2^{s+1} = b_{0,1}(0, s) f_2^s$, and this means that $b_{0,1}(0, s) = b_{f_2}(s)$ (see (10)).

We can compute $b_{1,0}(s_1, s_2)$ by using the ordinary Capelli identity (1) and representation theory.

Proposition 6. $b_{1,0}(s_1, s_2) = (s_1 + \frac{q+1}{2})^{((q))}(s_1 + s_2 + \frac{p+1}{2})^{((p-q))}$

Proof. The b-function $b_{1,0}(s_1, s_2)$ is defined as

$$f_1^*(\partial) . f_1^{s_1+1} f_2^{s_2} = b_{1,0}(s_1, s_2) f_1^{s_1} f_2^{s_2}.$$

and hence we can use the ordinary Capelli identity for f_1:

$$\det({}^tT') \det\left(\frac{\overline{\partial}}{\partial T'}\right) = \det\left({}^tT' \frac{\overline{\partial}}{\partial T'} + \begin{pmatrix} (p-1)/2 & & \\ & (p-2)/2 & \\ & & \ddots & \\ & & & 0 \end{pmatrix}\right), \tag{11}$$

where $T' = (t_{ij})_{1\le i,j\le p}$ is the same as in (9). Thus we need to consider the action of the subgroup $GL_p(\mathbb{C})$ of $G = GL_p(\mathbb{C}) \times GL_q(\mathbb{C})$ on the subspace $\mathrm{Sym}_p(\mathbb{C})$ of $V \simeq \mathrm{Sym}_p(\mathbb{C}) \oplus \mathrm{Mat}(p,q;\mathbb{C})$, and compute the weight of $f_1^{s_1+1}f_2^{s_2}$ with respect to this action. Note that monomials of f_2 do not have the equal weight.

We take the Cartan subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{gl}_p$ of $GL_p(\mathbb{C})$ as the diagonal matrices. Let ϵ_i $(i = 1,2,\dots,p)$ be the linear coordinate system on $\mathfrak{h}$. Then the weight of t_{ij} is equal to $\epsilon_i + \epsilon_j$ $(i \le p,\ j \le p)$, and zero (otherwise). It is clear that the weight of f_1 is equal to $2(\epsilon_1+\epsilon_2+\cdots+\epsilon_p)$. The monomials of f_2 which have the highest weight among the monomials of f_2 come from the product of the following three determinants

$$\det(t_{ij})_{\substack{1\le i\le p-q,\\ 1\le j\le p-q}},\quad \det(t_{ij})_{\substack{p-q<i\le p,\\ p<j\le p+q}},\quad \det(t_{ij})_{\substack{p<i\le p+q,\\ p-q<j\le p}}$$

up to sign. Therefore the highest weight among the monomials of f_2 is equal to $2(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{p-q})$. Finally it follows that the highest weight of the monomials of $f_1^{s_1+1}f_2^{s_2}$ is equal to

$$\begin{aligned}&2(\epsilon_1+\epsilon_2+\cdots+\epsilon_p)\cdot(s_1+1)+2(\epsilon_1+\epsilon_2+\cdots+\epsilon_{p-q})\cdot s_2\\ &\quad = 2(s_1+s_2+1)(\epsilon_1+\cdots+\epsilon_{p-q})+2(s_1+1)(\epsilon_{p-q+1}+\cdots+\epsilon_{p+q}).\end{aligned}$$

In computing $f_1^*(\partial).f_1^{s_1+1}f_2^{s_2}$, since the result is a scalar multiple of $f_1^{s_1}f_2^{s_2}$, we have only to know the scalar multiple by computing the differentiation on a monomial of the highest weight. We use (11) for this computation, and only the diagonal entries on the right-hand side of (11) have the contribution. The (i,i)-entry of the determinant has the same action as the action of $e_{ii} + (p-i)/2$, where e_{ii} is the unit matrix of $\mathfrak{h}$. Thus we can compute the desired b-function as follows.

$$\begin{aligned}&f_1^*(\partial).f_1^{s_1+1}f_2^{s_2}\\ &\quad = f_1^{-1}(f_1f_1^*(\partial)).f_1^{s_1+1}f_2^{s_2}\\ &\quad = f_1^{-1}\cdot(s_1+s_2+1+\frac{p-1}{2})(s_1+s_2+1+\frac{p-2}{2})\cdots(s_1+s_2+1+\frac{q}{2})\times\\ &\qquad\qquad (s_1+1+\frac{q-1}{2})(s_1+1+\frac{q-2}{2})\cdots(s_1+1+\frac{0}{2})\times f_1^{s_1+1}f_2^{s_2}.\end{aligned}$$

This shows the proposition. □

By using the explicit form of $b_{0,1}(0,s)$ and $b_{1,0}(s_1,s_2)$ we obtain the remaining b-function $b_{0,1}(s_1,s_2)$ of two variables.

Proposition 7. $b_{0,1}(s_1,s_2) = (s_2+\frac{p}{2})^{((q))}(s_2+\frac{q+1}{2})^{((q))}(s_1+s_2+\frac{p+1}{2})^{((p-q))}$

Proof. The b-function $b_{0,1}(s_1,s_2)$ is defined as

$$f_2^*(\partial).f_1^{s_1}f_2^{s_2+1} = b_{0,1}(s_1,s_2)f_1^{s_1}f_2^{s_2}.$$

We differentiate $f_1^{s_1} f_2^{s_2+1}$ by $f_1^*(\partial)^{s_1} f_2^*(\partial)$ in two different ways. First one is to differentiate by $f_1^*(\partial)^{s_1}$ and $f_2^*(\partial)$ in turn, and the other is to differentiate in reverse order. These two ways are illustrated as follows:

$$\begin{array}{ccccccccc}
f_1^{s_1} f_2^{s_2+1} & \xrightarrow{b_{1,0}(s_1-1,s_2+1)} & f_1^{s_1-1} f_2^{s_2+1} & \xrightarrow{b_{1,0}(s_1-2,s_2+1)} & \cdots & \xrightarrow{b_{1,0}(0,s_2+1)} & f_1^0 f_2^{s_2+1} \\
b_{0,1}(s_1,s_2) \downarrow & & & & & & b_{0,1}(0,s_2) \downarrow \\
f_1^{s_1} f_2^{s_2} & \xrightarrow[b_{1,0}(s_1-1,s_2)]{} & f_1^{s_1-1} f_2^{s_2} & \xrightarrow[b_{1,0}(s_1-2,s_2)]{} & \cdots & \xrightarrow[b_{1,0}(0,s_2)]{} & f_1^0 f_2^{s_2}
\end{array}$$

Horizontal arrows mean the differentiation by $f_1^*(\partial)$, two vertical arrows mean that by $f_2^*(\partial)$, and b-functions beside arrows are the scalar multiples which arise by the differentiations. Since the above diagram is commutative, we obtain the equation

$$\begin{aligned}
&b_{1,0}(s_1-1, s_2+1) b_{1,0}(s_1-2, s_2+1) \cdots b_{1,0}(0, s_2+1) \cdot b_{0,1}(0, s_2) \\
&\qquad\qquad = b_{0,1}(s_1, s_2) \cdot b_{1,0}(s_1-1, s_2) b_{1,0}(s_1-2, s_2) \cdots b_{1,0}(0, s_2).
\end{aligned}$$

In this equation b-functions except $b_{0,1}(s_1, s_2)$ are already known by Proposition 7 and $b_{0,1}(0, s) = b_{f_2}(s)$. Therefore we have

$$\begin{aligned}
&b_{0,1}(s_1, s_2) \\
&= b_{0,1}(0, s_2) \cdot \frac{\prod_{t=0}^{s_1-1} b_{1,0}(t, s_2+1)}{\prod_{t=0}^{s_1-1} b_{1,0}(t, s_2)} \\
&= (s + \frac{p+1}{2})^{((p))} (s + \frac{p}{2})^{((q))} \cdot \prod_{t=0}^{s_1-1} \frac{(t + s_2 + 1 + \frac{p+1}{2})^{((p-q))} (t + \frac{q+1}{2})^{((q))}}{(t + s_2 + \frac{p+1}{2})^{((p-q))} (t + \frac{q+1}{2})^{((q))}} \\
&= (s + \frac{p+1}{2})^{((p))} (s + \frac{p}{2})^{((q))} \cdot \prod_{t=0}^{s_1-1} \frac{(t + s_2 + \frac{p+3}{2})(t + s_2 + \frac{p+2}{2})}{(t + s_2 + \frac{q+3}{2})(t + s_2 + \frac{q+2}{2})} \\
&= (s_2 + \frac{p}{2})^{((q))} (s_2 + \frac{q+1}{2})^{((q))} (s_1 + s_2 + \frac{p+1}{2})^{((p-q))}.
\end{aligned}$$

This is the desired b-function. □

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