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Asymptotic analysis of positive solutions of a class of nonlinear differential equations in the framework of regular variation

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1 Introduction

The equation to be studied in this paper is

$$(\mathrm{A}_\pm) \qquad (p(t)\varphi(x'(t)))' \pm \sum_{i=1}^{m} q_i(t)\varphi(x(g_i(t))) \pm \sum_{j=1}^{n} r_j(t)\varphi(x(h_j(t))) = 0,$$

$$(\varphi(\xi) = |\xi|^{\alpha-1}\xi = |\xi|^{\alpha}\mathrm{sgn}\,\xi,\ \alpha > 0,\ \xi \in \mathbb{R},\ \text{Double-sign corresponding}),$$

where $p,\ q_i,\ r_j : [a,\infty) \to (0,\infty)$, $a \geqq 0$ are continuous functions, $g_i,\ h_j$ are continuous and increasing functions with $g_i(t) < t$ and $h_j(t) > t$ and $\lim_{t\to\infty} g_i(t) = \infty$ for $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$. In what follows we always assume that the function $p(t)$ satisfies

$$(1.1) \qquad \int_a^{\infty} \frac{dt}{p(t)^{\frac{1}{\alpha}}} = \infty.$$

It is shown in the monograph ([8]) that the class of regularly varying functions in the sense of Karamata is a well-suited framework for the asymptotic analysis of nonoscillatory solutions of second order linear differential equation of the form

$$x''(t) = q(t)x(t), \qquad q(t) > 0.$$

The study of asymptotic analysis of nonoscillatory solutions of functional differential equations with deviating arguments in the framework of regularly varying functions (called Karamata functions) was first attempted by Kusano and Marić ([5, 6]). They established a sharp condition for the existence of a slowly varying solution of second order functional differential equation with retarded argument of the form

$$(1.2) \qquad x''(t) = q(t)x(g(t)),$$

and the following functional differential equation of the form

$$(1.3) \qquad x''(t) \pm q(t)x(g(t)) \pm r(t)x(h(t)) = 0,$$

where $q,\ r : [a,\infty) \to (0,\infty)$, $a \geqq 0$ are continuous functions, $g,\ h$ are continuous and increasing with $g(t) < t$, $h(t) > t$ for $t \geqq a$, $\lim_{t\to\infty} g(t) = \infty$.

It is well known that there is the qualitative similarity between linear differential equations and half-linear differential equations (see the book Došlý and Řehák ([2])). Therefore, in our previous papers ([4, 7]) we proved how useful the regularly varying functions were for the study of nonoscillation and asymptotic analysis of the half-linear differentail equation involving nonlinear Sturm-Liouville type differential operator of the form

$$(\mathrm{B}_\pm) \qquad (p(t)\varphi(x'(t)))' \pm f(t)\varphi(x(t)) = 0, \quad p(t) > 0,$$

and the half-linear functional differential equation with both retarded and advanced arguments of the form

$$(1.4) \qquad (\varphi(x'(t)))' \pm q(t)\varphi(x(g(t))) \pm r(t)\varphi(x(h(t))) = 0,$$

where $f : [a, \infty) \to (0, \infty)$, $a \geqq 0$ is a continuous function, p, g, h are as in the above equations.

Theorem A (J. Jaroš, T. Kusano and T. Tanigawa ([4])) *Suppose that* (1.1) *holds. The equations* $(\mathrm{B}_\pm)$ *have a normalized slowly varying solution with respect to* $P(t)$ *and a normalized regularly varying solution of index* 1 *with respect to* $P(t)$ *if and only if*

$$(1.5) \qquad \lim_{t\to\infty} P(t)^\alpha \int_t^\infty f(s)\,ds = 0,$$

where the function $P(t)$ *is defined by*

$$(1.6) \qquad P(t) = \int_a^t \frac{ds}{p(s)^{\frac{1}{\alpha}}}$$

Theorem B (J. Manojlović and T. Tanigawa ([7])) *Suppose that*

$$\lim_{t\to\infty} \frac{g(t)}{t} = 1 \quad \textit{and} \quad \lim_{t\to\infty} \frac{h(t)}{t} = 1$$

hold. Then, the equations (1.4) *have a slowly varying solution and a regularly varying solution of index* 1 *if and only if*

$$\lim_{t\to\infty} t^\alpha \int_t^\infty q(s)\,ds = \lim_{t\to\infty} t^\alpha \int_t^\infty r(s)\,ds = 0.$$

The objective of this paper is to establish a sharp condition of the existence of a normalized slowly varying solution with respect to $P(t)$ and a normalized regularly varying solution of index 1 with respect to $P(t)$ of the equation $(\mathrm{A}_\pm)$. Our main result is the following.

Theorem 1.1 *Suppose that*

$$(1.7) \qquad \lim_{t\to\infty} \frac{P(g_i(t))}{P(t)} = 1 \quad \textit{for} \;\; i = 1, 2, \cdots, m$$

and

$$(1.8) \qquad \lim_{t\to\infty} \frac{P(h_j(t))}{P(t)} = 1 \quad \textit{for} \;\; i = 1, 2, \cdots, n$$

hold. The equation ($\mathrm{A}_{\pm}$) *possess a normalized slowly varying solution with respect to* $P(t)$ *and a normalized regularly varying solution of index* 1 *with respect to* $P(t)$ *if and only if*
(1.9)
$$\lim_{t\to\infty} P(t)^{\alpha}\int_t^{\infty} q_i(s)ds = \lim_{t\to\infty} P(t)^{\alpha}\int_t^{\infty} r_j(s)ds = 0 \ \ \textit{for}\ i = 1,2,\cdots,m \ \textit{and}\ j = 1,2,\cdots,n.$$

This paper is organized as follows. In Section 2 we briefly recall the definitions and properties of the slowly varying and regularly varying functions of index ρ with respect to $P(t)$ which are called the generalized regularly varying functions introduced by Jaroš and Kusano ([3]). An explicit expressions for the normalized slowly varying solution with respect to $P(t)$ and the normalized regularly varying solution of index 1 with respect to $P(t)$ of the equations ($\mathrm{B}_{\pm}$) obtained in ([4]) do not meet our need for application to the functional differential equations ($\mathrm{A}_{\pm}$), and so we present a modified proof of Theorem A in Section 3. Some examples illustrating our result wiil also be presented in Section 4.

2 Definitions and properties of the generalized regularly varying functions

For the reader's convenience we first state the definitions and some basic properties of the regularly varying functions and then refer to the generalized regularly varying functions. The generalized regularly varying functions are introduced for the first time by Jaroš and Kusano ([3]) in order to gain useful information about the asymptotic behavior of nonoscillatory solutions for the self-adjoint differential equations of the form

$$(p(t)x'(t))' + f(t)x(t) = 0.$$

(The definitions and properties of regularly varying functions):

Definition 2.1 A measurable function $f : [a,\infty) \to (0,\infty)$ is said to be a regularly varying of index ρ if it satisfies

$$\lim_{t\to\infty}\frac{f(\lambda t)}{f(t)} = \lambda^{\rho} \quad \text{for any} \ \ \lambda > 0, \ \ \rho\in\mathbb{R}.$$

Proposition 2.1 (Representation Theorem) *A measurable function* $f : [a,\infty) \to (0,\infty)$ *is regularly varying of index* ρ *if and only if it can be written in the form*

$$f(t) = c(t)\exp\left\{\int_{t_0}^{t}\frac{\delta(s)}{s}\,ds\right\}, \qquad t \geqq t_0,$$

for some $t_0 > a$ *where* $c(t)$ *and* $\delta(t)$ *are measurable functions such that*

$$\lim_{t\to\infty} c(t) = c \in (0,\infty) \qquad \textit{and} \qquad \lim_{t\to\infty}\delta(t) = \rho.$$

The totality of regularly varying functions of index ρ is denoted by $\mathrm{RV}(\rho)$. The symbol SV is used to denote $\mathrm{RV}(0)$ and a member of SV=RV(0) is referred to as a slowly varying function. If $f(t)\in\mathrm{RV}(\rho)$, then $f(t) = t^{\rho}L(t)$ for some $L(t)\in\mathrm{SV}$. Therefore, the class of slowly varying functions is of fundamental importance in the theory of regular variation. In addition to the functions tending to positive constants as $t \to \infty$, the following functions

$$\prod_{i=1}^{N}(\log_i t)^{m_i}\ (m_i \in \mathbb{R}), \qquad \exp\left\{\prod_{i=1}^{N}(\log_i t)^{n_i}\right\}\ (0 < n_i < 1), \qquad \exp\left\{\frac{\log t}{\log_2 t}\right\},$$

where $\log_1 t = \log t$ and $\log_k t = \log\log_{k-1} t$ for $k = 2, 3, \cdots, N$, also belong to the set of slowly varying functions.

Proposition 2.2 *Let $L(t)$ be any slowly varying function. Then, for any $\gamma > 0$,*

$$\lim_{t\to\infty} t^{\gamma}L(t) = \infty \quad \textit{and} \quad \lim_{t\to\infty} t^{-\gamma}L(t) = 0.$$

Proposition 2.3 (Karamata's integration theorem) *Let $L(t)\in$SV. Then,*
(i) *if $\gamma > -1$,*

$$\int_a^t s^{\gamma}L(s)ds \sim \frac{t^{\gamma+1}}{\gamma+1}L(t), \quad \textit{as } t \to \infty;$$

(ii) *if $\gamma < -1$,*

$$\int_t^{\infty} s^{\gamma}L(s)ds \sim -\frac{t^{\gamma+1}}{\gamma+1}L(t), \quad \textit{as } t \to \infty.$$

Here and hereafter the notation $\varphi(t) \sim \psi(t)$ as $t \to \infty$ is used to mean the asymptotic equivalence of $\varphi(t)$ and $\psi(t)$: $\lim_{t\to\infty} \psi(t)/\varphi(t) = 1$.

For an excellent explanation of the theory of regularly varying functions the reader is referred to the book ([1]).

(The definitions and properties of generalized regularly varying functions):

Definition 2.2 A measurable function $f : [a, \infty) \to (0, \infty)$ is said to be slowly varying with respect to $P(t)$ if the function $f \circ P(t)^{-1}$ is slowly varying in the sense of Karamata, where the function $P(t)$ is defined by (1.6) and $P(t)^{-1}$ denotes the inverse function of $P(t)$. The totality of slowly varying function with respect to $P(t)$ is denoted by SV_P.

Definition 2.3 A measurable function $g : [a, \infty) \to (0, \infty)$ is said to be regularly varying function of index ρ with respect to $P(t)$ if the function $g \circ P(t)^{-1}$ is regularly varying of index ρ in the sense of Karamata. The set of all regularly varying functions of index ρ with respect to $P(t)$ is denoted by $\mathrm{RV}_P(\rho)$.

Of fundamental importance is the following representation theorem for the generalized slowly and regularly varying functions, which is an immediate consequence of Proposition 2.1.

Proposition 2.4 (i) *A function $f(t)$ is slowly varying with respect to $P(t)$ if and only if it can be expressed in the form*

$$f(t) = c(t)\exp\left\{\int_{t_0}^{t} \frac{\delta(s)}{p(s)^{\frac{1}{\alpha}}P(s)}ds\right\}, \quad t \geqq t_0 \tag{2.1}$$

for some $t_0 > a$, *where* $c(t)$ *and* $\delta(t)$ *are measurable functions such that*

$$\lim_{t\to\infty} c(t) = c \in (0,\infty) \quad \textit{and} \quad \lim_{t\to\infty} \delta(t) = 0.$$

(ii) *A function* $g(t)$ *is regularly varying of index* ρ *with respect to* $P(t)$ *if and only if it has the representation*

$$g(t) = c(t)\exp\left\{\int_{t_0}^{t} \frac{\delta(s)}{p(s)^{\frac{1}{\alpha}}P(s)}ds\right\}, \quad t \geqq t_0 \tag{2.2}$$

for some $t_0 > a$, *where* $c(t)$ *and* $\delta(t)$ *are measurable functions such that*

$$\lim_{t\to\infty} c(t) = c \in (0,\infty) \quad \textit{and} \quad \lim_{t\to\infty} \delta(t) = \rho.$$

If the function $c(t)$ in (2.1) (or (2.2)) is identically a constant on $[t_0,\infty)$, then the function $f(t)$ (or $g(t)$) is called normalized slowly varying (or normalized regulalrly varying of index ρ) with respect to $P(t)$. The totality of such functions is denoted by n-SV_P (or n-RV_P).

It is easy to see that if $g(t) \in \mathrm{RV}_P(\rho)$ (n-$\mathrm{RV}_P(\rho)$), then $g(t) = P(t)^\rho f(t)$ for some $f(t) \in \mathrm{SV}_P$ (or n-SV_P).

Proposition 2.5 *Let* $f(t) \in \mathrm{SV}_P$. *Then, for any* $\gamma > 0$,

$$\lim_{t\to\infty} P(t)^\gamma f(t) = \infty \quad \textit{and} \quad \lim_{t\to\infty} P(t)^{-\gamma} f(t) = 0. \tag{2.3}$$

Karamata's integration theorem is generalized in the following manner.

Proposition 2.6 (Generalized Karamata's integration theorem) *Let* $f(t) \in$n-SV_P. *Then,*
(i) *If* $\gamma > -1$,

$$\int_{t_0}^{t} \frac{P(s)^\gamma}{p(s)^{\frac{1}{\alpha}}} f(s)ds \sim \frac{P(t)^{\gamma+1}}{\gamma+1} f(t) \quad \textit{as } t\to\infty; \tag{2.4}$$

(ii) *If* $\gamma < -1$, $\int_{t_0}^{\infty} P(t)^\gamma f(t)/p(t)^{\frac{1}{\alpha}}dt < \infty$ *and*

$$\int_{t}^{\infty} \frac{P(s)^\gamma}{p(s)^{\frac{1}{\alpha}}} f(s)ds \sim -\frac{P(t)^{\gamma+1}}{\gamma+1} f(t) \quad \textit{as } t\to\infty. \tag{2.5}$$

3 The existence of generalized regularly varying solution of self-adjoint differential equation without deviating arguments

Theorem 3.1 *Put* $F(t) = P(t)^\alpha \int_t^\infty f(s)ds$, $\widehat{F}(t) = \sup_{s\geqq t} F(s)$,

$$F_+(t,w) = |1 + F(t) - w|^{1+\frac{1}{\alpha}} + \left(1+\frac{1}{\alpha}\right) w - 1, \tag{3.1}$$

and

$$F_{-}(t,w) = 1 + \left(1 + \frac{1}{\alpha}\right) w - |1 + F(t) - w|^{1+\frac{1}{\alpha}}. \tag{3.2}$$

(i) *The equation* (B$_+$) *possesses a* n-SV$_P$ *solution* $x(t)$ *having the expression*

$$x(t) = \exp\left\{\int_{t_0}^{t} \left(\frac{v(s)+F(s)}{p(s)P(s)^{\alpha}}\right)^{\frac{1}{\alpha}} ds\right\}, \quad t \geqq t_0 \tag{3.3}$$

for some $t_0 > a$, *in which* $v(t)$ *satisfies*

$$v(t) = \alpha P(t)^{\alpha} \int_t^{\infty} \frac{(v(s)+F(s))^{1+\frac{1}{\alpha}}}{p(s)^{\frac{1}{\alpha}} P(s)^{\alpha+1}} ds, \quad t \geqq t_0 \tag{3.4}$$

and

$$0 \leqq v(t) \leqq \widehat{F}(t_0) \quad \textit{for} \ \ t \geqq t_0 \tag{3.5}$$

if and only if (1.5) *holds.*

(ii) *The equation* (B$_+$) *possesses a* n-RV$_P(1)$ *solution* $x(t)$ *having the expression*

$$x(t) = \exp\left\{\int_{t_1}^{t} \left(\frac{1+F(s)-w(s)}{p(s)P(s)^{\alpha}}\right)^{\frac{1}{\alpha}} ds\right\}, \quad t \geqq t_1 \tag{3.6}$$

for some $t_1 > a$, *in which* $w(t)$ *satisfies*

$$w(t) = \frac{\alpha}{P(t)} \int_t^{\infty} F_{+}(s, w(s)) ds, \quad t \geqq t_1 \tag{3.7}$$

and

$$0 \leqq w(t) \leqq \sqrt{\widehat{F}(t_1)} \quad \textit{for} \ \ t \geqq t_1 \tag{3.8}$$

if and only if (1.5) *holds.*

(iii) *The equation* (B$_-$) *possesses a* n-SV$_P$ *solution* $x(t)$ *having the expression*

$$x(t) = \exp\left\{\int_{t_0}^{t} \left(\frac{v(s)-F(s)}{p(s)P(s)^{\alpha}}\right)^{\frac{1}{\alpha}*} ds\right\}, \quad t \geqq t_0 \tag{3.9}$$

for some $t_0 > a$, *in which* $v(t)$ *satisfies*

$$v(t) = \alpha P(t)^{\alpha} \int_t^{\infty} \frac{|v(s)-F(s)|^{1+\frac{1}{\alpha}}}{p(s)^{\frac{1}{\alpha}} P(s)^{\alpha+1}} ds, \quad t \geqq t_0 \tag{3.10}$$

and (3.5) *if and only if* (1.5) *holds. Here the meaning of the asterisk notation is defined by* $\xi^{\gamma *} = |\xi|^{\gamma} \operatorname{sgn} \xi, \ \gamma > 0, \ \xi \in \mathbb{R}$.

(iv) *The equation* (B$_-$) *possesses a* n-RV$_P(1)$ *solution* $x(t)$ *having the expression*

$$x(t) = \exp\left\{\int_{t_1}^{t} \left(\frac{1-F(s)+w(s)}{p(s)P(s)^{\alpha}}\right)^{\frac{1}{\alpha}} ds\right\}, \quad t \geqq t_1 \tag{3.11}$$

for some $t_1 > a$, *in which* $w(t)$ *satisfies*

$$w(t) = \frac{\alpha}{P(t)} \int_t^{\infty} F_{-}(s, w(s)) ds, \quad t \geqq t_1 \tag{3.12}$$

and (3.8) *if and only if* (1.5) *holds.*

4 Examples

We here present four examples illustrating application of Theorem 1.1 to the functional differential equations of the type (A_+) and (A_-), respectively. We begin with two examples of the existence of n-SV_P and n-$RV_P(1)$ solutions of the type (A_+) with the case $i = 1,\ 2$ and $j = 1$.

Example 4.1 Consider the following functional differential equation with both retarded and advanced arguments

$$\begin{aligned}(4.1)\qquad &(e^{-\alpha t}\varphi(x'(t)))' + q_1(t)\varphi\left(x\left(t-\frac{1}{\log t}\right)\right) + \\ &+ q_2(t)\varphi\left(x\left(t-\frac{1}{\log t}-\frac{1}{\log_2 t}\right)\right) + r(t)\varphi\left(x\left(t+\frac{1}{\log t}\right)\right) = 0, \quad t \geqq e,\end{aligned}$$

where the functions $q_i(t)$, $i = 1,\ 2$ and $r(t)$ are given by

$$\begin{aligned} q_1(t) = \frac{\alpha}{3t^\alpha e^{\alpha t}}\left(1+\frac{\lambda}{\log t}\right)^{\alpha-1}&\left[1-\frac{\lambda}{t\log t}-\frac{\lambda(\lambda-1)}{t(\log t)^2}+\frac{\lambda}{\log t}\right]\times \\ &\times\left(1-\frac{1}{t\log t}\right)^{-\alpha}\left\{1+\frac{\log\left(1-\frac{1}{t\log t}\right)}{\log t}\right\}^{-\alpha\lambda}, \end{aligned}$$

$$\begin{aligned} q_2(t) = \frac{\alpha}{3t^\alpha e^{\alpha t}}\left(1+\frac{\lambda}{\log t}\right)^{\alpha-1}&\left[1-\frac{\lambda}{t\log t}-\frac{\lambda(\lambda-1)}{t(\log t)^2}+\frac{\lambda}{\log t}\right]\times \\ &\times\left(1-\frac{1}{t\log t}-\frac{1}{t\log_2 t}\right)^{-\alpha}\left\{1+\frac{\log\left(1-\frac{1}{t\log t}-\frac{1}{t\log_2 t}\right)}{\log t}\right\}^{-\alpha\lambda} \end{aligned}$$

and

$$\begin{aligned} r(t) = \frac{\alpha}{3t^\alpha e^{\alpha t}}\left(1+\frac{\lambda}{\log t}\right)^{\alpha-1}&\left[1-\frac{\lambda}{t\log t}-\frac{\lambda(\lambda-1)}{t(\log t)^2}+\frac{\lambda}{\log t}\right]\times \\ &\times\left(1+\frac{1}{t\log t}\right)^{-\alpha}\left\{1+\frac{\log\left(1+\frac{1}{t\log t}\right)}{\log t}\right\}^{-\alpha\lambda} \end{aligned}$$

for λ being a positive constant. The function $p(t) = e^{-\alpha t}$ satisfies (1.1) and that the function $P(t)$ given by (1.6) is $P(t) \sim e^t$. Moreover, the functions

$$g_1(t) = t - \frac{1}{\log t}, \quad g_2(t) = t - \frac{1}{\log t} - \frac{1}{\log_2 t} \quad \text{and} \quad h(t) = t + \frac{1}{\log t}$$

satisfy conditions (1.7) and (1.8). The condition (1.9) is satisfied for this equation since

$$\int_t^\infty q_i(s)ds \sim \frac{\alpha}{3t^\alpha e^{\alpha t}}, \quad i = 1,\ 2 \quad \text{and} \quad \int_t^\infty h(s)ds \sim \frac{\alpha}{3t^\alpha e^{\alpha t}} \quad \text{as } t \to \infty.$$

Therefore, equation (4.1) has a n-SV_{e^t} solution $x(t)$ by Theorem 1.1. One such solution is $x(t) = t(\log t)^\lambda$.

Example 4.2 Consider the following functional differential equation

$$(4.2)\quad (t^{\alpha}\varphi(x'(t)))' + q_1(t)\varphi(x(te^{-\frac{1}{t}})) + q_2(t)\varphi(x(te^{-\frac{1}{t}-\frac{1}{\log t}})) + r(t)\varphi(x(te^{\frac{1}{t}})) = 0, \quad t \geqq e^e,$$

where the functions $q_i(t)$, $i = 1, 2$ and $r(t)$ are given by

$$q_1(t) = \frac{\alpha\mu}{3t(\log t)^{\alpha+1}\log_2 t}\left(1 - \frac{\mu}{\log_2 t}\right)^{\alpha-1}\left(1 - \frac{\mu+1}{\log_2 t}\right)\left(1 - \frac{1}{t\log t}\right)^{-\alpha} \times \\ \times \left\{1 + \frac{\log\left(1 - \frac{1}{t\log t}\right)}{\log_2 t}\right\}^{\alpha\mu},$$

$$q_2(t) = \frac{\alpha\mu}{3t(\log t)^{\alpha+1}\log_2 t}\left(1 - \frac{\mu}{\log_2 t}\right)^{\alpha-1}\left(1 - \frac{\mu+1}{\log_2 t}\right)\left(1 - \frac{1}{t\log t} - \frac{1}{(\log t)^2}\right)^{-\alpha} \times \\ \times \left\{1 + \frac{\log\left(1 - \frac{1}{t\log t} - \frac{1}{(\log t)^2}\right)}{\log_2 t}\right\}^{\alpha\mu}$$

and

$$r(t) = \frac{\alpha\mu}{3t(\log t)^{\alpha+1}\log_2 t}\left(1 - \frac{\mu}{\log_2 t}\right)^{\alpha-1}\left(1 - \frac{\mu+1}{\log_2 t}\right)\left(1 + \frac{1}{t\log t}\right)^{-\alpha} \times \\ \times \left\{1 + \frac{\log\left(1 + \frac{1}{t\log t}\right)}{\log_2 t}\right\}^{\alpha\mu},$$

respectively, and μ is a positive constant. The function $p(t) = t^{\alpha}$ satisfies (1.1) and the function $P(t)$ reduces to $P(t) \sim \log t$, while the functions $g_1(t) = te^{-\frac{1}{t}}$, $g_2(t) = te^{-\frac{1}{t}-\frac{1}{\log t}}$ and $h(t) = te^{\frac{1}{t}}$ satisfy conditions (1.7) and (1.8). Moreover, since

$$\int_t^{\infty} q_i(s)ds \sim \frac{\alpha\mu}{3t(\log t)^{\alpha+1}\log_2 t}, \quad i = 1,\ 2 \quad \text{and} \quad \int_t^{\infty} h(s)ds \sim \frac{\alpha\mu}{3t(\log t)^{\alpha+1}\log_2 t}$$

as $t \to \infty$, condition (1.9) is satisfied and thus, the equation (4.2) possesses a n-$\mathrm{RV}_{\log t}$ solution by Theorem 1.1. One such solution is $\log t/(\log_2 t)^{\mu}$.

Next, two examples illustrating application of Theorem 1.1 to the functional differential equation of the type (A_-) with the case $i = 1,\ 2$ and $j = 1$ will be presented below.

Example 4.3 We consider the functional differential equation with both retarded and advanced arguments

$$(4.3)\quad (e^{-\alpha t}\varphi(x'(t)))' = q_1(t)\varphi\left(x\left(t - \frac{1}{\log t}\right)\right) + q_2(t)\varphi\left(x\left(t - \frac{1}{\log t} - \frac{1}{\log_2 t}\right)\right) + \\ + r(t)\varphi\left(x\left(t + \frac{1}{\log t}\right)\right), \quad t \geqq e,$$

where the functions $q_i(t)$, $i = 1,\ 2$ and $r(t)$ are given by

$$q_1(t) = \frac{\alpha}{3t^\alpha e^{\alpha t}}\left(1-\frac{\lambda}{\log t}\right)^{\alpha-1}\left[\left(1+\frac{2}{t}\right)\left(1-\frac{\lambda}{\log t}\right)+\frac{\lambda}{t\log t}\left(1-\frac{\lambda}{\log t}\right)+\frac{\lambda}{t(\log t)^2}\right]\times$$
$$\times\left(1-\frac{1}{t\log t}\right)^{\alpha}\left\{1+\frac{\log\left(1-\frac{1}{t\log t}\right)}{\log t}\right\}^{-\alpha\lambda},$$

$$q_2(t) = \frac{\alpha}{3t^\alpha e^{\alpha t}}\left(1-\frac{\lambda}{\log t}\right)^{\alpha-1}\left[\left(1+\frac{2}{t}\right)\left(1-\frac{\lambda}{\log t}\right)-\frac{\lambda}{t\log t}+\frac{\lambda(\lambda-1)}{t(\log t)^2}\right]\times$$
$$\times\left(1-\frac{1}{t\log t}-\frac{1}{t\log_2 t}\right)^{\alpha}\left\{1+\frac{\log\left(1-\frac{1}{t\log t}-\frac{1}{t\log_2 t}\right)}{\log t}\right\}^{-\alpha\lambda}$$

and

$$r(t) = \frac{\alpha}{3t^\alpha e^{\alpha t}}\left(1-\frac{\lambda}{\log t}\right)^{\alpha-1}\left[\left(1+\frac{2}{t}\right)\left(1-\frac{\lambda}{\log t}\right)-\frac{\lambda}{t\log t}+\frac{\lambda(\lambda-1)}{t(\log t)^2}\right]\times$$
$$\times\left(1+\frac{1}{t\log t}\right)^{\alpha}\left\{1+\frac{\log\left(1+\frac{1}{t\log t}\right)}{\log t}\right\}^{-\alpha\lambda},$$

for λ being a positive constant. As in Example 4.1 it could be shown without difficulty that all conditions of Theorem 1.1 are satisfied, so that the equation (4.3) has a n-SV_{e^t} solution $x(t)$ by Theorem 1.1. One such solution is $(\log t)^\lambda/t$.

Example 4.4 Consider the following functional differential equation

$$(4.4)\qquad (t^\alpha\varphi(x'(t)))' = q_1(t)\varphi(x(te^{-\frac{1}{t}})) + q_2(t)\varphi(x(te^{-\frac{1}{t}-\frac{1}{\log t}})) + r(t)\varphi(x(te^{\frac{1}{t}})),\qquad t \geqq e^e,$$

where the function $q_i(t)$, $i = 1,\ 2$ and $r(t)$ are given by

$$q_1(t) = \frac{\alpha\mu}{3t(\log t)^{\alpha+1}\log_2 t}\left(1+\frac{\mu}{\log_2 t}\right)^{\alpha-1}\left(1+\frac{\mu-1}{\log_2 t}\right)\left(1-\frac{1}{t\log t}\right)^{-\alpha}\times$$
$$\times\left\{1+\frac{\log\left(1-\frac{1}{t\log t}\right)}{\log_2 t}\right\}^{-\alpha\mu}$$

$$q_2(t) = \frac{\alpha\mu}{3t(\log t)^{\alpha+1}\log_2 t}\left(1+\frac{\mu}{\log_2 t}\right)^{\alpha-1}\left(1+\frac{\mu-1}{\log_2 t}\right)\left(1-\frac{1}{t\log t}-\frac{1}{(\log t)^2}\right)^{-\alpha}\times$$
$$\times\left\{1+\frac{\log\left(1-\frac{1}{t\log t}-\frac{1}{(\log t)^2}\right)}{\log_2 t}\right\}^{-\alpha\mu}$$

and

$$r(t) = \frac{\alpha\mu}{3t(\log t)^{\alpha+1}\log_2 t}\left(1+\frac{\mu}{\log_2 t}\right)^{\alpha-1}\left(1+\frac{\mu-1}{\log_2 t}\right)\left(1+\frac{1}{t\log t}\right)^{-\alpha}\times$$
$$\times\left\{1+\frac{\log\left(1+\frac{1}{t\log t}\right)}{\log_2 t}\right\}^{-\alpha\mu}$$

respectively, and μ is a positive constant. As in Example 4.2 it can be verified that all conditions of Theorem 1.1 are satisfied. Therefore, the equation (4.4) possesses a n-$\mathrm{RV}_{\log t}$ solution $x(t)$. One such solution is $x(t) = \log t(\log_2 t)^{\mu}$.

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