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A rigorous proof of a conjecture for the one-dimensional perturbed Gelfand problem from combustion theory

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1. Introduction

We study the global bifurcation curves and exact multiplicity of positive solutions for the two-point boundary value problem

$$\begin{cases} u''(x) + \lambda \exp\left(\frac{au}{a+u}\right) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases} \tag{1.1}$$

which is the one-dimensional case of a problem arising in the study of standard models of ignition in a context of thermal combustion, cf. [1, 23]. In (1.1), $\lambda > 0$ is the Frank–Kamenetskii parameter or ignition parameter, $a > 0$ is the activation energy parameter, u is the dimensionless temperature of the medium, and the reaction term

$$f_a(u) \equiv \exp\left(\frac{au}{a+u}\right)$$

is the temperature dependence obeying the simple Arrhenius reaction-rate law in irreversible chemical reaction kinetics, see, e.g. Boddington et al. [2]. Notice that nonlinearity $f_a \in C^\infty[0, \infty)$ satisfies $f_a(u)$, $f_a'(u) > 0$ for $u \geq 0$ and $a > 0$. In addition, $f_a''(u)$ is negative (concave) for $0 < a \leq 2$, and $f_a''(u)$ is positive and then negative (convex-concave) for $a > 2$.

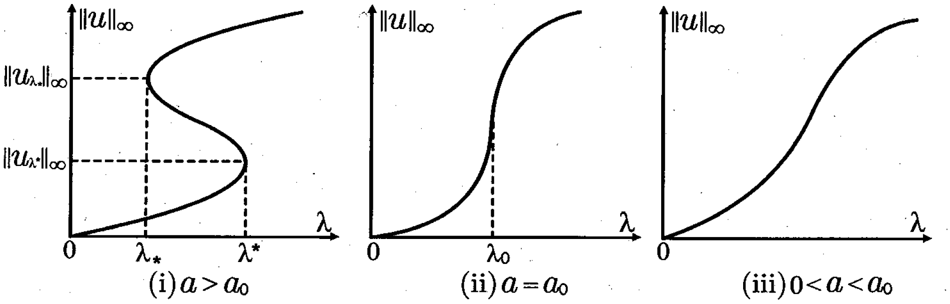

Figure 1.1: The global bifurcation of bifurcation curves S_a with varying $a > 0$.

For any $a > 0$, on the $(\lambda, \|u\|_\infty)$-plane, we study the shape and structure of bifurcation curves S_a of positive solutions of (1.1), defined by

$$S_a \equiv \{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.1)}\}.$$

We say that, on the $(\lambda, \|u_\lambda\|_\infty)$-plane, the bifurcation curve S_a is S-shaped if S_a has *exactly two* turning points at some points $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ and $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ where $\lambda_* < \lambda^*$ are two positive numbers such that

(i) $\|u_{\lambda^*}\|_\infty < \|u_{\lambda_*}\|_\infty$,

(ii) at $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ the bifurcation curve S_a turns to the left,

(iii) at $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ the bifurcation curve S_a turns to the right.

See Figure 1.1(i).

It is important to notice that, substituting $a = 1/\varepsilon$ (ε is the reciprocal activation energy parameter) into (1.1), we obviously obtain

$$\begin{cases} u''(x) + \lambda \exp\left(\frac{u}{1+\varepsilon u}\right) = 0, \quad -1 < x < 1, \\ u(-1) = u(1) = 0. \end{cases} \tag{1.2}$$

This problem (1.2) is the famous one-dimensional perturbed Gelfand problem, cf. [1, 6, 9]. It has been a long-standing conjecture ([5, 6, 15, 16, 20, 21]) about the shapes of evolutionary bifurcation curves and the exact multiplicity of positive solutions of (1.2) with varying $\varepsilon > 0$. This problem is obviously equivalent to study the shapes of evolutionary bifurcation curves and the exact multiplicity of positive solutions of (1.1) with varying $a > 0$. The conjecture for one-dimensional perturbed Gelfand problem (1.2) is stated in the form of (1.1) as follows.

Conjecture 1.1. *Consider (1.1) with varying $a > 0$. There exists a critical bifurcation value $a_0 > 4$ such that the following assertions (i)–(iii) hold:*

(i) (See Figure 1.1(i).) For $a > a_0$, the bifurcation curve S_a is S-shaped on the $(\lambda, \|u\|_\infty)$-plane. More precisely, there exist two positive numbers $\lambda_* < \lambda^*$ such that (1.1) has exactly three positive solutions for $\lambda_* < \lambda < \lambda^*$, exactly two positive solutions for $\lambda = \lambda_*$ and $\lambda = \lambda^*$, and exactly one positive solution for $0 < \lambda < \lambda_*$ and $\lambda > \lambda^*$. Furthermore, all positive solutions u_λ are nondegenerate except that u_{λ_*} and u_{λ^*} are degenerate.

(ii) (See Figure 1.1(ii).) For $a = a_0$, the bifurcation curve S_a of (1.1) is monotone increasing on the $(\lambda, \|u\|_\infty)$-plane. More precisely, for all $\lambda > 0$, (1.1) has exactly one positive solution u_λ. Furthermore, all positive solutions u_λ are nondegenerate except that u_{λ_0} is degenerate for some $\lambda_0 > 0$.

(iii) (See Figure 1.1(iii).) For $0 < a < a_0$, the bifurcation curve S_a of (1.1) is monotone increasing on the $(\lambda, \|u\|_\infty)$-plane. More precisely, for all $\lambda > 0$, (1.1) has exactly one positive solution u_λ. Furthermore, all positive solutions u_λ are nondegenerate.

Note that Korman, Li and Ouyang [16] gave a computer-assisted proof of this conjecture. Many researchers devoted to solve this conjecture since the 1980s. For $0 < a \leq 4$, it is easy to prove that the bifurcation curve S_a is monotone increasing and all positive solutions of (1.1) are nondegenerate, and hence $a_0 > 4$ under Conjecture 1.1, see e.g. [3]. In 1981, using quadratures, Brown et. al [3] showed that, for $a > \breve{a}_1 \approx 4.25$ for some $\breve{a}_1$, the bifurcation curve S_a is S-like shaped (i.e., S_a has *at least two* turning points). In 1985, using quadratures, Hastings and McLeod [8] proved that the bifurcation curve S_a is S-shaped for sufficiently large a. In 1994, again using quadratures, Wang [22] proved that the bifurcation curve S_a is S-shaped for $a \geq \breve{a}_2 \approx 4.4967$ for some $\breve{a}_2$, and hence

$a_0 < \breve{a}_2 \approx 4.4967$ under Conjecture 1.1. In 1999, Korman and Li [15] reduced the upper bound $\breve{a}_2$ of a_0 to $\breve{a}_3 \approx 4.35$ for some $\breve{a}_3$. They used tools from bifurcation theory, particularly the Crandall-Rabinowitz bifurcation theorem [4], and used quadratures. In 2011, again using quadratures, Hung and Wang [12] proved that the bifurcation curve S_a is S-shaped for $a \geq a^*$ where

$$a^* \equiv \inf \left\{ a > 4 : \int_0^{\frac{a(a-2)}{2}} \left[u f_a(u) - u^2 f_a'(u) \right] du < 0 \right\} \approx 4.166, \tag{1.3}$$

and hence $a_0 < a^* \approx 4.166$ under Conjecture 1.1. Very recently in 2015, using quadratures again together with Sturm's theorem, Huang and Wang [10] proved that the bifurcation curve S_a is S-shaped for $a \geq \tilde{a}$ where

$$\tilde{a} \equiv \inf \left\{ a > 4 : \int_0^{\frac{a(a-2)+a\sqrt{a(a-4)}}{2}} \left[u f_a(u) - u^2 f_a'(u) \right] du < 0 \right\} \approx 4.107, \tag{1.4}$$

and hence $a_0 < \tilde{a} \approx 4.107$ under Conjecture 1.1. So by above, we have the following theorem.

Theorem 1.2. *Consider (1.1) with $a > 0$. Then the following assertions (i) and (ii) hold:*

(i) For $0 < a \leq 4$, the bifurcation curve S_a is monotone increasing on the $(\lambda, \|u\|_\infty)$-plane. Furthermore, all positive solutions u_λ are nondegenerate.

(ii) For $a \geq \tilde{a} \approx 4.107$, the bifurcation curve S_a is S-shaped on the $(\lambda, \|u\|_\infty)$-plane.

Write $\exp\left(\frac{au}{a+u}\right) = \exp\left(\frac{u}{1+\varepsilon u}\right)$ with $\varepsilon = 1/a$. Thus, for fixed u, $\exp\left(\frac{u}{1+\varepsilon u}\right) \to \exp(u)$ as $\varepsilon \to 0^+$ (i.e., $a \to \infty$). In that case problem (1.1) and problem (1.2) reduce to the one-dimensional Gelfand problem (or called Liouville-Gelfand problem)

$$\begin{cases} u''(x) + \lambda \exp(u) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0. \end{cases} \tag{1.5}$$

In 1853, Liouville [18] first studied (1.5) and found an explicit solution. In 1959, Gelfand [7] observed that problem (1.5) can be solved by integration exactly, with positive solution

$$u_\lambda(x) = \alpha + \ln\left(\operatorname{sech}^2\left(\frac{\sqrt{2\lambda}}{2} x e^{\alpha/2}\right)\right),$$

where $\alpha \equiv \|u_\lambda\|_\infty = u_\lambda(0)$. This enabled him to deduce that (1.5) has either two, one, or zero solutions, depending on λ, see [9, p. 208] and [1, p. 34].

Define

$$S_\infty \equiv \{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.5)}\}.$$

Then by the quadrature method (time-map method) and the fact that the nonlinearity $\exp(u)$ has an *elementary* antiderivative $\exp(u)$, one finds that

$$\lambda = \lambda(\alpha) = \left[\frac{1}{\sqrt{2}} \int_0^\alpha \frac{1}{\sqrt{e^\alpha - e^u}} du\right]^2 = \frac{1}{2e^\alpha}\left[\ln\left(2e^\alpha + 2\sqrt{e^\alpha(e^\alpha - 1)} - 1\right)\right]^2 \quad \text{for } \alpha > 0 \tag{1.6}$$

after some simple computation, see e.g. [14, Eq. (5)]. It is easy to show that $\lim_{\alpha \to 0^+} \lambda(\alpha) = \lim_{\alpha \to \infty} \lambda(\alpha) = 0$ and $\lambda(\alpha)$ has exact one critical (maximum) value

$$\lambda_\infty \equiv \max_{\alpha \in (0,\infty)} \frac{1}{2e^\alpha}\left[\ln\left(2e^\alpha + 2\sqrt{e^\alpha(e^\alpha - 1)} - 1\right)\right]^2 \approx 0.878 \tag{1.7}$$

at some critical point

$$\alpha_\infty = \ln\left(\frac{2+\lambda_\infty}{\lambda_\infty}\right) \approx 1.187 \tag{1.8}$$

after some simple computation; we omit the proofs. Thus the bifurcation curve S_∞ is a $\supset$-shaped curve on the $(\lambda, \|u\|_\infty)$-plane and the next theorem follows.

Theorem 1.3. *Consider (1.5). There exist a critical (maximal) value $\lambda_\infty \approx 0.878$ and a critical point $\alpha_\infty = \ln\left(\frac{2+\lambda_\infty}{\lambda_\infty}\right) \approx 1.187$ for $\lambda(\alpha)$ in (1.6) such that the bifurcation curve S_∞ is a $\supset$-shaped curve on the $(\lambda, \|u\|_\infty)$-plane and satisfies (1.6)–(1.8). More precisely, (1.5) has exactly two positive solutions for $0 < \lambda < \lambda_\infty$, exactly one positive solution for $\lambda = \lambda_\infty$, and no positive solution for $\lambda > \lambda_\infty$. In addition, $\|u_{\lambda_\infty}\|_\infty = \alpha_\infty$.*

2. Main result

Theorem 2.1. *Consider (1.1) with varying $a > 0$. There exists a critical bifurcation value $a_0 \approx 4.069$ satisfying $4 < a_0 < \tilde{a} \approx 4.107$ such that the following assertions (i)–(iii) hold:*

(i) (See Figure 1.1(i).) For $a > a_0$, the bifurcation curve S_a is S-shaped on the $(\lambda, \|u\|_\infty)$-plane. More precisely, there exist two positive numbers $\lambda_* < \lambda^*$ such that (1.1) has:

- *(a) exactly three positive solutions w_λ, u_λ, v_λ with $w_\lambda < u_\lambda < v_\lambda$ for $\lambda_* < \lambda < \lambda^*$,*
- *(b) exactly two positive solutions w_{λ_*}, u_{λ_*} with $w_{\lambda_*} < u_{\lambda_*}$ for $\lambda = \lambda_*$, and exactly two positive solutions u_{λ^*}, v_{λ^*} with $u_{\lambda^*} < v_{\lambda^*}$ for $\lambda = \lambda^*$,*
- *(c) exactly one positive solution w_λ for $0 < \lambda < \lambda_*$, and exactly one positive solution v_λ for $\lambda > \lambda^*$.*

Furthermore,

- *(d) $\lim_{\lambda\to 0^+} \|w_\lambda\|_\infty = 0$ and $\lim_{\lambda\to\infty} \|v_\lambda\|_\infty = \infty$.*
- *(e) All positive solutions u_λ are nondegenerate except that u_{λ_*} and u_{λ^*} are degenerate.*
- *(f) $\|u_{\lambda^*}\|_\infty < \|u_{\lambda_*}\|_\infty$, $\lim_{a\to\infty} \|u_{\lambda_*}\|_\infty = \infty$ and $\lim_{a\to\infty} \|u_{\lambda^*}\|_\infty = \alpha_\infty \approx 1.187$.*

(ii) (See Figure 1.1(ii).) If $a = a_0$, then the bifurcation curve S_{a_0} is monotone increasing on the $(\lambda, \|u\|_\infty)$-plane. More precisely, for all $\lambda > 0$, (1.1) has exactly one positive solution u_λ satisfying $\lim_{\lambda\to 0^+} \|u_\lambda\|_\infty = 0$ and $\lim_{\lambda\to\infty} \|u_\lambda\|_\infty = \infty$. Furthermore, all positive solutions u_λ are nondegenerate except that u_{λ_0} is a degenerate solution for some $\lambda = \lambda_0 > 0$.

(iii) (See Figure 1.1(iii).) If $0 < a < a_0$, then the bifurcation curve S_a is monotone increasing on the $(\lambda, \|u\|_\infty)$-plane. More precisely, for all $\lambda > 0$, (1.1) has exactly one positive solution u_λ satisfying $\lim_{\lambda\to 0^+} \|u_\lambda\|_\infty = 0$ and $\lim_{\lambda\to\infty} \|u_\lambda\|_\infty = \infty$. Furthermore, all positive solutions u_λ are nondegenerate.

3. Lemmas

To prove Theorem 2.1, we develop some new time-map techniques and apply Sturm's theorem stated in [11]. The time-map formula which we apply to study (1.1) takes the form as follows:

$$\sqrt{\lambda} = \frac{1}{\sqrt{2}} \int_0^\alpha [F_a(\alpha) - F_a(u)]^{-1/2}\, du \equiv T_a(\alpha) \quad \text{for } 0 < \alpha < \infty, \tag{3.1}$$

where $F_a(u) \equiv \int_0^u f_a(t)dt$. So positive solutions u of (1.1) correspond to

$$\|u\|_\infty = \alpha \text{ and } T_a(\alpha) = \sqrt{\lambda}.$$

Thus, studying the exact number of positive solutions of (1.1) is equivalent to studying the shape of the time map $T_a(\alpha)$ on $(0,\infty)$, cf. [10]. In addition, proving that the bifurcation curve S_a is S-shaped on the $(\lambda, \|u\|_\infty)$-plane is equivalent to proving that $T_a(\alpha)$ has *exactly two* critical points, a local maximum and a local minimum, on $(0,\infty)$. We recall that a positive solution u_λ of (1.1) is *degenerate* if $T_a'(\|u_\lambda\|_\infty) = 0$ and is *nondegenerate* if $T_a'(\|u_\lambda\|_\infty) \neq 0$.

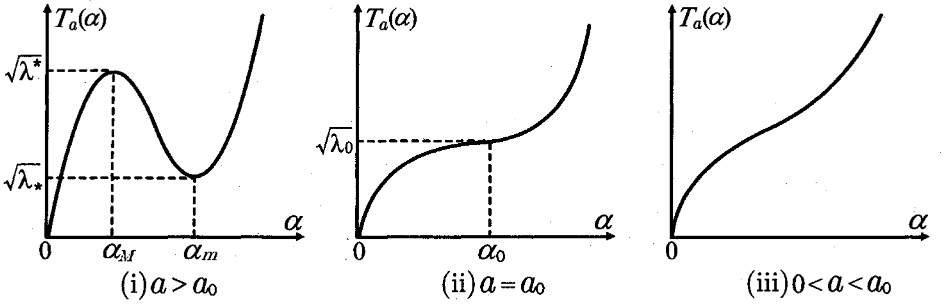

Figure 3.1: Graphs of $T_a(\alpha)$ $(= \sqrt{\lambda})$ on $(0,\infty)$ with varying $a > 0$, cf. Figure 1.1.

By Theorem 1.2, we see that $T_a(\alpha)$ is strictly increasing and has no critical points on $(0,\infty)$ for $0 < a \leq 4$, and $T_a(\alpha)$ has exactly two critical points, a local maximum and a local minimum, on $(0,\infty)$ for $a \geq \tilde{a}$. So by above, to prove Theorem 2.1(i), (ii) and (iii) which solves Conjecture 1.1, it is sufficient to prove that there exists a number $a_0 \approx 4.069$ satisfying $4 < a_0 < \tilde{a} \approx 4.107$ such that the following parts (M1), (M2) and (M3) hold, respectively:

(M1) (See Figure 3.1(i).) For $a > a_0$, on $(0,\infty)$, $T_a(\alpha)$ has *exactly two* critical points, a local maximum at some $\alpha_M(a)$ and a local minimum at some $\alpha_m(a)$ $(> \alpha_M(a))$, satisfying $\lambda^* = T_a^2(\alpha_M(a))$, $\lambda_* = T_a^2(\alpha_m(a))$. In addition, $\lim_{\alpha\to 0^+} T_a(\alpha) = 0$, $\lim_{\alpha\to\infty} T_a(\alpha) = \infty$, $\lim_{a\to\infty} \alpha_m(a) = \alpha_\infty$, and $\lim_{a\to\infty} \alpha_M(a) = \infty$ where α_∞ is defined in (1.8).

(M2) (See Figure 3.1(ii).) For $a = a_0$, $T_{a_0}(\alpha)$ is a strictly increasing function on $(0,\infty)$ and has *exactly one* critical point at some α_0 on $(0,\infty)$. Moreover, $T_{a_0}'(\alpha_0) = 0$ and $T_{a_0}'(\alpha) > 0$ for $\alpha \in (0,\infty) \setminus \{\alpha_0\}$. In addition, $\lim_{\alpha\to 0^+} T_{a_0}(\alpha) = 0$ and $\lim_{\alpha\to\infty} T_{a_0}(\alpha) = \infty$,

(M3) (See Figure 3.1(iii).) For $0 < a < a_0$, $T_a(\alpha)$ is a strictly increasing function and has no critical points on $(0,\infty)$. Moreover, $T_a'(\alpha) > 0$ on $(0,\infty)$. In addition, $\lim_{\alpha\to 0^+} T_a(\alpha) = 0$ and $\lim_{\alpha\to\infty} T_a(\alpha) = \infty$.

To prove parts (M1)–(M3), we need the following Lemmas 3.1–3.3.

Lemma 3.1. *Consider (1.1) with $a > 0$. Then $\lim_{\alpha\to 0^+} T_a(\alpha) = 0$ and $\lim_{\alpha\to\infty} T_a(\alpha) = \infty$.*

Lemma 3.2. *Consider (1.1) with $a > 0$. The set Ω defined by*

$$\Omega = \left\{ \begin{array}{l} a > 0 : T_a(\alpha) \text{ has exactly two critical points } \alpha_M(a) < \alpha_m(a), \\ \text{a local maximum and a local minimum, on } (0,\infty) \end{array} \right\}$$

is nonempty, open and connected. Moreover, $\Omega = (a_0, \infty)$ *for some number* a_0 $(\approx 4.069) \in (4, \tilde{a})$ *where* $\tilde{a}$ *is defined in (1.4).*

Lemma 3.3. *Consider (1.1) with* $4 < a \le a_0$*. The following assertions (i)–(ii) hold:*

(i) *For* $4 < a < a_0$, $T_a'(\alpha) > 0$ *for* $\alpha > 0$.

(ii) *For* $a = a_0$*, there exists* $\alpha_0 \in (\gamma(a_0), \kappa(a_0))$ *such that* $T_{a_0}'(\alpha_0) = 0$ *and* $T_{a_0}'(\alpha) > 0$ *for* $\alpha > 0$ *and* $\alpha \neq \alpha_0$*, where* $\kappa(a_0)$ *is defined in Lemma 3.5 stated below. Moreover,* $4 < a_0 < \tilde{a} \approx 4.107$.

First of all, Lemma 3.1 follows easily from [17, Theorems 2.6 and 2.9]. Before proving Lemmas 3.2 and 3.3, we need to investigate some properties of $T_a(\alpha)$ on $(0, \infty)$. In fact, we apply the next Lemmas 3.5–3.7 and 3.12 to prove Lemma 3.2, and apply Lemma 3.2 and the next Lemmas 3.5–3.7, 3.11, 3.14 and 3.16 to prove Lemma 3.3.

Next, we divide this section into three subsections.

3.1. Basic functions estimates

We first compute and obtain that, for $u > 0$,

$$f_a'(u) = \frac{1}{(a+u)^2} a^2 f_a(u) > 0, \tag{3.2}$$

$$f_a''(u) = -\frac{2a^2\left[u - a(a-2)/2\right]}{(a+u)^4} \exp\left(\frac{au}{a+u}\right) \begin{cases} > 0 & \text{for } u < \gamma(a), \\ = 0 & \text{for } u = \gamma(a) \equiv \frac{a(a-2)}{2}, \\ < 0 & \text{for } u > \gamma(a). \end{cases} \tag{3.3}$$

For the sake of convenience, we let $\gamma = \gamma(a)$ for $a > 2$. We let

$$\theta_a(u) \equiv 2F_a(u) - uf_a(u) = 2\int_0^u f_a(t)dt - uf_a(u). \tag{3.4}$$

Lemma 3.4(ii) follows easily from [10, Lemma 2.1].

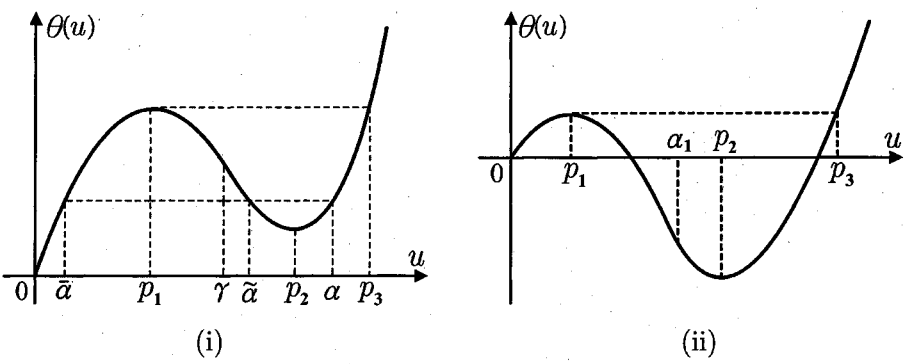

Figure 3.2: Graphs of $\theta_a(u)$ on $[0, \infty)$. (i) $\theta_a(p_2(a)) \ge 0$. (ii) $\theta_a(p_2(a)) < 0$.

Lemma 3.4. *Consider (1.1) with $a > 4$. Define positive numbers*

$$p_1(a) \equiv \frac{a\,(a-2) - a\sqrt{a(a-4)}}{2} < p_2(a) \equiv \frac{a\,(a-2) + a\sqrt{a(a-4)}}{2}. \tag{3.5}$$

Then

$$0 < p_1(a) < \gamma(a) = \frac{a(a-2)}{2} < p_2(a), \tag{3.6}$$

and the following assertions (i)–(ii) hold:

(i) (See Figure 3.2.) $\theta_a(0) = 0$, $\lim_{u\to\infty} \theta_a(u) = \infty$, *and*

$$\theta_a'(u) \begin{cases} > 0 & \text{for } u \in (0, p_1(a)) \cup (p_2(a), \infty)\,, \\ = 0 & \text{for } u \in \{p_1(a), p_2(a)\}\,, \\ < 0 & \text{for } u \in (p_1(a), p_2(a))\,. \end{cases} \tag{3.7}$$

Furthermore, there exists a unique number $p_3(a) > p_2(a)$ such that $\theta_a(p_3(a)) = \theta_a(p_1(a))$. In addition, $p_3(a) < 12$ for $4 < a \le \tilde{a} \approx 4.107$.

(ii) (See Figure 3.2(i).) For $4 < a \le \frac{417}{100}$ and $\alpha \in [\gamma(a), p_3(a))$, $\theta_a(u) > 0$ for $u > 0$ and there exist two numbers $\bar{\alpha} \in (0, p_1(a))$ and $\tilde{\alpha} \in (p_1(a), p_2(a)]$ such that $\theta_a(\bar{\alpha}) = \theta_a(\tilde{\alpha}) = \theta_a(\alpha)$. (Notes that we choose $\tilde{\alpha} = \alpha$ if $\alpha \in [\gamma(a), p_2(a)]$.)

Proof of Lemma 3.4. Since Lemma 3.4(ii) follows easily from [10, Lemma 2.1], it is sufficient to prove Lemma 3.4(i). It is easy to see that (3.6) holds for $a > 4$. Secondly, we compute and observe that

$$\begin{aligned} \theta_a'(u) &= f_a(u) - u f_a'(u) \\ &= \frac{\left[u^2 - a(a-2)u + a^2\right]}{(a+u)^2} f_a(u) \begin{cases} > 0 & \text{for } u \in (0, p_1(a)) \cup (p_2(a), \infty)\,, \\ = 0 & \text{for } u \in \{p_1(a), p_2(a)\}\,, \\ < 0 & \text{for } u \in (p_1(a), p_2(a))\,. \end{cases} \end{aligned} \tag{3.8}$$

Then we simply prove that $p_3(a) < 12$ for $4 < a \le \tilde{a}$ in part (i) because the remainder parts follow easily from [3, p. 482, lines 29–30] and [12, p. 228]. Since $\tilde{a}$ (≈ 4.107) $< 4.108 = \frac{1027}{250}$ by (1.4), it is sufficient to prove that $p_3(a) < 12$ for $4 < a < \frac{1027}{250}$. Clearly,

$$p_2'(a) = \frac{(a-1)\sqrt{a^2-4a} + a\,(a-3)}{\sqrt{a^2-4a}} > 0 \quad \text{for } a > 4. \tag{3.9}$$

So we see that

$$p_1(a) < p_2(a) < p_2\left(\frac{1027}{250}\right) = \frac{3081\sqrt{3081}}{125000} + \frac{541229}{125000} \; (\approx 5.698) < 12 \quad \text{for } 4 < a < \frac{1027}{250}. \tag{3.10}$$

We assert that

$$\theta_a(12) - \theta_a(p_3(a)) > 0 \quad \text{for } 4 < a < \frac{1027}{250}. \tag{3.11}$$

So by (3.8), $p_3(a) < 12$ for $4 < a < \frac{1027}{250}$. It implies that $p_3(a) < 12$ for $4 < a \le \tilde{a}$.

Next, we prove assertion (3.11). We observe that

$$\frac{\alpha^3 f_a(\alpha)}{(a+\alpha)^2} - \frac{u^3 f_a(u)}{(a+u)^2} = \int_u^\alpha \left[\frac{d}{dt}\frac{t^3 f_a(t)}{(a+t)^2}\right] dt = \int_u^\alpha \frac{\left[t^2 + a\,(a+4)\,t + 3a^2\right] t^2 f_a(t)}{(a+t)^4} dt. \tag{3.12}$$

Since $t^2 - 16t - 16 < 0$ for $0 \le t \le 12$, and by Lemma 3.4(i), (3.10) and (3.12), we compute and obtain that, for $4 < a < \frac{1027}{250}$,

$$\begin{aligned}
\frac{\partial}{\partial a}\left[\theta_a(12) - \theta_a(p_1(a))\right] &= \frac{\partial}{\partial a}\theta_a(12) - \frac{\partial}{\partial a}\theta_a(u)\Big|_{u=p_1(a)} \\
&= 2\int_{p_1(a)}^{12} \frac{t^2 f_a(t)}{(a+t)^2}dt + \frac{p_1^3(a) f_a(p_1(a))}{[a+p_1(a)]^2} - \frac{(12)^3 f_a(12)}{(a+12)^2} \\
&= \int_{p_1(a)}^{12} \frac{t^2 f_a(t)}{(a+t)^4}\left(t^2 - a^2 t - a^2\right) dt \quad (\text{since } a > 4) \\
&< \int_{p_1(a)}^{12} \frac{t^2 f_a(t)}{(a+t)^4}\left(t^2 - 16t - 16\right) dt < 0.
\end{aligned} \tag{3.13}$$

So by (3.4), (3.8) and (3.10), we compute and obtain that, for $4 < a < \frac{1027}{250}$,

$$\begin{aligned}
&\theta_a(12) - \theta_a(p_1(a)) \\
=\ & \int_{p_1(a)}^{12} \theta_a'(u)du \\
=\ & \int_{p_1(a)}^{12} \frac{\left[u^2 - a(a-2)u + a^2\right]}{(a+u)^2} f_a(u)du \\
\ge\ & f_a(p_2(a))\left\{\int_{p_1(a)}^{p_2(a)} \frac{u^2 - a(a-2)u + a^2}{(a+u)^2}du + \int_{p_2(a)}^{12} \frac{\left[u^2 - a(a-2)u + a^2\right]}{(a+u)^2}du\right\} \\
=\ & f_a(p_2(a))\int_{p_1(a)}^{12} \frac{u^2 - a\,(a-2)\,u + a^2}{(a+t)^2}du = \frac{f_a(p_2(a))}{(a+12)\,[a+p_1(a)]}\Phi_a,
\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}
\Phi_a \equiv\ & [12 - p_1(a)]\left\{a^3 + a^2 + [12 + p_1(a)]\,a + 12p_1(a)\right\} \\
& + a^2(a+12)\,[a + p_1(a)] \ln\left(\frac{a + p_1(a)}{a+12}\right).
\end{aligned}$$

Since $\Phi_{\frac{1027}{250}}$ (≈ 73.2) > 0, and by (3.13) and (3.14), we see that, for $4 < a < \frac{1027}{250}$,

$$\begin{aligned}
\theta_a(12) - \theta_a(p_3(a)) &= \theta_a(12) - \theta_a(p_1(a)) \ge \theta_{\frac{1027}{250}}(12) - \theta_{\frac{1027}{250}}\left(p_1\left(\frac{1027}{250}\right)\right) \\
&\ge \frac{f_{\frac{1027}{250}}\left(p_2\left(\frac{1027}{250}\right)\right)}{\left(\frac{1027}{250} + 12\right)\left[\frac{1027}{250} + p_1\left(\frac{1027}{250}\right)\right]}\Phi_{\frac{1027}{250}} > 0.
\end{aligned}$$

So assertion (3.11) holds. The proof of Lemma 3.4 is complete. ■

Lemma 3.5. *Consider (1.1) with $a > 4$. Then the following assertions (i)–(ii) hold:*

(i) There exists a continuous function $\kappa(a) \in (\gamma, \infty)$ of a on $(4, \infty)$ such that

$$G_a(u) \equiv -\int_0^u t^2\theta_a''(t)dt \begin{cases} > 0 & \text{for } 0 < u < \kappa(a), \\ = 0 & \text{for } u = \kappa(a), \\ < 0 & \text{for } u > \kappa(a). \end{cases} \tag{3.15}$$

Furthermore, $\kappa(a)$ is a strictly increasing function of a on $(4, \frac{417}{100}]$ and $\kappa(a) < 8$ for $4 < a \le \frac{417}{100}$.

(ii) There exists a strictly decreasing, continuous function $\rho(a) \in (0, \kappa(a)]$ of $a \geq \tilde{a}$ such that

$$H_a(u) \equiv \int_0^u t\theta_a'(t)dt \begin{cases} > 0 & \text{for } 0 < u < \rho(a), \\ = 0 & \text{for } u = \rho(a), \\ < 0 & \text{for } \rho(a) < u < p_2(a). \end{cases} \tag{3.16}$$

Furthermore,

$$\gamma < \rho(a) = \kappa(a) = p_2(a) \quad \text{for } a = \tilde{a}, \tag{3.17}$$

$$\gamma < \rho(a) < \kappa(a) < p_2(a) \quad \text{for } \tilde{a} < a < a^*, \tag{3.18}$$

$$\gamma = \rho(a) < \kappa(a) < p_2(a) \quad \text{for } a = a^*, \tag{3.19}$$

$$p_1(a) < \rho(a) < \gamma < \kappa(a) < p_2(a) \quad \text{for } a > a^*. \tag{3.20}$$

Proof of Lemma 3.5. We divide this proof into the next Steps 1–2.

Step1. We prove assertion (i). Clearly, $G_a(0) = 0$. Since $\lim_{u\to\infty} f_a(u) = \exp(a)$ and by (3.3), we compute and find that, for $a > 4$,

$$G_a'(u) = -u^2\theta_a''(u) = u^3 f_a''(u) \begin{cases} > 0 & \text{for } 0 < u < \gamma, \\ = 0 & \text{for } u = \gamma, \\ < 0 & \text{for } u > \gamma, \end{cases}$$

$$\begin{aligned} \lim_{u\to\infty} G_a(u) &= \lim_{u\to\infty} \left[\int_0^\gamma t^3 f_a''(t)dt + \int_\gamma^u t^3 f_a''(t)dt \right] < \gamma^3 \lim_{u\to\infty} \int_0^u f_a''(t)dt \\ &= \gamma^3 \left[\lim_{u\to\infty} f_a'(u) - f_a'(0) \right] = \gamma^3 \left[\lim_{u\to\infty} \frac{a^2 f_a(u)}{(a+u)^2} - f_a'(0) \right] \\ &= -\gamma^3 f_a'(0) < 0. \end{aligned}$$

So for any $a > 4$, there exists a unique number $\kappa(a) > \gamma$ such that (3.15) holds. Since $G_a'(\kappa(a)) = [\kappa(a)]^3 f_a''(\kappa(a)) < 0$ by (3.3), and by the Implicit Function Theorem, $\kappa(a)$ is a continuous function of a on $(4, \infty)$. In addition, it is easy to observe that $\gamma'(a) = a - 1 > 0$ for $a > 4$, and

$$\frac{\partial}{\partial t} \frac{t^2 f_a(t)}{(a+t)^3} = \frac{\left[-t^2 + (a^2 + a)\, t + 2a^2\right] t f_a(t)}{(a+t)^5} > 0 \quad \text{for } 0 \leq t \leq 8 \text{ and } a > 4.$$

Thus $0 < \gamma(a) < \gamma(5) = 7.5 < 8$ for $4 < a \leq \frac{417}{100}$. Then we compute that

$$\begin{aligned} G_a(8) &= -\int_0^8 t^2\theta_a''(t)dt = \int_0^8 t^3 f_a''(t)dt = 2a^2 \int_0^8 \left[\frac{t^2 f_a(t)}{(a+t)^3} \left(\frac{\gamma t - t^2}{a+t} \right) \right] dt \\ &\leq \frac{2a^2\gamma^2 f_a(\gamma)}{(a+\gamma)^3} \left[\int_0^\gamma \left(\frac{\gamma t - t^2}{a+t} \right) dt + \int_\gamma^8 \left(\frac{\gamma t - t^2}{a+t} \right) dt \right] \\ &= \frac{2a^2\gamma^2 f_a(\gamma)}{(a+\gamma)^3} \int_0^8 \left(\frac{\gamma t - t^2}{a+t} \right) dt = \frac{a^5\gamma^2 f_a(\gamma)}{(a+\gamma)^3} \left[\frac{8a^2 - 64}{a^3} + \ln\left(\frac{a}{a+8} \right) \right] \\ &= \frac{a^5\gamma^2 f_a(\gamma)}{(a+\gamma)^3} \Psi_a, \end{aligned} \tag{3.21}$$

where $\Psi_a \equiv \left(8a^2 - 64\right)/a^3 + \ln\left(a/\left(a+8\right)\right)$. Since

$$\frac{d}{da}\Psi_a = \frac{64\left(-a^2 + 3a + 24\right)}{a^4\left(a+8\right)} > 0 \quad \text{for } 4 < a \leq \frac{417}{100},$$

we see that $\Psi_a \leq \Psi_{\frac{417}{100}}$ (≈ -0.03) < 0 for $4 < a \leq \frac{417}{100}$. So by (3.21), $G_a(8) < 0$ for $4 < a \leq \frac{417}{100}$. It implies that $\kappa(a) < 8$ for $4 < a \leq \frac{417}{100}$.

In addition, by (3.3), we compute and find that, for $4 < a \leq \frac{417}{100}$ and $0 \leq u \leq 8$,

$$\begin{aligned}
\frac{\partial}{\partial a} u^3 f_a''(u) &= \frac{u^3 a f_a(u)}{(a+u)^6}\left[(-4-2a)u^3 + (a^3+2a^2-6a)u^2 + 4a^3u + 2a^3\right] \\
&\geq \frac{u^3 a f_a(u)}{(a+u)^6}[(-4-2\times\frac{417}{100})u^3 + (4^3+2\times 4^2 - 6\times\frac{417}{100})u^2 + 4\times 4^3 u + 2\times 4^3] \\
&= \frac{u^3 a f_a(u)}{50\,(a+u)^6}\left[2u\,(309u+800)\,(8-u) + u^3 + 205u^2 + 6400\right] \\
&> 0.
\end{aligned}$$

So for $4 < a_1 < a_2 \leq \frac{417}{100}$, $G_{a_1}(\kappa(a_2)) = \int_0^{\kappa(a_2)} t^3 f_{a_1}''(t)dt < \int_0^{\kappa(a_2)} t^3 f_{a_2}''(t)dt = G_{a_2}(\kappa(a_2)) = 0$. It follows that $\kappa(a_1) < \kappa(a_2)$ for $4 < a_1 < a_2 \leq \frac{417}{100}$. Therefore, $\kappa(a)$ is a strictly increasing function of $a \in (4, \frac{417}{100}]$.

Step 2. We prove assertion (ii). It is easy to observe that $t^2 - a^2 t - a^2 < 0$ for $0 \leq t \leq p_2(a) < a\left(a+\sqrt{a^2+4}\right)/2$ and $a > 4$. So we further observe that

$$\frac{\partial}{\partial a} H_a(u) = \int_0^u \left(t^2 - a^2 t - a^2\right)\frac{t^3 f_a(t)}{(a+t)^4}dt < 0 \quad \text{for } 0 < u \leq p_2(a) \text{ and } a > 4. \tag{3.22}$$

Clearly, $H_a(0) = 0$. By Lemma 3.4(i), we see that, for $a > 4$,

$$H_a'(u) = u\theta_a'(u)\begin{cases} > 0 & \text{for } u \in (0, p_1(a)) \cup (p_2(u), \infty), \\ = 0 & \text{for } u \in \{p_1(a), p_2(a)\}, \\ < 0 & \text{for } u \in (p_1(a), p_2(a)). \end{cases} \tag{3.23}$$

So by (3.22) and (3.23), we find that

$$\frac{\partial}{\partial a} H_a(p_2(a)) = \left.\frac{\partial}{\partial a} H_a(u)\right|_{u=p_2(a)} + H_a'(p_2(a))p_2'(a) = \left.\frac{\partial}{\partial a} H_a(u)\right|_{u=p_2(a)} < 0. \tag{3.24}$$

Since $H_a(p_2(a)) = 0$ for $a = \tilde{a}$ by (1.4), and by (3.24), we observe that $H(p_2(a)) < 0$ for $a > \tilde{a}$. So by (3.23), there exists a unique number $\rho(a) \in (p_1(a), p_2(a)]$ such that (3.16) holds. Furthermore, $\rho(a) = p_2(a)$ for $a = \tilde{a}$ and $p_1(a) < \rho(a) < p_2(a)$ for $a > \tilde{a}$. In addition, by integration by parts, we have

$$H_a(u) = \frac{1}{2}\left[u^2\theta_a'(u) + G_a(u)\right]. \tag{3.25}$$

By (3.23) and (3.25),

$$G_a(\rho(a)) = -\left[\rho(a)\right]^2\theta_a'(\rho(a))\begin{cases} = 0 & \text{for } a = \tilde{a}, \\ > 0 & \text{for } a > \tilde{a}, \end{cases} \quad G_a(p_2(a)) = 2H_a(p_2(a))\begin{cases} = 0 & \text{for } a = \tilde{a}, \\ < 0 & \text{for } a > \tilde{a}. \end{cases}$$

By (3.15), we see that $\rho(a) = p_2(a) = \kappa(a)$ for $a = \tilde{a}$ and $\rho(a) < \kappa(a) < p_2(a)$ for $a > \tilde{a}$. Since $\gamma'(a) > 0$ for $a > 4$, and by (3.6), (3.22) and (3.23), we find that

$$\frac{\partial}{\partial a} H_a(\gamma) = \left.\frac{\partial}{\partial a} H_a(u)\right|_{u=\gamma} + H_a'(\gamma)\gamma'(a) < \left.\frac{\partial}{\partial a} H_a(u)\right|_{u=\gamma} < 0 \quad \text{for } a > 4. \tag{3.26}$$

So we observe that $H_a(\gamma)$ is a strictly decreasing continuous function of $a > 4$. By (1.3), we see that $a^* = \inf\{a > 4 : H_a(\gamma) < 0\}$. Thus $H_a(\gamma) = 0$ for $a = a^*$ and $H_a(\gamma) < 0$ for $a > a^*$. It follows that $\rho(a) = \gamma$ for $a = a^*$, and $\rho(a) < \gamma$ for $a > a^*$. Thus, by above discussion, (3.17)–(3.20) hold. Next, we prove that $\rho(a)$ is a strictly decreasing function of $a \geq \tilde{a}$. Let numbers $a_2 > a_1 \geq \tilde{a}$ be given. Then $\rho(a_1) \leq p_2(a_1) < p_2(a_2)$ by (3.9). So by (3.22), we observe that $H_{a_2}(\rho(a_1)) < H_{a_1}(\rho(a_1)) = 0$. It follows that $\rho(a_2) < \rho(a_1)$. It implies that $\rho(a)$ is strictly decreasing for $a \geq \tilde{a}$.

Finally, the proof of the fact that $\rho(a)$ is a continuous function of $a \geq \tilde{a}$ is omitted.

The proof of Lemma 3.5 is complete. ■

3.2. Estimates of T_a and its derivatives

For $T_a(\alpha)$ in (3.1), we compute that

$$T_a'(\alpha) = \frac{1}{2\sqrt{2}\alpha}\int_0^\alpha \frac{\theta_a(\alpha) - \theta_a(u)}{[F_a(\alpha) - F_a(u)]^{3/2}}du \quad \text{for } 0 < \alpha < \infty, \tag{3.27}$$

where $\theta_a(u)$ is defined by (3.4). The next Lemma 3.6(i) follows easily from [17, Lemma 3.2] and (3.3). And the proof of next Lemma 3.6(ii) is easy but tedious and hence we omit it.

Lemma 3.6. *Consider (1.1) with fixed $a > 0$. The following assertions (i)–(ii) hold:*

(i) For any fixed $a > 4$, either $T_a(\alpha)$ is strictly increasing on $(0, \gamma]$, or $T_a(\alpha)$ is strictly increasing and then strictly decreasing on $(0, \gamma]$.

(ii) For any fixed $\alpha > 0$, $T_a'(\alpha)$ is a continuously differentiable function of $a > 0$.

Lemma 3.7. *Consider (1.1) with $a > 4$. Then*

$$T_a''(\alpha) + \frac{2}{\alpha}T_a'(\alpha) > 0 \quad \text{for } \alpha \geq \kappa(a).$$

Proof of Lemma 3.7. By (3.27), we compute that

$$\begin{aligned} T_a''(\alpha) + \frac{2}{\alpha}T_a'(\alpha) &= \frac{1}{2\sqrt{2}\alpha^2}\int_0^\alpha \frac{\frac{3}{2}[\theta_a(\alpha) - \theta_a(u)]^2 + [F_a(\alpha) - F_a(u)][\phi_a(\alpha) - \phi_a(u)]}{[F_a(\alpha) - F_a(u)]^{5/2}}du \\ &\geq \frac{1}{2\sqrt{2}\alpha^2}\int_0^\alpha \frac{\phi_a(\alpha) - \phi_a(u)}{[F_a(\alpha) - F_a(u)]^{3/2}}du, \end{aligned} \tag{3.28}$$

where $\phi_a(u) \equiv u\theta_a'(u) - \theta_a(u)$, see [12, (3.12)]. We obtain that, by (3.3),

$$\phi_a(0) = 0 \quad \text{and} \quad \phi_a'(u) = u\theta_a''(u) = -u^2 f_a''(u) \begin{cases} < 0 & \text{for } 0 < u < \gamma, \\ = 0 & \text{for } u = \gamma, \\ > 0 & \text{for } u > \gamma. \end{cases} \tag{3.29}$$

We note that $\kappa(a) > \gamma$ for $a > 4$ by Lemma 3.5(i). We fix $\alpha \geq \kappa(a)$. If $\phi_a(\alpha) \geq 0$, by (3.29), we see that $\phi_a(\alpha) - \phi_a(u) > 0$ for $0 < u < \alpha$, and hence $T_a''(\alpha) + \frac{2}{\alpha}T_a'(\alpha) > 0$ by (3.28). While if $\phi_a(\alpha) < 0$, since $\alpha \geq \kappa(a) > \gamma$, there exists $\xi_\alpha \in (0, \gamma)$ such that $\phi_a(\xi_\alpha) = \phi_a(\alpha)$. See Figure 3.3. So by [12, (3.15)] and Lemma 3.5(i),

$$T_a''(\alpha) + \frac{2}{\alpha}T_a'(\alpha) > \frac{-1}{2\sqrt{2}\alpha^2[F_a(\alpha) - F_a(\xi_\alpha)]^{3/2}}\int_0^\alpha u^3 f_a''(u)du \geq 0.$$

The proof of Lemma 3.7 is complete. ■

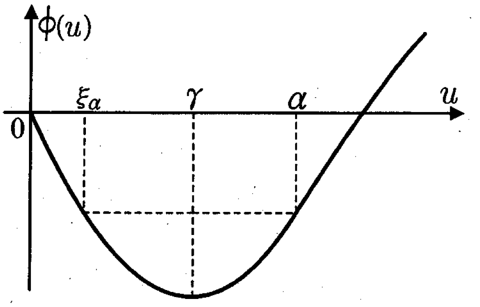

Figure 3.3: The graph of $\phi_a(u)$ with $\phi_a(\alpha) < 0$ and $\alpha > \gamma > \xi_\alpha > 0$.

Lemma 3.8. *Consider (1.1) with $a \geq \tilde{a}$. Then $T_a'(\alpha) < 0$ for $\rho(a) \leq \alpha \leq p_2(a)$. In particular, $\rho(a) < 3$ for $a \geq 6$.*

Proof of Lemma 3.8. Let $\alpha \in [\rho(a), p_2(a)]$ be given. By Lemmas 3.4 and 3.5(ii), we observe that

$$0 < \bar{\alpha} < p_1(a) < \rho(a) \leq \alpha \leq p_2(a) \quad \text{for } a \geq \tilde{a}.$$

Moreover, $\theta_a(\alpha) - \theta_a(u) > 0$ for $0 < u < \bar{\alpha}$ and $\theta_a(\alpha) - \theta_a(u) < 0$ for $\bar{\alpha} < u < \alpha$. Then by Lemma 3.5(ii), we obtain that

$$\begin{aligned}
T_a'(\alpha) &= \frac{1}{2\sqrt{2}\alpha}\left\{\int_0^{\bar{\alpha}} \frac{\theta_a(\alpha) - \theta_a(u)}{[F_a(\alpha) - F_a(u)]^{3/2}} du + \int_{\bar{\alpha}}^{\alpha} \frac{\theta_a(\alpha) - \theta_a(u)}{[F_a(\alpha) - F_a(u)]^{3/2}} du\right\} \\
&< \frac{1}{2\sqrt{2}\alpha}\left\{\int_0^{\bar{\alpha}} \frac{\theta_a(\alpha) - \theta_a(u)}{[F_a(\alpha) - F_a(\bar{\alpha})]^{3/2}} du + \int_{\bar{\alpha}}^{\alpha} \frac{\theta_a(\alpha) - \theta_a(u)}{[F_a(\alpha) - F_a(\bar{\alpha})]^{3/2}} du\right\} \\
&= \frac{1}{2\sqrt{2}\alpha\,[F_a(\alpha) - F_a(\bar{\alpha})]^{3/2}} \int_0^{\alpha} [\theta_a(\alpha) - \theta_a(u)]\, du \\
&= \frac{1}{2\sqrt{2}\alpha\,[F_a(\alpha) - F_a(\bar{\alpha})]^{3/2}} \left[\alpha\theta_a(\alpha) - \int_0^{\alpha} \theta_a(u) du\right] \\
&= \frac{1}{2\sqrt{2}\alpha\,[F_a(\alpha) - F_a(\bar{\alpha})]^{3/2}} \int_0^{\alpha} u\theta_a'(u) du \\
&= \frac{1}{2\sqrt{2}\alpha\,[F_a(\alpha) - F_a(\bar{\alpha})]^{3/2}} \int_0^{\alpha} \left[u f_a(u) - u^2 f_a'(u)\right] du < 0.
\end{aligned}$$

Next, we prove that $\rho(a) < 3$ for $a \geq 6$. It is easy to compute and find that

$$p_1'(a) = \frac{(a-1)\sqrt{a^2 - 4a} - a(a-3)}{\sqrt{a^2 - 4a}} < 0 \quad \text{for } a > 4. \tag{3.30}$$

So by (3.9), we compute and find that

$$p_2(a) \geq p_2(6) \ (\approx 22.3) > 3 > p_1(6) \ (\approx 1.6) > p_1(a) \quad \text{for } a \geq 6. \tag{3.31}$$

By Lemma 3.4(i) and (3.31),

$$u\theta_a'(u)\begin{cases} >0 & \text{for } 0<u<p_1(a),\\ =0 & \text{for } u=p_1(a),\\ <0 & \text{for } p_1(a)<u<3. \end{cases} \tag{3.32}$$

We compute that

$$\int \frac{u\,(a+u)^2-a^2u^2}{(a+u)^2}du=\frac{u^2}{2}-a^2u+\frac{a^4}{a+u}+2a^3\ln(a+u). \tag{3.33}$$

By (3.2) and (3.31)–(3.33), we compute and observe that, for $a\geq 6$,

$$\begin{aligned} H_a(3) &= \int_0^3 u\theta_a'(u)du=\int_0^{p_1(a)} u\theta_a'(u)du+\int_{p_1(a)}^3 u\theta_a'(u)du\\ &\leq f_a(p_1(a))\left[\int_0^{p_1(a)}\frac{u\,(a+u)^2-a^2u^2}{(a+u)^2}du+\int_{p_1(a)}^3\frac{u\,(a+u)^2-a^2u^2}{(a+u)^2}du\right]\\ &= f_a(p_1(a))\int_0^3\frac{u\,(a+u)^2-a^2u^2}{(a+u)^2}du=\frac{f_a(p_1(a))}{8a^3(a+3)^2}\Lambda_a, \end{aligned} \tag{3.34}$$

where

$$\Lambda_a\equiv\frac{-12a^3-18a^2+9a+27}{4a^3(a+3)}-\ln\left(\frac{a}{a+3}\right).$$

We find that $\Lambda_6=-\frac{13}{32}-\ln\frac{2}{3}$ ($\approx -7.85\times 10^{-4}$) <0, $\lim_{a\to\infty}\Lambda_a=0$, and

$$\Lambda'(a)=\frac{27}{4a^4\,(a+3)^2}\left(a^2-6a-9\right)\begin{cases} <0 & \text{for } 6\leq a<3+3\sqrt{2},\\ =0 & \text{for } a=3+3\sqrt{2},\\ >0 & \text{for } a>3+3\sqrt{2}. \end{cases}$$

So by (3.34),

$$H_a(3)\leq\frac{f_a(p_1(a))}{8a^3(a+3)^2}\Lambda_a<0 \quad \text{for } a\geq 6,$$

which implies that $\rho(a)<3$ for $a\geq 6$. The proof of Lemma 3.8 is complete. ■

Lemma 3.9. *Consider (1.1) with $a>0$. Then*

$$\alpha f_a(\alpha)-uf_a(u)\leq M_a(u,\alpha)\left[F_a(\alpha)-F_a(u)\right] \quad \text{for } 0\leq u\leq\alpha,$$

where

$$M_a(u,\alpha)\equiv\begin{cases} 1+\frac{a^2\alpha}{(a+\alpha)^2} & \text{for } 0\leq u\leq\alpha\leq a,\\ 1+\frac{a}{4} & \text{for } 0\leq u\leq a<\alpha,\\ 1+\frac{a^2u}{(a+u)^2} & \text{otherwise} \end{cases}$$

satisfies $M_a(u,\alpha)\leq 1+\frac{a}{4}$ for $u\geq 0$ and $\alpha\geq 0$.

Proof of Lemma 3.9 Since we compute that, for any $a>0$,

$$\frac{\partial}{\partial u}\frac{a^2u}{(a+u)^2}=\frac{a^2\,(a-u)}{(a+u)^3}\begin{cases} >0 & \text{if } 0<u<a,\\ =0 & \text{if } u=a,\\ <0 & \text{if } a>u, \end{cases} \tag{3.35}$$

we see that

$$M_a(u,\alpha) \leq 1 + \frac{a^2 \times a}{(a+a)^2} = 1 + \frac{a}{4} \quad \text{for } u \geq 0 \text{ and } \alpha \geq 0.$$

We let

$$W_a(u,\alpha) \equiv M_a(u,\alpha)\,[F_a(\alpha) - F_a(u)] - [\alpha f_a(\alpha) - u f_a(u)] \quad \text{for } u \geq 0.$$

To complete the proof, it is sufficient to prove that $W_1(u,\alpha) \geq 0$ for $0 \leq u \leq \alpha$ and $a > 0$. We compute and obtain that

$$\begin{aligned}
&\frac{\partial}{\partial u} W_a(u,\alpha) \\
= &\left[\frac{\partial}{\partial u} M_a(u,\alpha)\right][F_a(\alpha) - F_a(u)] + \left[1 + \frac{a^2 u}{(a+u)^2} - M_a(u,\alpha)\right] f_a(u) \\
= &\begin{cases}
\left[1 + \frac{a^2u}{(a+u)^2} - M_a(u,\alpha)\right] f_a(u) = \left[\frac{a^2u}{(a+u)^2} - \frac{a^2\alpha}{(a+\alpha)^2}\right] f_a(u) < 0 & \text{if } 0 \leq u < \alpha \leq a, \\
\left[1 + \frac{a^2u}{(a+u)^2} - M_a(u,\alpha)\right] f_a(u) = \left[\frac{a^2u}{(a+u)^2} - \frac{a}{4}\right] f_a(u) < 0 & \text{if } 0 \leq u \leq a < \alpha, \\
\left[\frac{\partial}{\partial u} M_a(u,\alpha)\right][F_a(\alpha) - F_a(u)] = \frac{a^2(a-u)}{(a+u)^4}[F_a(\alpha) - F_a(u)] < 0 & \text{if } 0 < a < u < \alpha.
\end{cases}
\end{aligned} \tag{3.36}$$

Since $W_a(\alpha,\alpha) = 0$ and by (3.36), we see that $W_a(u,\alpha) \geq 0$ for $0 \leq u \leq \alpha$. The proof of Lemma 3.9 is complete. ■

The proof of the following Lemma 3.10 is rather lengthy, and hence it is given in [11].

Lemma 3.10. *Consider (1.1) with $4 < a < \tilde{a}$. Then $[\alpha T_a''(\alpha)]' > 0$ for $\gamma(a) \leq \alpha \leq \kappa(a) = \eta(a)$.*

Lemma 3.11. *Consider (1.1) with $4 < a < a^*$. Then $[\alpha T_a''(\alpha)]' > 0$ for*

$$\gamma(a) \leq \alpha \leq \eta(a) \equiv \begin{cases} \kappa(a) & \text{for } 4 < a < \tilde{a}, \\ \rho(a) & \text{for } a \geq \tilde{a}. \end{cases} \tag{3.37}$$

Moreover, one of the following assertions (a)–(c) holds:

(a) $T_a'(\alpha)$ is a strictly increasing function of α on $[\gamma(a), \eta(a)]$.

(b) $T_a'(\alpha)$ is a strictly decreasing function of α on $[\gamma(a), \eta(a)]$.

(c) $T_a'(\alpha)$ is a strictly decreasing and then strictly increasing function of α on $[\gamma(a), \eta(a)]$.

Proof of Lemma 3.11. By [10, Lemma 2.6], we obtain that $[\alpha T_a''(\alpha)]' > 0$ for $\gamma(a) \leq \alpha \leq \rho(a) = \eta(a)$ and $4 < \tilde{a} \leq a < a^*$. By Lemma 3.10, we see that $[\alpha T_a''(\alpha)]' > 0$ for $\gamma(a) \leq \alpha \leq \eta(a)$ and $4 < a < \tilde{a}$. So $[\alpha T_a''(\alpha)]' > 0$ for $\gamma(a) \leq \alpha \leq \eta(a)$ and $4 < a < a^*$. By Lemma 3.5, we see that $\gamma(a) < \eta(a)$ for $4 < a < a^*$. Since $\alpha T_a''(\alpha)$ is a strictly increasing function of $\alpha \in [\gamma(a), \eta(a)]$ for $4 < a < a^*$, we observe that there are three possible cases:

Case 1 $T_a''(\alpha) > 0$ for $\alpha \in (\gamma(a), \eta(a)]$.

Case 2 $T_a''(\alpha) < 0$ for $\alpha \in [\gamma(a), \eta(a))$.

Case 3 $T_a''(\alpha) < 0$ for $\alpha \in [\gamma(a), \alpha')$, $T_a''(\alpha) > 0$ for $\alpha \in (\alpha', \eta(a)]$, and $T_a''(\alpha') = 0$ for some $\alpha' \in (\gamma(a), \eta(a))$.

So if Case 1 (Case 2 and Case 3 respectively) holds, then assertion (a) ((b) and (c) respectively) holds.

The proof of Lemma 3.11 is complete. ■

Remark 1. *By (3.37) and Lemma 3.5, we see that $\eta(a)$ is a continuous function of $a > 4$.*

Lemma 3.12. *Consider (1.1) with $a > 4$. The following assertions (i)–(ii) hold:*

(i)

$$\eta(a) < \omega(a) \equiv \begin{cases} 12 & \text{if } 4 < a < 6, \\ 3 & \text{if } a \geq 6. \end{cases}$$

(ii) $\partial T_a'(\alpha)/\partial a < 0$ for $0 < \alpha \leq \omega(a)$.

Proof of Lemma 3.12. By (3.9), Lemmas 3.5 and 3.8, we see that

$$\eta(a) = \begin{cases} \kappa(a) < 8 < 12 = \omega(a) & \text{for } 4 < a < \tilde{a}, \\ \rho(a) \leq p_2(a) < p_2(a^*) < 12 = \omega(a) & \text{for } \tilde{a} \leq a < a^*, \\ \rho(a) \leq \gamma(a) < \gamma(6) = 12 = \omega(a) & \text{for } a^* \leq a < 6, \\ \rho(a) < 3 = \omega(a) & \text{for } a \geq 6. \end{cases} \tag{3.38}$$

So $\eta(a) < \omega(a)$, and hence assertion (i) holds.

We compute that

$$\frac{\partial}{\partial a} T_a'(\alpha) = \frac{1}{2\sqrt{2}\alpha} \int_0^\alpha \frac{N_a(u,\alpha)}{\left[F_a(\alpha) - F_a(u)\right]^{3/2}} du,$$

where

$$\begin{aligned} N_a(u,\alpha) &\equiv \left[F_a(\alpha) - F_a(u)\right] \left[\frac{u^3 f_a(u)}{(a+u)^2} - \frac{\alpha^3 f_a(\alpha)}{(a+\alpha)^2} - \int_u^\alpha \frac{t^2 f_a(t)}{(a+t)^2} dt\right] \\ &\quad + \frac{3}{2}\left[\alpha f_a(\alpha) - u f_a(u)\right] \int_u^\alpha \frac{t^2 f_a(t)}{(a+t)^2} dt. \end{aligned} \tag{3.39}$$

By (3.12), (3.39) and Lemma 3.9, we obtain that

$$\begin{aligned} \frac{N_a(u,\alpha)}{\left[\alpha f_a(\alpha) - u f_a(u)\right]} &\leq \frac{3}{2} \int_u^\alpha \frac{t^2 f_a(t)}{(a+t)^2} dt - \frac{1}{M_a(u,\alpha)} \int_u^\alpha \frac{\left[2t^2 + a(a+6)t + 4a^2\right] t^2 f_a(t)}{(a+t)^4} dt \\ &\equiv N_{a,1}(u,\alpha). \end{aligned} \tag{3.40}$$

Thus, to prove assertion (ii), it is sufficient to prove the next parts (a) and (b):

(a) For $4 < a < 6$, $N_{a,1}(u,\alpha) < 0$ for $0 < u < \alpha \leq 12$.

(b) For $a \geq 6$, $N_{a,1}(u,\alpha) < 0$ for $0 < u < \alpha \leq 3$.

(I) We prove part (a). Assume that $4 < a < 6$. By Lemma 3.9, we have that $M_a(u,\alpha) \leq (a+4)/4$ for $0 \leq u \leq \alpha$. We compute and find that, for $4 < a < 6$,

$$(3a-4)\,12^2 - 2a(a-12)\,12 + a^2(3a-20) = -(12-a)\left[3a^2 + 8(6-a)\right] < 0.$$

It follows that $(3a-4)t^2 - 2a(a-12)t + a^2(3a-20) < 0$ for $0 \le t \le 12$ and $4 < a < 6$. So by (3.40), we obtain that, for $0 \le u < \alpha \le 12$ and $4 < a < 6$,

$$\begin{aligned} N_{a,1}(u,\alpha) &\le \frac{3}{2}\int_u^\alpha \frac{t^2 f_a(t)}{(a+t)^2}dt - \frac{4}{a+4}\int_u^\alpha \frac{\left[2t^2 + a(a+6)t + 4a^2\right]t^2 f_a(t)}{(a+t)^4}dt \\ &= \int_u^\alpha \frac{t^2 f_a(t)}{2(a+4)(a+t)^4}\left[(3a-4)t^2 - 2a(a-12)t + a^2(3a-20)\right]dt < 0. \end{aligned}$$

So part (a) holds.

(II) We prove part (b). Assume that $a \ge 6$. Since $a > 3 \ge \alpha > 0$ and by Lemma 3.9, we observe that

$$\frac{1}{M_a(u,\alpha)} = \frac{(a+\alpha)^2}{(a+\alpha)^2 + a^2\alpha} \quad \text{for } 0 \le u \le \alpha.$$

So by (3.40), we obtain that

$$N_{a,1}(u,\alpha) = \int_u^\alpha \frac{t^2 f_a(t)}{(a+t)^2} N_{a,2}(t,\alpha)dt, \tag{3.41}$$

where

$$N_{a,2}(t,\alpha) \equiv \frac{3}{2} - \frac{(a+\alpha)^2}{(a+\alpha)^2 + a^2\alpha}\left[\frac{2t^2 + a(a+6)t + 4a^2}{(a+t)^2}\right].$$

We compute and observe that, for $a > 6$ and $0 \le t \le 3$,

$$\begin{aligned} \frac{\partial}{\partial t}N_{a,2}(t,\alpha) &= \frac{a(a+\alpha)^2}{\left[(a+\alpha)^2 + a^2\alpha\right](a+t)^3}\left[(a+2)t - a^2 + 2a\right] \\ &\le \frac{a(a+\alpha)^2}{\left[(a+\alpha)^2 + a^2\alpha\right](a+t)^3}\left[(a+2)\times 3 - a^2 + 2a\right] \\ &= \frac{a(a+\alpha)^2(a+1)(6-a)}{\left[(a+\alpha)^2 + a^2\alpha\right](a+t)^3} < 0. \end{aligned}$$

So $N_{a,2}(t,\alpha)$ is a strictly decreasing function of $t \in [0,3]$ for $a \ge 6$. Since

$$N_{a,2}(\alpha,\alpha) = \frac{(\alpha-5)a^2 - 6a\alpha - \alpha^2}{2(a+\alpha)^2 + 2a^2\alpha} < 0 \quad \text{for } 0 < \alpha \le 3,$$

we see that either $N_{a,2}(t,\alpha) < 0$ for $0 < t \le \alpha$, or

$$N_{a,2}(t,\alpha) = \begin{cases} > 0 & \text{if } 0 < t < r_1, \\ = 0 & \text{if } t = r_1, \\ < 0 & \text{if } r_2 < t \le \alpha \end{cases} \quad \text{for some } r_1 \in (0,\alpha).$$

So by (3.41), we further see that, for $0 < u < \alpha \le 3$,

$$\text{either } \frac{\partial}{\partial u}N_{a,1}(u,\alpha) > 0 \text{ or } \frac{\partial}{\partial u}N_{a,1}(u,\alpha)\begin{cases} < 0 & \text{if } 0 < t < r_1, \\ = 0 & \text{if } t = r_1, \\ > 0 & \text{if } r_2 < t \le \alpha. \end{cases} \tag{3.42}$$

In addition, since $3\alpha^2 - 8\alpha - 5 < 0$ for $0 < \alpha \leq 3$, and by (3.41), we compute and find that, for $a \geq 6$,

$$
\begin{aligned}
\frac{\partial}{\partial \alpha}\left[\frac{2(a+\alpha)^2 + 2a^2\alpha}{(a+\alpha)^2 + 3a^2\alpha} N_{a,1}(0,\alpha)\right] &= \frac{\alpha^2 f_a(\alpha)}{\left[(a+\alpha)^2 + 3a^2\alpha\right]^2 (a+\alpha)^2}[(3\alpha^2 - 8\alpha - 5)a^4 \\
&\quad -16(\alpha^2+\alpha)a^3 - (8\alpha+18)\alpha^2 a^2 - 8\alpha^3 a - \alpha^4] \\
&< 0.
\end{aligned}
$$

It follows that

$$
\frac{2(a+\alpha)^2 + 2a^2\alpha}{(a+\alpha)^2 + 3a^2\alpha} N_{a,1}(0,\alpha) < \left.\frac{2(a+\alpha)^2 + 2a^2\alpha}{(a+\alpha)^2 + 3a^2\alpha} N_{a,1}(0,\alpha)\right|_{\alpha=0} = 0 \quad \text{for } 0 < \alpha \leq 3.
$$

Thus, $N_{a,1}(0,\alpha) < 0$ for $0 < \alpha \leq 3$. Clearly, $N_{a,1}(\alpha,\alpha) = 0$. So by (3.42), $N_{a,1}(u,\alpha) < 0$ for $0 \leq u < \alpha$ and $0 < \alpha \leq 3$. So part (b) holds.

The proof of Lemma 3.12 is complete. ■

3.3. Statements and proofs of main lemmas

Lemma 3.13. *Consider (1.1) with fixed $a > 4$. Either one of the following assertions (i)–(ii) holds:*

(i) $T_a(\alpha)$ is a strictly increasing function on $(0,\infty)$.

(ii) $T_a(\alpha)$ has exactly one local maximum and exactly one local minimum on $(0,\infty)$.

Proof of Lemma 3.13. Assume that assertion (i) does not hold. Then by Lemma 3.1, $T_a(\alpha)$ has a local maximum and a local minimum on $(0,\infty)$.

Assume that $T_a(\alpha)$ has two local maxima at some $\alpha_{M_1} < \alpha_{M_2}$. Then there exists $\alpha_m \in (\alpha_{M_1}, \alpha_{M_2})$ such that $T_a(\alpha_m)$ is a local minimum value. If $4 < a < \tilde{a}$, and by Lemmas 3.5(ii) and 3.6–3.8, we observe that $\gamma(a) \leq \alpha_m < \alpha_{M_2} < \kappa(a) = \eta(a)$. It is a contradiction by Lemma 3.11. If $\tilde{a} \leq a < a^*$, and by Lemmas 3.5(ii) and 3.6–3.8, we observe that $\gamma(a) \leq \alpha_m < \alpha_{M_2} < \rho(a) = \eta(a)$. It is a contradiction by Lemma 3.11. If $a \geq a^*$, and by Lemmas 3.5(ii), 3.7 and 3.8 , we observe that $\alpha_m < \alpha_{M_2} \leq \gamma(a)$. It is a contradiction by Lemma 3.6(i). So by above discussions, $T_a(\alpha)$ has exactly one local maximum on $(0,\infty)$.

In addition, if $T_a(\alpha)$ has two local minima at some $\alpha_{m_1} < \alpha_{m_2}$, then by Lemma 3.1, $T_a(\alpha)$ has two local maxima at some $\alpha_{M_1} \in (0, \alpha_{m_1})$ and $\alpha_{M_2} \in (\alpha_{m_1}, \alpha_{m_2})$. It is a contradiction. So $T_a(\alpha)$ has exactly one local minimum on $(0,\infty)$.

The proof of Lemma 3.13 is complete. ■

Lemma 3.14. *Consider (1.1) with $a > 4$. Either one of the following two assertions holds:*

(i) $T_a(\alpha)$ is a strictly increasing function on $(0,\infty)$ and $T_a(\alpha)$ has at most one critical point on $(0,\infty)$.

(ii) $T_a(\alpha)$ has exactly two critical points, a local maximum at some α_M and a local minimum at some $\alpha_m > \alpha_M$ on $(0,\infty)$.

Proof of Lemma 3.14. By Lemma 3.13, either one of the following two cases holds:

(a) $T_a(\alpha)$ is a strictly increasing function on $(0,\infty)$.

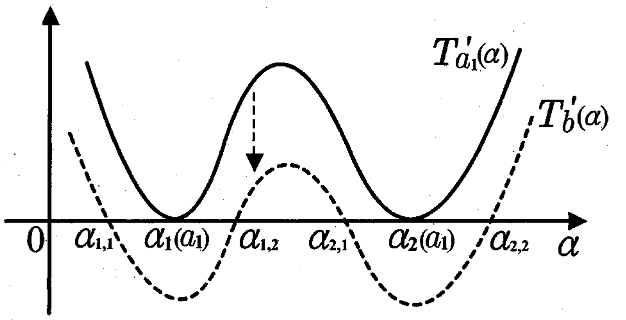

Figure 3.4: Graphs of $T'_{a_1}(\alpha)$ and $T''_b(\alpha)$ with $b < a_1$ sufficiently close to a_1.

(b) $T_a(\alpha)$ has exactly one local maximum at some α_M and exactly one local minimum at some α_m $(> \alpha_M)$ on $(0, \infty)$.

(I) We prove assertion (i) under Case (a). We fix $a_1 > 4$. Assume that $T_{a_1}(\alpha)$ has two critical points $\alpha_1(a_1) < \alpha_2(a_1)$ on $(0, \infty)$. We obtain that

$$T'_{a_1}(\alpha_1(a_1)) = T'_{a_1}(\alpha_2(a_1)) = T''_{a_1}(\alpha_1(a_1)) = T''_{a_1}(\alpha_2(a_1)) = 0. \tag{3.43}$$

So by Lemmas 3.7 and 3.8, we observe that $0 < \alpha_1(a_1) < \alpha_2(a_1) < \eta(a_1) < \omega(a_1)$. By (3.38) and continuities of $\eta(a)$, $\omega(a)$ and $p_2(a)$, we observe that

$$0 < \alpha_1(a_1) < \alpha_2(a_1) < \min_{a \in [a_1 - \delta, a_1]} \{\omega(a)\} \quad \text{for some } \delta > 0. \tag{3.44}$$

Let $b \in (a_1 - \delta, a_1)$ be given. By Lemma 3.12, (3.43) and (3.44), we observe that

$$T'_b(\alpha_1(a_1)) < T'_{a_1}(\alpha_1(a_1)) = 0 \quad \text{and} \quad T'_b(\alpha_2(a_1)) < T'_{a_1}(\alpha_2(a_1)) = 0. \tag{3.45}$$

In addition, we assume that there exists an open interval I such that $T'_a(\alpha) \equiv 0$ on I. It implies that $T''_a(\alpha) = T'''_a(\alpha) = 0$ on I. It is a contradiction by Lemmas 3.6–3.8 and 3.11. So there exist three numbers $\beta_1 \in (0, \alpha_1(a_1))$, $\beta_2 \in (\alpha_1(a_1), \alpha_2(a_1))$ and $\beta_3 \in (\alpha_2(a_1), \eta(a_1))$ such that $T'_{a_1}(\beta_i) > 0$ for $i = 1, 2, 3$. So by Lemma 3.6(ii) and (3.45), we choose $b < a_1$ sufficiently close to a_1 such that $T'_b(\alpha)$ has four positive zeros $\alpha_{1,1}$, $\alpha_{1,2}$, $\alpha_{2,1}$, $\alpha_{2,2}$ satisfying

$$\alpha_{1,1} < \alpha_1(a_1) < \alpha_{1,2} < \alpha_{2,1} < \alpha_2(a_1) < \alpha_{2,2}.$$

See Figure 3.4. Furthermore, $T_b(\alpha_{1,1})$ and $T_b(\alpha_{2,1})$ are local maximum values, and $T_b(\alpha_{1,2})$ and $T_b(\alpha_{2,2})$ are local minimum values. It is a contradiction by Lemma 3.13. Therefore, assertion (i) holds under Case (a).

(II) We prove assertion (ii) under Case (b). We fix $a_2 > 4$. Assume that $T_{a_2}(\alpha)$ has a critical point $\alpha_3(a_2)$ on $(0, \infty)$, distinct from α_M and α_m. It follows that $T'_{a_2}(\alpha_3(a_2)) = T''_{a_2}(\alpha_3(a_2)) = 0$. By Lemmas 3.7 and 3.8, we obtain that $0 < \alpha_3(a_2) < \eta(a_2) < \omega(a_2)$. Similarly, we have that

$$0 < \alpha_3(a_2) < \min_{a \in [a_2 - \delta, a_2 + \delta]} \{\omega(a)\} \quad \text{for some } \delta > 0.$$

So by Lemma 3.12, we observe that

$$T_b'(\alpha_3(a_2)) < T_{a_2}'(\alpha_3(a_2)) = 0 \quad \text{for } a_2 - \delta < b < a_2, \tag{3.46}$$

$$T_b'(\alpha_3(a_2)) > T_{a_2}'(\alpha_3(a_2)) = 0 \quad \text{for } a_2 < b < a_2 + \delta. \tag{3.47}$$

Similarly, by Lemma 3.6(ii), (3.46) and (3.47), there exists $b > 0$ sufficiently close to a_2 such that $T_b(\alpha)$ has two local extrema at $\alpha_{3,1} \in (0, \alpha_3(a_2))$ and $\alpha_{3,2} \in (\alpha_3(a_2), \eta(a_2))$, distinct from α_M and α_m. See Figure 3.5. It is a contradiction by Lemma 3.13. Therefore, assertion (ii) holds under

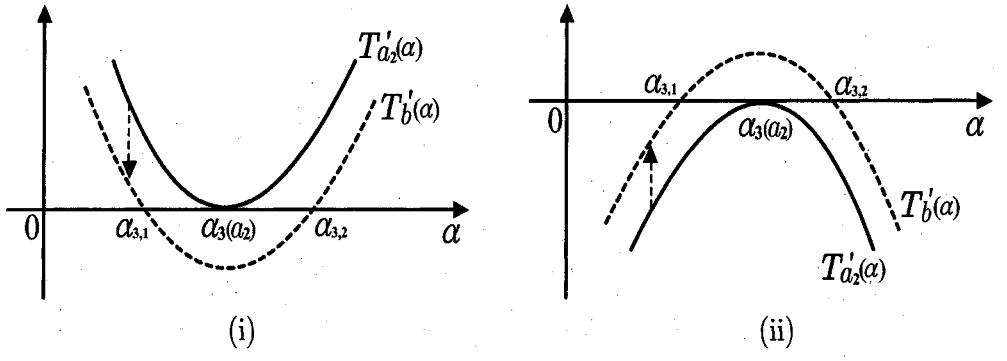

Figure 3.5: Local graphs of $T_{a_2}'(\alpha)$ and $T_b'(\alpha)$ for α near $\alpha_3(a_2)$ and $b > 0$ sufficiently close to a_2. (i) $T_{a_2}''(\alpha_3(a_2)) \geq 0$. (ii) $T_{a_2}''(\alpha_3(a_2)) \leq 0$.

Case (b).

The proof of Lemma 3.14 is complete. ■

We are in a position to prove Lemma 3.2 by applying Lemmas 3.5(i), 3.6(ii), 3.7 and 3.12.

Proof of Lemma 3.2. By Theorem 1.2 and Lemma 3.14, we obtain that

$$\begin{aligned} \Omega &= \left\{ \begin{array}{l} a > 0 : T_a(\alpha) \text{ has exactly two critical points,} \\ \text{a local maximum and a local minimum, on } (0,\infty) \end{array} \right\} \\ &= \left\{ a > 4 : T_a'(\alpha) < 0 \text{ for some } \alpha \in (0,\infty) \right\}. \end{aligned} \tag{3.48}$$

(I) It is obvious that Ω is nonempty because $[\tilde{a}, \infty) \subset \Omega$ by [12, Theorem 2.2] and Lemma 3.14.

(II) We show that Ω is open. If $b \in \Omega$, then $T_b'(\beta) < 0$ for some $\beta \in (0,\infty)$. By Lemma 3.6(ii), we observe that $T_a'(\beta) < 0$ for a belonging to some open neighborhood of b. So Ω is open.

(III) We then show that Ω is connected. First, we see that $[\tilde{a}, \infty) \subset \Omega$ by [12, Theorem 2.2(i)]. Suppose to the contrary that the set $\Omega \cap (4, \tilde{a}]$ is not connected, then there exist positive numbers a_1 and a_2 satisfying $4 < a_1 < a_2 < \tilde{a}$ such that $a_1 \in \Omega$ and $a_2 \notin \Omega$. Hence $T_{a_2}'(\alpha) \geq 0$ on $(0,\infty)$ by Lemma 3.14 and (3.48). Since $\tilde{a}$ (≈ 4.166) $< \frac{417}{100}$ by (1.4), and by Lemma 3.5(i), we have that $\kappa(a_1) < \kappa(a_2) < 8$. So by Lemma 3.12,

$$T_{a_1}'(\alpha) > T_{a_2}'(\alpha) \geq 0 \quad \text{for all } \alpha \in (0, \kappa(a_1)) \subset (0, \kappa(a_2)). \tag{3.49}$$

Since $a_1 \in \Omega \cap (4, \tilde{a})$ and by (3.49), there exists a number $\alpha_1 \geq \kappa(a_1)$ such that $T_{a_1}'(\alpha_1) = 0$ and $T_{a_1}''(\alpha_1) \leq 0$. It is a contradiction by Lemma 3.7. So $\Omega \cap (4, \tilde{a}]$ is connected. It implies that Ω is connect.

(IV) Since Ω is open and connect and $[\tilde{a},\infty)\subset\Omega$, there exists $a_0\in(4,\tilde{a})$ such that $\Omega=(a_0,\infty)$. Moreover, by numerical simulation, we find that $a_0\approx 4.069$.

The proof of Lemma 3.2 is complete. ■

In addition to $T_a(\alpha)$ with $0<a<\infty$ defined in (3.1) to problem (1.1) with $f_a(u)=\exp\left(\frac{au}{a+u}\right)$, we define two time-map functions $T_0(\alpha)$ and $T_\infty(\alpha)$ for corresponding nonlinearities $f_0(u)\equiv 1$ and $f_\infty(u)=\exp u$ by

$$T_0(\alpha)\equiv\frac{1}{\sqrt{2}}\int_0^\alpha\frac{1}{\sqrt{\alpha-u}}du=\sqrt{2\alpha}\quad\text{for }\alpha>0, \tag{3.50}$$

$$T_\infty(\alpha)\equiv\frac{1}{\sqrt{2}}\int_0^\alpha\frac{1}{\sqrt{e^\alpha-e^u}}du=\frac{1}{\sqrt{2e^\alpha}}\ln\left(2e^\alpha-1+2\sqrt{e^\alpha\left(e^\alpha-1\right)}\right)\quad\text{for }\alpha>0, \tag{3.51}$$

respectively, see (3.1) and (1.6). The following Lemma 3.15(i) determines the shape of $T_\infty(\alpha)$ on $(0,\infty)$, and Lemma 3.15(ii) is a basic comparison theorem for the time map formula T_0, T_a and T_∞. Lemma 3.15(i) is obvious, cf. Theorem 1.3. In addition, for fixed $u>0$, since $\exp\left(\frac{au}{a+u}\right)$ is a strictly increasing function of $a>0$, we obtain that

$$f_0(u)=1<f_a(u)=\exp\left(\frac{au}{a+u}\right)=\exp\left(\frac{u}{1+\frac{u}{a}}\right)<\exp u=f_\infty(u)\quad\text{for }a>0.$$

Thus Lemma 3.15(ii) follows by modification of the proofs of [17, Theorems 2.3–2.4]; we omit the proof.

Lemma 3.15. *Consider (3.1), (3.50) and (3.51). Let* $\lambda_\infty\approx 0.878$ *and* $\alpha_\infty=\ln\left(\frac{2+\lambda_\infty}{\lambda_\infty}\right)\approx 1.187$ *be two numbers defined in (1.7) and (1.8), respectively. The following assertions (i)–(ii) hold:*

(i) $\lim_{\alpha\to0^+}T_\infty(\alpha)=\lim_{\alpha\to\infty}T_\infty(\alpha)=0$. *In addition,* $T'_\infty(\alpha)>0$ *on* $(0,\alpha_\infty)$, $T'_\infty(\alpha_\infty)=0$ *and* $T'_\infty(\alpha)<0$ *on* (α_∞,∞).

(ii) For any fixed $\alpha>0$ *such that* $T_a(\alpha)$ *is a continuous, strictly decreasing function of* $a>0$. *Moreover,*

$$\sqrt{2\alpha}=T_0(\alpha)=\lim_{a\to0^+}T_a(\alpha)>T_a(\alpha)>\lim_{a\to\infty}T_a(\alpha)=T_\infty(\alpha)\quad\textit{for }\alpha>0\textit{ and }a>0.$$

Throughout this paper, for $a>a_0$, by Lemma 3.2, let $\alpha_M(a)$ and $\alpha_m(a)$ denote the local maximum and the local minimum points of $T_a(\alpha)$ on $(0,\infty)$ where $\alpha_M<\alpha_m$, respectively. See Figure 3.1(i).

Lemma 3.16. *Consider (1.1) with* $a>a_0$. *Then the following assertions (i)–(iii) hold:*

(i) $\alpha_M(a)$ *is a strictly decreasing and continuous function of* $a>a_0$. *Furthermore,* $\alpha_M(a)<\omega(a)$ *for* $a>a_0$.

(ii) $\alpha_m(a)$ *is a continuous function of* $a>a_0$. *Furthermore,* $\alpha_m(a)$ *is a strictly increasing function on* $(a_0,\tilde{a}]$ *and* $p_2(a)\le\alpha_m(a)$ *for* $a\ge\tilde{a}$.

(iii) For $a>a_0$,

$$\alpha_\infty=\lim_{a\to\infty}\alpha_M(a)<\alpha_M(a)<\lim_{a\to a_0^+}\alpha_M(a)=\lim_{a\to a_0^+}\alpha_m(a)<\alpha_m(a)<\lim_{a\to\infty}\alpha_m(a)=\infty.$$

Proof of Lemma 3.16. We divide this proof into next Steps 1–6.

Step 1. We prove that $\alpha_M(a) < \omega(a)$ for $a > a_0$, and $\alpha_M(a)$ is a strictly decreasing function of a both on $(a_0, 6)$ and $[6, \infty)$. Let $a_1 > a_0$ be given. Since $T'_{a_1}(\alpha_M(a_1)) = 0$ and $T''_{a_1}(\alpha_M(a_1)) \le 0$, and by Lemmas 3.7, 3.8 and 3.12, we observe that $\alpha_M(a_1) < \eta(a_1) < \omega(a_1)$. Then we see that $\alpha_M(a_1) < \omega(a_1) = \omega(a_2)$ for either $a_0 < a_1 < a_2 < 6$ or $6 \le a_1 < a_2$. So by Lemma 3.12, we see that $T'_{a_2}(\alpha_M(a_1)) < T'_{a_1}(\alpha_M(a_1)) = 0$ for either $a_0 < a_1 < a_2 < 6$ or $6 \le a_1 < a_2$. So by Lemma 3.14, $\alpha_M(a_2) < \alpha_M(a_1)$ for either $a_0 < a_1 < a_2 < 6$ or $6 \le a_1 < a_2$. Thus, $\alpha_M(a)$ is a strictly decreasing function both on $(a_0, 6)$ and $[6, \infty)$.

Step 2. We prove that $\alpha_m(a)$ is a strictly increasing function of a on $(a_0, \tilde{a}]$. By Lemma 3.4(i), we see that $\theta_a(\alpha) > \theta_a(u)$ for $0 < u < \alpha$ and $\alpha > p_3(a)$. It follows that $T'_a(\alpha) > 0$ for $\alpha > p_3(a)$ by (3.27). So by Lemma 3.4(i), we further see that $\alpha_m(a) \le p_3(a) < 12 = \omega(a)$ for $a \in (a_0, \tilde{a}]$. Assume that $a_0 < a_1 < a_2 \le \tilde{a}$. By Lemma 3.12, we observe that $T'_{a_2}(\alpha_m(a_1)) < T'_{a_1}(\alpha_m(a_1)) = 0$. It follows that $\alpha_m(a_1) < \alpha_m(a_2)$ by Lemma 3.14. Thus, $\alpha_m(a)$ is a strictly increasing function on $(a_0, \tilde{a}]$.

Step 3. We prove that $\lim_{a\to a_0^+} \alpha_M(a) \le \lim_{a\to a_0^+} \alpha_m(a)$ and

$$\alpha_\infty = \lim_{a\to\infty} \alpha_M(a) < \alpha_M(a) < \lim_{a\to a_0^+} \alpha_M(a) \le \alpha_m(a) \quad \text{for } a > a_0. \tag{3.52}$$

Since $0 < \alpha_M(a) < \omega(a) \le 12$ for $a > a_0$, and by Step 1, we see that $\lim_{a\to a_0^+} \alpha_M(a)$ and $\lim_{a\to\infty} \alpha_M(a)$ both exist. Suppose to the contrary that $\lim_{a\to\infty} \alpha_M(a) \ne \alpha_\infty$. Then we have two cases. Case 1: $\lim_{a\to\infty} \alpha_M(a) < \alpha_\infty$ and Case 2: $\lim_{a\to\infty} \alpha_M(a) > \alpha_\infty$.

Assume that Case 1 holds. Then $\alpha_M(a_1) < \alpha_\infty$ for some $a_1 > 6$ by Step 1. We let

$$\delta \equiv \frac{1}{2} \min\{\alpha_\infty - \alpha_M(a_1), \alpha_m(a_1) - \alpha_M(a_1)\}.$$

Clearly, $\delta > 0$. We let $\alpha \in (\alpha_M(a_1), \alpha_M(a_1) + \delta)$ be given. Then

$$\alpha_M(a_1) < \alpha < \alpha_M(a_1) + \delta < \alpha_m(a_1),$$

$$\alpha_M(a_1) < \alpha < \alpha_M(a_1) + \delta < \alpha_\infty \;\; (\approx 1.187) < \omega(a) \;\; \text{for } a > a_1. \tag{3.53}$$

So by Lemma 3.15, we have that $T'_{a_1}(\alpha) < 0$ and $T'_\infty(\alpha) > 0$. Then by (3.53), Lemmas 3.6 and 3.12, we find that

$$0 < T'_\infty(\alpha) = \lim_{a\to\infty} T'_a(\alpha) < T'_{a_1}(\alpha) < 0,$$

which is a contradiction.

Assume that Case 2 holds. We let $\beta \in (\alpha_\infty, \lim_{a\to\infty} \alpha_M(a))$ be given. By Step 1 and Lemma 3.15, we see that $T'_a(\beta) > 0$ and $T'_\infty(\beta) < 0$ for $a > 6$. So by Lemma 3.6(ii) we find that $0 > T'_\infty(\beta) = \lim_{a\to\infty} T'_a(\beta) \ge 0$, which is a contradiction.

Thus, by above discussions, we obtain that $\lim_{a\to\infty} \alpha_M(a) = \alpha_\infty$. In addition, by Lemma 3.4(i), we see that $\theta_a(\alpha) > \theta_a(u)$ for $0 < u < \alpha \le p_1(a)$ and $a > 4$. It follows that $T'_a(\alpha) > 0$ for $0 < \alpha \le p_1(a)$ and $a > 4$ by (3.27). So $\alpha_M(a) > p_1(a)$ for $a > a_0$. Since $a_0 < 4.08$ by Lemma 3.2, and by Lemma 3.15 and (3.30), we observe that

$$\lim_{a\to a_0^+} \alpha_M(a) \ge p_1(a_0) \ge p_1(4.08) = \frac{2652}{625} - \frac{102}{625}\sqrt{51} \;\; (\approx 3.078) > \omega_6 > \alpha_M(6), \tag{3.54}$$

$$\lim_{a\to 6^-} \alpha_M(a) \ge p_1(6) = 12 - 6\sqrt{3} \;\; (\approx 1.608) > \alpha_\infty \;\; (\approx 1.187). \tag{3.55}$$

Since $\lim_{a\to\infty}\alpha_M(a)=\alpha_\infty$, and by Step 1, (3.54) and (3.55), we further observe that

$$\alpha_\infty=\lim_{a\to\infty}\alpha_M(a)<\alpha_M(a)<\lim_{a\to a_0^+}\alpha_M(a)\quad\text{for } a>a_0.$$

Next, we prove that $\lim_{a\to a_0^+}\alpha_M(a)\le\alpha_m(a)$ for $a>a_0$. By Lemmas 3.7 and 3.8, we obtain that $\alpha_M(a)\le\eta(a)$ for $a>a_0$. So by (3.9) and Lemmas 3.5 and 3.8, we observe that

$$\lim_{a\to a_0^+}\alpha_M(a)\le\eta(a_0)=\kappa(a_0)\le\kappa(\tilde a)=p_2(\tilde a)<p_2(a)<\alpha_m(a)\quad\text{for } a\ge\tilde a. \tag{3.56}$$

Assume that there exists $a_2\in(a_0,\tilde a)$ such that $\alpha_m(a_2)<\lim_{a\to a_0^+}\alpha_M(a)$. Then there exists $a_3\in(a_0,a_2)$ such that $\alpha_m(a_2)<\alpha_M(a_3)$. Since $\alpha_m(a_2)<\alpha_M(a_3)<\omega(a)$ for $a_3\le a\le a_2$, and by Lemma 3.12, we observe that

$$0=T'_{a_2}(\alpha_m(a_2))<T'_{a_2}(\alpha_M(a_3))<T'_{a_3}(\alpha_M(a_3))=0,$$

which is a contradiction. So by (3.56), $\lim_{a\to a_0}\alpha_M(a)\le\alpha_m(a)$ for $a>a_0$. So (3.52) holds. In addition, by Step 2 and (3.52), we see that $\lim_{a\to a_0^+}\alpha_m(a)$ exists and $\lim_{a\to a_0^+}\alpha_M(a)\le\lim_{a\to a_0^+}\alpha_m(a)$.

Step 4. We prove that

$$\alpha_M:(a_0,\infty)\to\left(\alpha_\infty,\lim_{a\to a_0^+}\alpha_M(a)\right)\ \text{is surjective,} \tag{3.57}$$

$$\alpha_m:(a_0,\infty)\to\left(\lim_{a\to a_0^+}\alpha_m(a),\infty\right)\ \text{is surjective.} \tag{3.58}$$

The proofs of (3.57) are (3.58) are omitted.

Step 5. We prove assertions (i) and (ii). By Step 1 and (3.57), we have that $\alpha_M(a)<\omega(a)$ for $a>a_0$, and $\alpha_M(a)$ is a strictly decreasing and continuous function of a both on $(a_0,6)$ and $[6,\infty)$. To prove assertion (i), it is sufficient to prove that $\alpha_M(6)=\lim_{a\to6^-}\alpha_M(a)$. By (3.57), we see that $\alpha_M(6)\ge\lim_{a\to6^-}\alpha_M(a)$. Suppose to the contrary that $\alpha_M(6)>\lim_{a\to6^-}\alpha_M(a)$. Then there exists $\delta>0$ such that $\alpha_M(6)>\alpha_M(a)$ for $6-\delta\le a<6$. Furthermore,

$$\alpha_M(a)<\alpha_M(6)<\omega_6=3<\omega(a)\quad\text{for } 6-\delta\le a<6. \tag{3.59}$$

By (3.59) and Lemma 3.12, we further see that $T'_6(\alpha_M(a))<T'_a(\alpha_M(a))=0$ for $6-\delta\le a<6$. It follows that $\alpha_M(6)<\alpha_M(a)$ for $6-\delta\le a<6$. It is a contradiction. Thus $\lim_{a\to6^-}\alpha_M(a)=\alpha_M(6)$. It implies that $\alpha_M(a)$ is a strictly decreasing and continuous function of a on (a_0,∞). Thus assertion (i) holds.

By Step 2 and (3.58), $\alpha_m(a)$ is a continuous function on $(a_0,\tilde a]$. By Lemmas 3.2, 3.5(ii) and 3.8, we see that

$$\alpha_M(a)<\rho(a)\le\kappa(a)\le p_2(a)<\alpha_m(a)\quad\text{for } a\ge\tilde a. \tag{3.60}$$

So we further see that $T'_a(\alpha)$ has a unique zero $\alpha_m(a)$ on $[p_2(a),\infty)$ for $a\ge\tilde a$ and

$$T''_a(\alpha_m(a))=T''_a(\alpha_m(a))+\frac{2}{\alpha_m(a)}T'_a(\alpha_m(a))>0\quad\text{for } a\ge\tilde a$$

by Lemma 3.7. Then by the Implicit Function Theorem, $\alpha_m(a)$ is a continuous function on $[\tilde a,\infty)$. It follows that $\alpha_m(a)$ is a continuous function on (a_0,∞). Thus by Step 2 and (3.60) assertion (ii) holds.

Step 6. We prove assertion (iii). By (3.52), it is sufficient to prove that

$$\lim_{a\to a_0^+} \alpha_M(a) = \lim_{a\to a_0^+} \alpha_m(a) < \alpha_m(a) < \lim_{a\to\infty} \alpha_m(a) = \infty \quad \text{for } a > a_0. \tag{3.61}$$

By [12, Theorem 2.2], we see that

$$p_1(a) < \|u_{\lambda^*}\|_\infty < \gamma(a) = \frac{a\,(a-2)}{2} < p_2(a) < \|u_{\lambda_*}\|_\infty \quad \text{for } a \geq a^*, \tag{3.62}$$

where a^* is defined in (1.3). So we observe that $\lim_{a\to\infty} \alpha_m(a) \geq \lim_{a\to\infty} p_2(a) = \infty$. It follows that $\alpha_m(a) < \lim_{a\to\infty} \alpha_m(a) = \infty$ for $a > a_0$.

For the sake of convenience, we let $\alpha^+ \equiv \lim_{a\to a_0^+} \alpha_M(a)$ and $\alpha^- \equiv \lim_{a\to a_0^+} \alpha_m(a)$. By Step 3, $\alpha^+ \leq \alpha^-$. Suppose to the contrary that $\alpha^+ < \alpha^-$. Then we assert that $T'_{a_0}(\beta) > 0$ for some $\beta \in (\alpha^+, \alpha^-)$. Otherwise, $T'_{a_0}(\alpha) = 0$ for all $\alpha \in (\alpha^+, \alpha^-)$. It is a contradiction by Lemmas 3.6, 3.7 and 3.11. By (3.52), we find that

$$\alpha_M(a) < \alpha^+ < \beta < \alpha^- < \alpha_m(a) \quad \text{for } a_0 < a \leq \tilde{a},$$

which implies that $T'_a(\beta) < 0 < T'_{a_0}(\beta)$ for $a_0 < a \leq \tilde{a}$. It is a contradiction by Lemma 3.6(ii). So $\alpha^+ = \alpha^-$. Then (3.61) holds. It implies that assertion (iii) holds.

The proof of Lemma 3.16 is complete. ■

We are in a position to prove Lemma 3.3 by applying Lemmas 3.2, 3.5–3.7, 3.11, 3.14 and 3.16.

Proof of Lemma 3.3. Since $\kappa(a) < 8 < \omega(a)$ for $4 < a \leq a_0$ by Lemma 3.5(i), we see that $T'_a(\alpha) > T'_{a_0}(\alpha) \geq 0$ for $0 < \alpha \leq \kappa(a)$ and $4 < a < a_0$. Suppose to the contrary that there exists $\beta_a > \kappa(a)$ for some $a \in (4, a_0)$ such that $T'_a(\beta_a) = 0$. So by Lemma 3.7, $T''_a(\beta_a) > 0$. It implies that $T_a(\alpha)$ has a local minimum point at $\beta_a > \kappa(a)$. It is a contradiction by Lemma 3.2. Thus, $T'_a(\alpha) > 0$ for $\alpha > 0$ and $4 < a < a_0$. So assertion (i) holds.

Next, we prove assertion (ii) of Lemma 3.3. By Lemma 3.16(iii), we obtain that $\alpha_M(a) < \alpha_0 < \alpha_m(a)$ for $a > a_0$ where $\alpha_0 \equiv \lim_{a\to a_0^+} \alpha_M(a) = \lim_{a\to a_0^+} \alpha_m(a)$. It follows that $T'_a(\alpha_0) < 0$ for $a > a_0$. Moreover, $T'_{a_0}(\alpha_0) \leq 0$ by Lemma 3.6(ii). By Lemmas 3.2 and 3.14, $T'_{a_0}(\alpha_0) = 0$ and $T'_{a_0}(\alpha) > 0$ for $\alpha \in (0, \infty)\backslash\{\alpha_0\}$. We assert that $\gamma(a_0) \leq \alpha_0 < \kappa(a_0)$. Indeed, if $\alpha_0 < \gamma(a_0)$, and by Lemma 3.16(i)–(ii), then there exists $a > a_0$ such that $\alpha_M(a) < \alpha_0 < \alpha_m(a) < \gamma(a_0)$. It is a contradiction by Lemma 3.6(i). So $\gamma(a_0) \leq \alpha_0$. Since $T'_{a_0}(\alpha_0) = T''_{a_0}(\alpha_0)$, and by Lemma 3.7, we find that $\alpha_0 < \kappa(a_0)$. So $\gamma(a_0) \leq \alpha_0 < \kappa(a_0)$.

The proof of Lemma 3.3 is complete. ■

4. Proof of The Main Result

Proof of Theorem 2.1. As mentioned in Section 3, to prove Theorem 2.1(i), (ii) and (iii), it is sufficient to prove that there exists a number $a_0 \approx 4.069$ satisfying $4 < a_0 < \tilde{a} \approx 4.107$ such that parts (M1), (M2) and (M3) hold, respectively. Notice that ordering properties of positive solutions of (1.1) in Theorem 2.1(i) can be obtained easily. We have that part (M1) holds immediately by Lemmas 3.1, 3.2 and 3.16(iii); part (M2) holds immediately by Lemmas 3.1 and 3.3(ii); and part (M3) holds immediately by Lemmas 3.1 and 3.3(i). Furthermore, by numerical simulation, we find that $a_0 \approx 4.069$.

The proof of Theorem 2.1 is complete. ■

References

[1] J. Bebernes, D. Eberly, Mathematical problems from combustion theory, Springer-Verlag, New York, 1989.

[2] T. Boddington, P. Gray, C. Robinson, Thermal explosion and the disappearance of criticality at small activation energies: exact results for the slab, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 368 (1979) 441–461.

[3] K.J. Brown, M.M.A. Ibrahim, R. Shivaji, S-shaped bifurcation curves, Nonlinear Anal. 5 (1981) 475–486.

[4] M.G. Crandall, P.H. Rabinowitz, Bifurcation, perturbation of simple eigenvalues and linearized stability, Arch. Rational Mech. Anal. 52 (1973) 161–180.

[5] Y. Du, Exact multiplicity and S-shaped bifurcation curve for some semilinear elliptic problems from combustion theory, SIAM J. Math. Anal. 32 (2000) 707–733.

[6] Y. Du, Y. Lou, Proof of a conjecture for the perturbed Gelfand equation from combustion theory, J. Differential Equations 173 (2001) 213–230.

[7] I.M. Gelfand, Some problems in the theory of quasilinear equations, Amer. Math. Soc. Transl. (2) 29 (1963) 295–381.

[8] S.P. Hastings, J.B. McLeod, The number of solutions to an equation from catalysis, Proc. Roy. Soc. Edinburgh Sect. A 101 (1985) 15–30.

[9] S.P. Hastings, J.B. McLeod, Classical methods in ordinary differential equations: With applications to boundary value problems. Graduate Studies in Mathematics, 129. American Mathematical Society, Providence, RI, 2012.

[10] S.-Y. Huang, S.-H. Wang, On S-shaped bifurcation curves for a two-point boundary value problem arising in a theory of thermal explosion, Discrete Contin. Dyn. Syst. 35 (2015) 4839–4858.

[11] S.-Y. Huang, S.-H. Wang, Tasks in some proofs (2016). Available form http://mx.nthu.edu.tw/~sy-huang/Gelfand3/Proof.pdf.

[12] K.-C. Hung, S.-H. Wang, A theorem on S-shaped bifurcation curve for a positone problem with convex-concave nonlinearity and its applications to the perturbed Gelfand problem, J. Differential Equations 251 (2011) 223–237.

[13] K.-C. Hung, S.-H. Wang, Global bifurcation and exact multiplicity of positive solutions for a positone problem with cubic nonlinearity and their applications, Trans. Amer. Math. Soc. 365 (2013) 1933–1956.

[14] J. Jacobsen, K. Schmitt, The Liouville-Bratu-Gelfand problem for radial operators, J. Differential Equations 184 (2002) 283–298.

[15] P. Korman, Y. Li, On the exactness of an S-shaped bifurcation curve, Proc. Amer. Math. Soc. 127 (1999) 1011–1020.

[16] P. Korman, Y. Li, T. Ouyang, Computing the location and direction of bifurcation, Math. Res. Lett. 12 (2005) 933–944.

[17] T. Laetsch, The number of solutions of a nonlinear two point boundary value problem, Indiana Univ. Math. J. 20 (1970) 1–13.

[18] J. Liouville, Sur l'équation aux différences partielles $\frac{d^2 \log \lambda}{dudv} \pm \lambda 2a^2 = 0$, J. Math. Pures Appl. 18 (1853) 71–72.

[19] M. Mimura, K. Sakamoto, Multi-dimensional transition layers for an exothermic reaction-diffusion system in long cylindrical domains, J. Math. Sci. Univ. Tokyo 3 (1996) 109–179.

[20] J. Shi, Persistence and bifurcation of degenerate solutions, J. Funct. Anal. 169 (1999) 494–531.

[21] R. Shivaji, Remarks on an S-shaped bifurcation curve, J. Math. Anal. Appl. 111 (1985) 374–387.

[22] S.-H. Wang, On S-shaped bifurcation curves, Nonlinear Anal. TMA 22 (1994) 1475–1485.

[23] Ya.B. Zeldovich, G.I. Barenblatt, V.B. Librovich, G.M. Makhviladze, The mathematical theory of combustion and explosions. Consultants Bureau [Plenum], New York, 1985.

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