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Arithmetic properties of the generalized trigonometric functions *

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1 Introduction

Let $p,\ q\in(1,\infty)$ be any constants. We define $\sin_{p,q}x$ by the inverse function of

$$\sin_{p,q}^{-1}x:=\int_{0}^{x}\frac{dt}{(1-t^{q})^{1/p}},\quad 0\leq x\leq 1,$$

and

$$\pi_{p,q}:=2\sin_{p,q}^{-1}1=2\int_{0}^{1}\frac{dt}{(1-t^{q})^{1/p}}=\frac{2}{q}B\left(\frac{1}{p^{*}},\frac{1}{q}\right), \tag{1.1}$$

where $p^{*}:=p/(p-1)$ and B denotes the beta function. The function $\sin_{p,q}x$ is increasing in $[0,\pi_{p,q}/2]$ onto $[0,1]$. We extend it to $(\pi_{p,q}/2,\pi_{p,q}]$ by $\sin_{p,q}(\pi_{p,q}-x)$ and to the whole real line $\mathbb{R}$ as the odd $2\pi_{p,q}$-periodic continuation of the function. Since $\sin_{p,q}x\in C^{1}(\mathbb{R})$, we also define $\cos_{p,q}x$ by $\cos_{p,q}x:=(\sin_{p,q}x)'$. Then, it follows that

$$|\cos_{p,q}x|^{p}+|\sin_{p,q}x|^{q}=1.$$

In case $p=q=2$, it is obvious that $\sin_{p,q}x,\ \cos_{p,q}x$ and $\pi_{p,q}$ are reduced to the ordinary $\sin x,\ \cos x$ and π, respectively. This is a reason why these functions and the constant are called *generalized trigonometric functions* (with parameter (p,q)) and *the generalized* π, respectively.

Originally E. Lundberg introduced the generalized trigonometric functions in 1879; see [32] for details. After his work, there are a lot of literature on the generalized trigonometric functions and related functions. See [11, 12, 19, 21, 23, 28, 29, 31, 32, 35] for general properties as functions; [18, 19, 20, 28, 33, 38] for applications to differential equations involving p-Laplacian; [6, 12, 13, 21, 22, 28, 39] for basis properties for sequences of these functions.

In particular, let us explain the work [20] of Drábek and Manásevich. They reintroduced the generalized trigonometric functions with two parameters to study an inhomogeneous

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eigenvalue problem of p-Laplacian. They gave a closed form of solutions (λ, u) of the eigenvalue problem

$$-(|u'|^{p-2}u')' = \lambda|u|^{q-2}u, \quad u(0) = u(L) = 0.$$

Indeed, for any $n = 1, 2, \ldots$, there exists a curve of solutions $(\lambda_{n,R}, u_{n,R})$ with a parameter $R \in \mathbb{R} \setminus \{0\}$ such that

$$\lambda_{n,R} = \frac{q}{p^*}\left(\frac{n\pi_{p,q}}{L}\right)^p |R|^{p-q}, \tag{1.2}$$

$$u_{n,R}(x) = R\sin_{p,q}\left(\frac{n\pi_{p,q}}{L}x\right) \tag{1.3}$$

(Figure 1). Conversely, there exists no other solution of the eigenvalue problem. In this sense, the generalized sine function $\sin_{p,q} x$ is also called *the (p,q)-eigenfunction of the p-Laplacian.* Thus, the generalized trigonometric functions play important roles to study problems of the p-Laplacian.

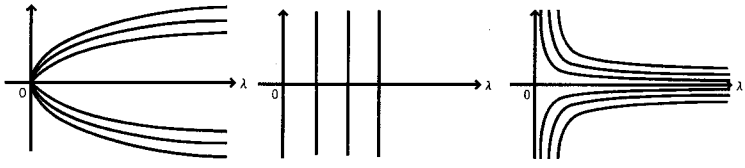

Figure 1: The bifurcation diagrams in cases $p > q$, $p = q$ and $p < q$.

As above, there are many works in which the generalized trigonometric functions are used to study problems of existence, bifurcation and oscillation. However, any arithmetic properties are almost unknown though they are generalizations of the classical trigonometric functions.

This is a survey of author's recent studies [27, 40, 41, 42, 43] about arithmetic properties of the generalized trigonometric functions.

This paper is organized as follows. Section 2 is devoted to prepare basic properties of the generalized trigonometric functions. In Section 3, we will present new multiple-angle formulas which are established between two kinds of the generalized trigonometric functions, and apply the formulas to generalize classical topics related to the trigonometric functions and the lemniscate function. Concerning these functions, no multiple-angle formula has been known except for the classical cases and a special case discovered by Edmunds, Gurka and Lang, not to mention addition theorems. In Section 4, the generalized trigonometric functions are applied to the Legendre form of complete elliptic integrals, and a new form of the generalized complete elliptic integrals of the Borweins [7] is presented. According to the form, it can be easily shown that these integrals have similar properties to the classical ones. In particular, it is possible to establish a computation formula of the generalized π in terms of the arithmetic-geometric mean, in the classical way as the Gauss-Legendre algorithm for π by Salamin and Brent. Moreover, an alternative proof of Ramanujan's cubic transformation can be also given. In Section 5, Legendre's relation for the complete elliptic integrals of the first and second kinds is generalized. The proof depends on an application of the generalized trigonometric functions and is alternative to the proof for Elliott's identity.

Finally, in Section 6, we give a proof of Legendre's relation of the incomplete elliptic integrals for our future works.

2 Preparation

Let $p,\ q \in (1,\infty)$ and $x \in (0, \pi_{p,q}/2)$. It is easy to see that

$$\cos_{p,q}^p x + \sin_{p,q}^q x = 1,$$
$$(\sin_{p,q} x)' = \cos_{p,q} x, \quad (\cos_{p,q} x)' = -\frac{q}{p} \sin_{p,q}^{q-1} x \cos_{p,q}^{2-p} x,$$
$$(\cos_{p,q}^{p-1} x)' = -\frac{q}{p^*} \sin_{p,q}^{q-1} x.$$

If we extend to these formulas for any $x \in \mathbb{R}$, then the last one, for example, corresponds to

$$(|\cos_{p,q} x|^{p-2} \cos_{p,q} x)' = -\frac{q}{p^*} |\sin_{p,q} x|^{q-2} \sin_{p,q} x. \tag{2.1}$$

From the differentiation of inverse functions,

$$(\cos_{p^*,p}^{p^*-1})^{-1}(y) = \int_y^1 \frac{dt}{(1-t^p)^{1/p^*}}, \quad 0 \le y \le 1,$$

hence

$$\sin_{p^*,p}^{-1} y + (\cos_{p^*,p}^{p^*-1})^{-1}(y) = \frac{\pi_{p^*,p}}{2}.$$

Therefore, for $x \in [0, \pi_{p^*,p}/2]$ we have

$$\sin_{p^*,p}\left(\frac{\pi_{p^*,p}}{2} - x\right) = \cos_{p^*,p}^{p^*-1} x, \tag{2.2}$$
$$\cos_{p^*,p}^{p^*-1}\left(\frac{\pi_{p^*,p}}{2} - x\right) = \sin_{p^*,p} x. \tag{2.3}$$

Also, the following function is useful:

$$\tau_{p,q}(x) := \frac{\sin_{p,q} x}{|\cos_{p,q} x|^{p/q-1} \cos_{p,q} x}, \quad x \ne \frac{2n+1}{2}\pi_{p,q},\ n \in \mathbb{Z}.$$

Then, it follows immediately from (2.2) that

Lemma 2.1. *For $x \in (0, \pi_{p^*,p}/2)$, $\tau_{p^*,p}(x) = 1$ implies $x = \pi_{p^*,p}/4$. Moreover, $\sin_{p^*,p}^{-1}(2^{-1/p}) = \cos_{p^*,p}^{-1}(2^{-1/p^*}) = \pi_{p^*,p}/4$.*

3 Multiple-angle formulas

For the details of this section, we refer the reader to [41].

It is of interest to know whether the generalized trigonometric functions have multiple-angle formulas unless $p = q = 2$. A few multiple-angle formulas seem to be known. Actually, in case $2p = q = 4$, the function $\sin_{p,q} x = \sin_{2,4} x$ coincides with the lemniscate sine function $\operatorname{sl} x$, whose inverse function is defined as

$$\operatorname{sl}^{-1} x := \int_0^x \frac{dt}{\sqrt{1-t^4}}.$$

Furthermore, $\pi_{2,4}$ is equal to the lemniscate constant $\varpi := 2\operatorname{sl}^{-1} 1 = 2.6220\cdots$. Concerning $\operatorname{sl} x$ and ϖ, we refer the reader to [37, p. 81], [44] and [45, §22.8]. Since $\operatorname{sl} x$ has the multiple-angle formula

$$\operatorname{sl}(2x) = \frac{2\operatorname{sl} x\sqrt{1-\operatorname{sl}^4 x}}{1+\operatorname{sl}^4 x}, \quad 0 \le x \le \frac{\varpi}{2}, \tag{3.1}$$

we see that

$$\sin_{2,4}(2x) = \frac{2\sin_{2,4} x \cos_{2,4} x}{1+\sin_{2,4}^4 x}, \quad 0 \le x \le \frac{\pi_{2,4}}{2}.$$

Also in case $p^* = q = 4$, it is possible to show that $\sin_{p,q} x = \sin_{4/3,4} x$ can be expressed in terms of the Jacobian elliptic function, whose multiple-angle formula yields

$$\sin_{4/3,4}(2x) = \frac{2\sin_{4/3,4} x \cos_{4/3,4}^{1/3} x}{\sqrt{1+4\sin_{4/3,4}^4 x \cos_{4/3,4}^{4/3} x}} \quad 0 \le x < \frac{\pi_{4/3,4}}{4}. \tag{3.2}$$

The formula (3.2) was investigated by Edmunds, Gurka and Lang [22, Proposition 3.4]. They also proved an addition theorem for $\sin_{4/3,4} x$ involving (3.2). Such reductions to the elliptic functions have previously been used by Cayley [16] and Lindqvist and Peetre [30].

3.1 Results

We will present multiple-angle formulas which are established between two kinds of the generalized trigonometric functions with parameters $(2,p)$ and (p^*,p).

Theorem 3.1 ([41]). *For $p \in (1,\infty)$ and $x \in [0, 2^{-2/p}\pi_{2,p}] = [0, \pi_{p^*,p}/2]$, we have*

$$\sin_{2,p}(2^{2/p}x) = 2^{2/p}\sin_{p^*,p} x \cos_{p^*,p}^{p^*-1} x \tag{3.3}$$

and

$$\begin{aligned}\cos_{2,p}(2^{2/p}x) &= \cos_{p^*,p}^{p^*} x - \sin_{p^*,p}^{p} x \\ &= 1 - 2\sin_{p^*,p}^{p} x = 2\cos_{p^*,p}^{p^*} x - 1.\end{aligned} \tag{3.4}$$

Moreover, for $x \in \mathbb{R}$, we have

$$\sin_{2,p}(2^{2/p}x) = 2^{2/p}\sin_{p^*,p} x |\cos_{p^*,p} x|^{p^*-2}\cos_{p^*,p} x \tag{3.5}$$

and

$$\begin{aligned}\cos_{2,p}(2^{2/p}x) &= |\cos_{p^*,p} x|^{p^*} - |\sin_{p^*,p} x|^{p} \\ &= 1 - 2|\sin_{p^*,p} x|^{p} = 2|\cos_{p^*,p} x|^{p^*} - 1.\end{aligned} \tag{3.6}$$

Proof. Let $x \in [0, \pi_{p^*,p}/4]$. Then, $y = \sin_{p^*,p} x \in [0, 2^{-1/p}]$ by Lemma 2.1. Setting $t^p = (1-(1-s^p)^{1/2})/2$ in

$$\sin_{p^*,p}^{-1} y = \int_0^y \frac{dt}{(1-t^p)^{1/p^*}},$$

we have

$$\begin{aligned}\sin_{p^*,p}^{-1} y &= \int_0^{y(4(1-y^p))^{1/p}} \frac{\dfrac{2^{-1-1/p}s^{p-1}}{(1-s^p)^{1/2}(1-(1-s^p)^{1/2})^{1-1/p}}\,ds}{2^{-1+1/p}(1+(1-s^p)^{1/2})^{1-1/p}} \\ &= 2^{-2/p}\int_0^{y(4(1-y^p))^{1/p}} \frac{ds}{(1-s^p)^{1/2}};\end{aligned}$$

that is,

$$\sin_{p^*,p}^{-1} y = 2^{-2/p}\sin_{2,p}^{-1}(y(4(1-y^p))^{1/p}). \tag{3.7}$$

Hence we obtain

$$\sin_{2,p}(2^{2/p}x) = 2^{2/p}\sin_{p^*,p} x \cos_{p^*,p}^{p^*-1} x,$$

and (3.3) is proved. In particular, letting $y = 2^{-1/p}$ in (3.7) and using Lemma 2.1, we get

$$\frac{\pi_{p^*,p}}{4} = 2^{-2/p}\sin_{2,p}^{-1} 1 = \frac{\pi_{2,p}}{2^{1+2/p}},$$

which implies

$$\frac{\pi_{2,p}}{2^{2/p}} = \frac{\pi_{p^*,p}}{2}. \tag{3.8}$$

Next, let $x \in (\pi_{p^*,p}/4, \pi_{p^*,p}/2]$ and $y := \pi_{p^*,p}/2 - x \in [0, \pi_{p^*,p}/4)$. By the symmetry properties (2.2) and (2.3), we obtain

$$2^{2/p}\sin_{p^*,p} x \cos_{p^*,p}^{p^*-1} x = 2^{2/p}\cos_{p^*,p}^{p^*-1} y \sin_{p^*,p} y.$$

According to the argument above, the right-hand side is identical to $\sin_{2,p}(2^{2/p}y)$. Moreover, (3.8) gives

$$\sin_{2,p}(2^{2/p}y) = \sin_{2,p}(\pi_{2,p} - 2^{2/p}x) = \sin_{2,p}(2^{2/p}x).$$

The formula (3.4) is deduced from differentiating both sides of (3.3). Moreover, (3.5) and (3.6) come from the periodicities of the functions. □

In Theorem 3.1, the fact (3.8) is the special case $n = 2$ of the following identity.

Theorem 3.2 ([41]). *Let* $2 \le n < p+1$. *Then*

$$\pi_{\frac{p}{p-1},p}\pi_{\frac{p}{p-2},p}\cdots\pi_{\frac{p}{p-n+1},p} = n^{1-n/p}\pi_{\frac{n}{n-1},p}\pi_{\frac{n}{n-2},p}\cdots\pi_{\frac{n}{1},p}.$$

Proof. Set $x = 1/n$ and $y = 1/p$ in the formula of the beta function (see [45, §12.15, Example])

$$B(nx, ny) = \frac{1}{n^{ny}}\frac{\prod_{k=0}^{n-1} B(x+k/n, y)}{\prod_{k=1}^{n-1} B(ky, y)}, \quad n \ge 2,\ x,\ y > 0$$

and use (1.1). We omit the details. □

We give a series expansion of $\pi_{p^*,p}$ as a counterpart of the Gregory-Leibniz series for π. It is worth pointing out that $\pi_{p^*,p}$ is the area enclosed by the p-circle $|x|^p + |y|^p = 1$ (see [28, 31]).

Theorem 3.3 ([41]).

$$\begin{aligned}\frac{\pi_{p^*,p}}{4} &= \sum_{n=0}^{\infty} \frac{(2/p)_n}{n!} \frac{(-1)^n}{pn+1} \\ &= 1 - \frac{2}{p(p+1)} + \frac{2+p}{p^2(2p+1)} - \frac{(2+p)(2+2p)}{3p^3(3p+1)} + \cdots,\end{aligned}$$

where $(a)_n := \Gamma(a+n)/\Gamma(a) = a(a+1)(a+2)\cdots(a+n-1)$ *and* Γ *denotes the gamma function.*

Proof. Let $x \in (0,1)$. Differentiating the inverse function of $\tau_{p^*,p}(x)$, we have

$$\tau_{p^*,p}^{-1}(x) = \int_0^x \frac{dt}{(1+t^p)^{2/p}}.$$

Hence

$$\tau_{p^*,p}^{-1}(x) = \int_0^x \sum_{n=0}^{\infty} \binom{-2/p}{n} t^{pn}\,dt = x \sum_{n=0}^{\infty} \frac{(2/p)_n}{n!} \frac{(-x^p)^n}{pn+1}. \tag{3.9}$$

By Abel's continuity theorem [45, §3.71], the series above converges to $\tau_{p^*,p}^{-1}(1)$ (see for instance [45, §2.31, Corollary (ii)]). From Lemma 2.1, we concludes the theorem. □

Remark 3.1. Combining (3.5) and (3.6), we can assert that $\tau_{2,p}$ and $\tau_{p^*,p}$ satisfy the multiple-angle formula

$$\tau_{2,p}(2^{2/p}x) = \frac{2^{2/p}\tau_{p^*,p}(x)}{1-|\tau_{p^*,p}(x)|^p},$$

which coincides with that of the tangent function if $p = 2$.

3.2 Appications

The following curious fact is the consequence of a straightforward calculation with (1.2), (1.3), (3.5) and (3.8).

Theorem 3.4 ([41]). *Let* $n \in \mathbb{N}$ *and* $p \in (1,\infty)$. *Let* u *be an eigenfunction with* $(n-1)$*-zeros in* $(0,L)$ *for an eigenvalue* $\lambda > 0$ *of the eigenvalue problem*

$$-(|u'|^{p-2}u')' = \lambda |u|^{p^*-2}u, \quad u(0) = u(L) = 0, \tag{3.10}$$

and v *an eigenfunction with* n*-zeros in* $(0,L)$ *for an eigenvalue* $\mu > 0$ *of the eigenvalue problem*

$$-(|v'|^{p-2}v')' = \mu |v|^{p^*-2}v, \quad v'(0) = v'(L) = 0. \tag{3.11}$$

Then, the product $w = uv$ *is an eigenfunction for the eigenvalue* $\xi = 2p^*(\lambda\mu)^{1/p}$ *with* $(2n-1)$*-zeros in* $(0,L)$ *of the eigenvalue problem*

$$-w'' = \xi |w|^{p^*-2}w, \quad w(0) = w(L) = 0. \tag{3.12}$$

Such a relation between the eigenvalue problems of the p-Laplacian and that of the Laplacian may be known. However, we can not find a literature proving it, while the assertion in case $p = 2$ is trivial because

$$w = \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right) = \frac{1}{2}\sin\left(\frac{2n\pi}{L}x\right).$$

Proof of Theorem 3.4. By (1.2) and (1.3), the solution (λ, u) of (3.10) can be expressed as follows:

$$\lambda = \left(\frac{n\pi_{p,p^*}}{L}\right)^p |R|^{p-p^*},$$
$$u(x) = R\sin_{p,p^*}\left(\frac{n\pi_{p,p^*}}{L}x\right), \quad R \neq 0.$$

Similarly, by the symmetry (2.2), the solution (μ, v) of (3.11) is represented as

$$\mu = \left(\frac{n\pi_{p,p^*}}{L}\right)^p |Q|^{p-p^*},$$
$$v(x) = Q\left|\cos_{p,p^*}\left(\frac{n\pi_{p,p^*}}{L}x\right)\right|^{p-2}\cos_{p,p^*}\left(\frac{n\pi_{p,p^*}}{L}x\right), \quad Q \neq 0.$$

Applying (3.5) in Theorem 3.1 and (3.8) to the product $w = uv$, we have

$$\begin{aligned} w(x) &= RQ\sin_{p,p^*}\left(\frac{n\pi_{p,p^*}}{L}x\right)\left|\cos_{p,p^*}\left(\frac{n\pi_{p,p^*}}{L}x\right)\right|^{p-2}\cos_{p,p^*}\left(\frac{n\pi_{p,p^*}}{L}x\right) \\ &= 2^{-2/p^*}RQ\sin_{2,p^*}\left(\frac{2n\pi_{2,p^*}}{L}x\right), \end{aligned}$$

which belongs to $C^2(\mathbb{R})$ and has $(2n-1)$-zeros in $(0, L)$. Therefore, by (2.1) with $p = 2$, a direct calculation shows

$$\begin{aligned} w'' &= -p^* 2^{1-2/p^*}\left(\frac{n\pi_{2,p^*}}{L}\right)^2 RQ\left|\sin_{2,p^*}\left(\frac{n\pi_{2,p^*}}{L}x\right)\right|^{p^*-2}\sin_{2,p^*}\left(\frac{n\pi_{2,p^*}}{L}x\right) \\ &= -p^* 2^{3-4/p^*}\left(\frac{n\pi_{2,p^*}}{L}\right)^2 |RQ|^{2-p^*}|w|^{p^*-2}w. \end{aligned} \tag{3.13}$$

On the other hand, (3.8) gives

$$(\lambda\mu)^{1/p} = 2^{2-4/p^*}\left(\frac{n\pi_{2,p^*}}{L}\right)^2 |RQ|^{2-p^*}. \tag{3.14}$$

Combining (3.13) and (3.14), we obtain (3.12). □

Moreover, we can also apply Theorems 3.1–3.3 to the following problems (I)–(IV).

(I) *An alternative proof of* (3.2). It should be noted that the multiple-angle formula (3.3) in Theorem 3.1 allows (3.2) to be rewritten in terms of the lemniscate function $\operatorname{sl} x = \sin_{2,4} x$:

$$\sin_{4/3,4}(2x) = \frac{\sqrt{2}\operatorname{sl}(\sqrt{2}x)}{\sqrt{1+\operatorname{sl}^4(\sqrt{2}x)}}, \quad 0 \leq x < \frac{\pi_{4/3,4}}{4} = \frac{\varpi}{2\sqrt{2}},$$

where the last equality above follows from (3.8) with $\pi_{2,4} = \varpi$. This indicates that it is possible to obtain (3.2) from the multiple-angle formula (3.1) for the lemniscate function.

(II) *A pendulum-type equation with the p-Laplacian.* It is possible to give a closed form of solutions of the pendulum-type equation

$$-(|\theta'|^{p-2}\theta')' = \lambda^p(p-1)|\sin_{2,p}\theta|^{p-2}\sin_{2,p}\theta.$$

In case $p = 2$, this equation is the ordinary pendulum equation $-\theta'' = \lambda^2 \sin\theta$ and it is well known that the solutions can be expressed in terms of the Jacobian elliptic function. We can obtain an expression of the solution for the pendulum-type equation above by using our special functions involving a generalization of the Jacobian elliptic function in [38, 39]. There are studies of other (forced) pendulum-type equations with p-Laplacian versus $\sin\theta$ in [36]; versus $\sin_{p,p}\theta$ in [1], for the purpose of finding periodic solutions.

(III) *Catalan-type constants.* Catalan's constant, which occasionally appears in estimates in combinatorics, is defined by

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.9159\cdots.$$

We can find a lot of representation of G in [9]; for a typical example,

$$\frac{1}{2}\int_0^{\pi/2} \frac{x}{\sin x}\,dx = G. \tag{3.15}$$

The multiple-angle formula (3.3) gives a generalization of (3.15) as

$$\frac{1}{2^{2/p}}\int_0^{\pi_{2,p}/2} \frac{x}{\sin_{2,p} x}\,dx = \sum_{n=0}^{\infty} \frac{(2/p)_n}{n!}\frac{(-1)^n}{(pn+1)^2}. \tag{3.16}$$

In case $p = 2$, the formula (3.16) coincides with (3.15). Moreover, for $p = 4$ we obtain the interesting formula for the lemniscate function:

$$\frac{1}{\sqrt{2}}\int_0^{\varpi/2} \frac{x}{\operatorname{sl} x}\,dx = \sum_{n=0}^{\infty} \frac{(1/2)_n}{n!}\frac{(-1)^n}{(4n+1)^2}.$$

(IV) *Series expansions of the lemniscate constant* ϖ. The lemniscate constant ϖ has the formula ([44, Theorem 5]):

$$\frac{\varpi}{2} = 1 + \frac{1}{10} + \frac{1}{24} + \frac{5}{208} + \cdots + \frac{(2n-1)!!}{(2n)!!}\frac{1}{4n+1} + \cdots,$$

where $(-1)!! := 1$. For this, using Theorem 3.3 with (3.8), we can obtain

$$\frac{\varpi}{2\sqrt{2}} = 1 - \frac{1}{10} + \frac{1}{24} - \frac{5}{208} + \cdots + \frac{(2n-1)!!}{(2n)!!}\frac{(-1)^n}{4n+1} + \cdots,$$

which does not appear in Todd [44] and seems to be new. We can also produce some other formulas of ϖ.

4 Gauss-Legendre algorithm for π_p

For the details of this section, we refer the reader to [40, 43].

The complete elliptic integrals of the first kind and of the second kind

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},$$
$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\, d\theta = \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}\, dt$$

are classical integrals which have helped us, for instance, to evaluate the length of curves and to express exact solutions of differential equations.

In this section we give a generalization of the complete elliptic integrals as an application of the generalized trigonometric functions. For this, we need the generalized sine function $\sin_p x$ and the generalized π denoted by π_p, where $\sin_p x$ is the inverse function of

$$\sin_p^{-1} x := \sin_{p,p}^{-1} x = \int_0^x \frac{dt}{(1-t^p)^{1/p}}, \quad 0 \le x \le 1,$$

and π_p is the number defined by

$$\pi_p := \pi_{p,p} = 2\sin_p^{-1} 1 = 2\int_0^1 \frac{dt}{(1-t^p)^{1/p}} = \frac{2\pi}{p\sin(\pi/p)}.$$

Clearly, $\sin_2 x = \sin x$ and $\pi_2 = \pi$. These two appear in the eigenvalue problem of one-dimensional p-Laplacian:

$$-(|u'|^{p-2}u')' = \lambda|u|^{p-2}u, \quad u(0) = u(1) = 0.$$

Indeed, the eigenvalues are given as $\lambda_n = (p-1)(n\pi_p)^p$, $n = 1, 2, 3, \ldots$, and the corresponding eigenfunction to λ_n is $u_n(x) = \sin_p(n\pi_p x)$ for each n.

Remark 4.1. The behavior of λ_n with respect to p is interesting; see [26].

4.1 Generalized elliptic integrals with one-parameter

Now, applying $\sin_p x$ and π_p to the complete elliptic functions, we define the *complete p-elliptic integrals of the first kind* $K_p(k)$ and *of the second kind* $E_p(k)$: for $p \in (1, \infty)$ and $k \in [0, 1)$

$$K_p(k) := \int_0^{\pi_p/2} \frac{d\theta}{(1-k^p\sin_p^p\theta)^{1-1/p}} = \int_0^1 \frac{dt}{(1-t^p)^{1/p}(1-k^pt^p)^{1-1/p}}, \tag{4.1}$$

$$E_p(k) := \int_0^{\pi_p/2} (1-k^p\sin_p^p\theta)^{1/p}\, d\theta = \int_0^1 \left(\frac{1-k^pt^p}{1-t^p}\right)^{1/p} dt. \tag{4.2}$$

Here, each second equality of the definitions is obtained by setting $\sin_p\theta = t$. It is easy to see that for $p = 2$ these integrals are equivalent to the classical complete elliptic integrals

$K(k)$ and $E(k)$. The complete p-elliptic integrals have similar properties to the complete elliptic integrals.

It is worth pointing out that the Borweins [7, Section 5.5] define the *generalized complete elliptic integrals of the first* and *of the second kind* by

$$\mathrm{K}_s(k) := \frac{\pi}{2} F\left(\frac{1}{2} - s, \frac{1}{2} + s; 1; k^2\right),$$
$$\mathrm{E}_s(k) := \frac{\pi}{2} F\left(-\frac{1}{2} - s, \frac{1}{2} + s; 1; k^2\right)$$

for $|s| < 1/2$ and $0 \leq k < 1$. Here, $F(a, b; c; x)$ denotes the Gaussian hypergeometric series defined for $|x| < 1$ as

$$F(a, b; c; x) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!},$$

where $a,\ b \in \mathbb{R},\ c \neq -1, -2, \ldots$ and

$$(a)_n := a(a+1)(a+2)\cdots(a+n-1), \quad (a)_0 := 1.$$

Note that $\mathrm{K}_0(k) = K(k)$ and $\mathrm{E}_0(k) = E(k)$. According to Euler's integral representation (see [3, Theorem 2.2.1] or [45, p. 293]), we have

$$\mathrm{K}_s(k) = \frac{\cos \pi s}{2s+1} \int_0^1 \frac{dt}{(1 - t^{\frac{2}{2s+1}})^{\frac{2s+1}{2}} (1 - k^2 t^{\frac{2}{2s+1}})^{1 - \frac{2s+1}{2}}},$$
$$\mathrm{E}_s(k) = \frac{\cos \pi s}{2s+1} \int_0^1 \left(\frac{1 - k^2 t^{\frac{2}{2s+1}}}{1 - t^{\frac{2}{2s+1}}}\right)^{\frac{2s+1}{2}} dt.$$

Thus

$$\mathrm{K}_s(k) = \frac{\pi}{\pi_p} K_p(k^{2/p}), \quad \mathrm{E}_s(k) = \frac{\pi}{\pi_p} E_p(k^{2/p}),$$

where $p = 2/(2s+1)$. We emphasize that the complete p-elliptic integrals (4.1) and (4.2) give representations of the generalized complete elliptic integrals in the Legendre form with the generalized trigonometric functions. The advantage of using the complete p-elliptic integrals lies in the fact that it is possible to prove formulas of the generalized complete elliptic integrals simply as well as that of the classical complete elliptic integrals. For example, we have known the following Legendre relation between $K(k)$ and $E(k)$ (see [3, 7, 25, 45]).

$$E(k)K'(k) + K(k)E'(k) - K(k)K'(k) = \frac{\pi}{2}, \tag{4.3}$$

where $k' := \sqrt{1-k^2}$, $K'(k) := K(k')$ and $E'(k) := E(k')$. For this we can show the following relation between $K_p(k)$ and $E_p(k)$.

Theorem 4.1 ([40]). *For* $k \in (0, 1)$

$$E_p(k)K_p'(k) + K_p(k)E_p'(k) - K_p(k)K_p'(k) = \frac{\pi_p}{2}, \tag{4.4}$$

where $k' := (1 - k^p)^{1/p}$, $K_p'(k) := K_p(k')$ *and* $E_p'(k) := E_p(k')$.

In fact, it is known in [7] that $\mathrm{K}_s(k)$ and $\mathrm{E}_s(k)$ also satisfy the similar relation

$$\mathrm{E}_s(k)\mathrm{K}_s'(k) + \mathrm{K}_s(k)\mathrm{E}_s'(k) - \mathrm{K}_s(k)\mathrm{K}_s'(k) = \frac{\pi \cos \pi s}{2(1+2s)}$$

to (4.4), which follows from Elliott's identity (5.3) below. In contrast to this, our approach with generalized trigonometric functions seems to be more elementary and self-contained. In fact, (4.4) is more extended to an equivalent one to Elliott's identity in Section 5.

4.2 Gauss-Legendre algorithm

We prepare the auxiliary integral

$$I_p(a,b) := \int_0^{\pi_p/2} \frac{d\theta}{(a^p \cos_p^p \theta + b^p \sin_p^p \theta)^{1-1/p}}.$$

Using I_p, we can write $K_p(k) = I_p(1,k')$, where $k' := (1-k^p)^{1/p}$.

4.2.1 Case $p=2$

In case $p=2$, all the objects above coincide with the classical ones. As far as the complete elliptic integrals are concerned, the following fact is well-known (see [3, 7] for more details): Let $a \geq b > 0$, and assume that $\{a_n\}$ and $\{b_n\}$ are the sequences satisfying $a_0 = a,\ b_0 = b$ and

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}, \quad n = 0,1,2,\ldots.$$

Both the sequences converge to the same limit as $n \to \infty$, denoted by $M_2(a,b)$, the *arithmetic-geometric mean of a and b*. It is surprising that

$$I_2(a_n,b_n) = I_2(a,b) \quad \text{for all } n = 0,1,2,\ldots,$$

so that we can obtain the celebrated *Gauss formula*

$$K_2(k) = \frac{\pi}{2}\frac{1}{M_2(1,\sqrt{1-k^2})}. \tag{4.5}$$

Combining (4.5) with $k = 1/\sqrt{2}$ and the Legendre relation (4.3), Brent [10] and Salamin [34] independently proved the following famous formula of π:

$$\pi = \frac{4M_2\left(1,\dfrac{1}{\sqrt{2}}\right)^2}{1 - \displaystyle\sum_{n=1}^{\infty} 2^{n+1}(a_n^2 - b_n^2)}. \tag{4.6}$$

We emphasize that (4.6) is known as a fundamental formula to Brent-Salamin's algorithm, or Gauss-Legendre's algorithm, for computing the value of π.

4.2.2 Case $p = 3$

We are interested in finding a formula as (4.6) of π_p for $p \neq 2$. We take the sequences

$$a_{n+1} = \frac{a_n + 2b_n}{3}, \quad b_{n+1} = \sqrt[3]{\frac{(a_n^2 + a_n b_n + b_n^2)b_n}{3}}, \quad n = 0, 1, 2, \ldots. \tag{4.7}$$

As in the case of $p = 2$, both the sequences converge to the same limit $M_3(a, b)$ as $n \to \infty$. The important point is that

$$a_n I_3(a_n, b_n) = a I_3(a, b) \quad \text{for all } n = 0, 1, 2, \ldots,$$

and hence

$$K_3(k) = \frac{\pi_3}{2} \frac{1}{M_3(1, \sqrt[3]{1 - k^3})}. \tag{4.8}$$

Then, by (4.8) with $k = 1/\sqrt[3]{2}$ and the Legendre relation (4.4) with $p = 3$, we obtain

$$\pi_3 = \frac{2M_3\left(1, \frac{1}{\sqrt[3]{2}}\right)^2}{1 - 2\sum_{n=1}^{\infty} 3^n (a_n + c_n) c_n}, \quad c_n := \sqrt[3]{a_n^3 - b_n^3}.$$

Actually, (4.8) is identical to the result of the Borweins [8, Theorem 2.1 (b)] (with some trivial typos). In either proof, it is essential to show Ramanujan's cubic transformation: for $k \in (0, 1]$

$$F\left(\frac{1}{3}, \frac{2}{3}; 1; 1 - k^3\right) = \frac{3}{1 + 2k} F\left(\frac{1}{3}, \frac{2}{3}; 1; \left(\frac{1 - k}{1 + 2k}\right)^3\right). \tag{4.9}$$

This identity has been proved by, for instance, the Borweins [8], Berndt et al. [5, Corollary 2.4] or [4, Corollary 2.4 and (2.25)], and Chan [17], though Ramanujan did not leave his proof. Moreover, the use of the generalized elliptic integrals can give an alternative proof with elementary calculation; see [40] for details.

By Theorem 4.1 and (4.8), we obtain the following formula of π_3.

Theorem 4.2 ([40]). *Let $a = 1$ and $b = 1/\sqrt[3]{2}$. Then*

$$\pi_3 = \frac{2M_3\left(1, \frac{1}{\sqrt[3]{2}}\right)^2}{1 - 2\sum_{n=1}^{\infty} 3^n (a_n + c_n) c_n},$$

where $\{a_n\}$ and $\{b_n\}$ are the sequences (4.7) and $c_n := \sqrt[3]{a_n^3 - b_n^3}$.

It is a simple matter to obtain other formulas for π_3 if we combine $\pi_3 = 4\sqrt{3}\pi/9 = 2.418\cdots$ with a formula as (4.6). The former converges quadratically to π_3 and the latter does cubically. On the other hand, our formula in Theorem 4.2 converges cubically to π_3 (Table 1). However, we are not interested in such trivial formulas obtained from those of π, and it is not our purpose to study the speed of convergence and we will not develop this point here.

	25 digits	Error
q_1	2.418399152309345558425031	2.9449×10^{-12}
q_2	2.418399152312290467458771	4.0425×10^{-40}
q_3	2.418399152312290467458771	1.0367×10^{-124}
q_4	2.418399152312290467458771	1.8728×10^{-379}

Table 1: Convergence of q_m to π_3, where $q_m := \frac{2a_{m+1}^2}{1-2\sum_{n=1}^{m} 3^n(a_n+c_n)c_n}$.

4.2.3 Case $p = 4$

We give the following result of π_p for $p = 4$.

Theorem 4.3 ([43]). *Let $a \geq b > 0$, and assume that $\{a_n\}$ and $\{b_n\}$ are the sequences satisfying $a_0 = a,\ b_0 = b$ and*

$$a_{n+1} = \sqrt{\frac{a_n^2 + 3b_n^2}{4}}, \quad b_{n+1} = \sqrt[4]{\frac{(a_n^2 + b_n^2)b_n^2}{2}}, \quad n = 0, 1, 2, \dots. \tag{4.10}$$

Then, both the sequences converge to the same limit $M_4(a, b)$ as $n \to \infty$, and π_4 can be represented as

$$\pi_4 = \frac{2M_4\left(1, \frac{1}{\sqrt[4]{2}}\right)^2}{1 - \sum_{n=1}^{\infty} 2^{n+1}\sqrt{a_n^4 - b_n^4}},$$

where $a_0 = a = 1$ and $b_0 = b = 1/\sqrt[4]{2}$.

To show Theorem 4.3, it is crucial to prove

$$a_n^2 I_4(a_n, b_n) = a^2 I_4(a, b) \quad \text{for all } n = 0, 1, 2, \dots,$$

which yields

$$K_4(k) = \frac{\pi_4}{2} \frac{1}{M_4(1, \sqrt[4]{1 - k^4})}$$

(cf. (4.5) and (4.8)). Here we rely on Ramanujan's transformation (see [4, Theorem 9.4, p. 146]):

$$F\left(\frac{1}{4}, \frac{3}{4}; 1; 1 - \left(\frac{1-x}{1+3x}\right)^2\right) = \sqrt{1+3x}\, F\left(\frac{1}{4}, \frac{3}{4}; 1; x^2\right).$$

4.2.4 Other cases

It would be desirable to establish a formula of π_p for any $p \neq 2$, 3, 4 but we have not been able to do this. Our ultimate goal of this study is to generalize the strategy of Brent and Salamin, based on the Legendre relation and the Gauss formula, to the case $p \neq 2$.

5 Legendre's relation

For the details of this section, we refer the reader to [42].

5.1 Generalized elliptic integrals with three-parameters

Let $k \in [0,1)$. We consider generalizations of $K(k)$ and $E(k)$ as

$$K_{p,q,r}(k) := \int_0^1 \frac{dt}{(1-t^q)^{1/p}(1-k^qt^q)^{1/r}},$$
$$E_{p,q,r}(k) := \int_0^1 \frac{(1-k^qt^q)^{1/r}}{(1-t^q)^{1/p}}\,dt,$$

where $p \in \mathbb{P}^* := (-\infty,0) \cup (1,\infty]$ and $q,\ r \in (1,\infty)$. In case $p=q=r=2$, $K_{p,q,r}(k)$ and $E_{p,q,r}(k)$ are reduced to the classical $K(k)$ and $E(k)$, respectively. For $p=\infty$ we regard $K_{p,q,r}$ and $E_{p,q,r}$ as

$$K_{\infty,q,r}(k) := \int_0^1 \frac{dt}{(1-k^qt^q)^{1/r}},$$
$$E_{\infty,q,r}(k) := \int_0^1 (1-k^qt^q)^{1/r}\,dt.$$

Let s^* be the number such that $1/s+1/s^*=1$ for s. Under the convention that $1/\infty=0$ and $1/0=\infty$, we should note that $s \in \mathbb{P}^*$ if and only if $s^* \in (0,\infty)$, particularly, $\infty^*=1$.

There is a lot of literature about the generalized complete elliptic integrals. $K_{p,q,p}$ is introduced in [38] with a generalization of the Jacobian elliptic function with a period of $4K_{p,q,p}$ to study a bifurcation problem of a bistable reaction-diffusion equation involving p-Laplacian. Relationship between $K_{p,q,p}$ and $E_{p,q,p}$ has been observed in [14, 46]. Regarding K_{p,q,p^*}, another generalization of Jacobian elliptic function with a period of K_{p,q,p^*} is given and the basis properties for the family of these functions are shown in [39]. Moreover, K_{p,q,p^*} is also applied to a problem on Bhatia-Li's mean and a curious relation between K_{p,q,p^*} and $E_{p,q,p}$ is given in [27].

Our purpose in the present section is to generalize Legendre's relation (4.3) to the generalized complete elliptic integrals above.

To state the results, we will give some notations. For $p \in \mathbb{P}^*$ and $q \in (1,\infty)$, let $\pi_{p,q}$ be the constant defined in (1.1). In particular, $\pi_{\infty,q}=2$ for any $q \in (1,\infty)$. We write $K_{p,q} := K_{p,q,q^*}$, $E_{p,q} := E_{p,q,q}$ for $p \in \mathbb{P}^*$ and $q \in (1,\infty)$.

Theorem 5.1 ([42]). *Let $p \in \mathbb{P}^*$, $q,\ r \in (1,\infty)$ and $k \in (0,1)$. Then*

$$E_{p,q,r}(k)K_{p,r,q^*}(k') + K_{p,q,r^*}(k)E_{p,r,q}(k') - K_{p,q,r^*}(k)K_{p,r,q^*}(k') = \frac{\pi_{p,q}\pi_{s,r}}{4}, \qquad (5.1)$$

where $k' := (1-k^q)^{1/r}$ and $1/s = 1/p - 1/q$.

Corollary 5.1 ([42], Case $q=r$). *Let $p \in \mathbb{P}^*$, $q \in (1,\infty)$ and $k \in (0,1)$. Then*

$$E_{p,q}(k)K_{p,q}(k') + K_{p,q}(k)E_{p,q}(k') - K_{p,q}(k)K_{p,q}(k') = \frac{\pi_{p,q}\pi_{s,q}}{4}, \qquad (5.2)$$

where $k' := (1-k^q)^{1/q}$ and $1/s = 1/p - 1/q$.

Remark 5.1. In particular, if $p=q$, then (5.2) coincides with (4.4) in Theorem 4.1 since $K_{p,p}=K_p$, $E_{p,p}=E_p$ and $\pi_{\infty,q}=2$.

In fact, (5.1) is equivalent to *Elliott's identity* (5.3) below. The advantage of our result lies in the facts that it is understandable without acknowledge of hypergeometric functions and that its proof gives an alternative proof for Elliott's identity with straightforward calculations.

5.2 Proof of Theorem 5.1

The following property immediately follows from the definitions of $K_{p,q,r}$ and $E_{p,q,r}$.

Proposition 5.1 ([42]). *Let $p \in \mathbb{P}^*$, q, $r \in (1,\infty)$. Then, $K_{p,q,r}(k)$ is increasing on $[0,1)$ and*

$$K_{p,q,r}(0) = \frac{\pi_{p,q}}{2},$$
$$\lim_{k\to 1-0} K_{p,q,r}(k) = \begin{cases} \infty & \text{if } 1/p+1/r \geq 1, \\ \pi_{u,q}/2 \ (1/u = 1/p+1/r) & \text{if } 1/p+1/r < 1; \end{cases}$$

and $E_{p,q,r}(k)$ is decreasing on $[0,1]$ and

$$E_{p,q,r}(0) = \frac{\pi_{p,q}}{2}, \quad E_{p,q,r}(1) = \frac{\pi_{v,q}}{2} \ (1/v = 1/p - 1/r).$$

Now, we apply the generalized trigonometric function to the generalized complete elliptic integrals. For $p \in \mathbb{P}^*$ and q, $r \in (1,\infty)$, using $\sin_{p,q}\theta$ ($\sin_{\infty,q}\theta = \theta$ for $p=\infty$) and $\pi_{p,q}$, we can express $K_{p,q,r}(k)$ and $E_{p,q,r}(k)$ as follows.

$$K_{p,q,r}(k) = \int_0^{\pi_{p,q}/2} \frac{d\theta}{(1-k^q \sin_{p,q}^q \theta)^{1/r}},$$
$$E_{p,q,r}(k) = \int_0^{\pi_{p,q}/2} (1-k^q \sin_{p,q}^q \theta)^{1/r}\, d\theta.$$

Then, we see that the functions $K_{p,q,r^*}(k)$ and $E_{p,q,r}(k)$ satisfy a system of linear differential equations.

Proposition 5.2 ([42]). *Let $p \in \mathbb{P}^*$, q, $r \in (1,\infty)$. Then,*

$$\frac{dE_{p,q,r}}{dk} = \frac{q(E_{p,q,r}-K_{p,q,r^*})}{rk},$$
$$\frac{dK_{p,q,r^*}}{dk} = \frac{aE_{p,q,r}-(a-k^q)K_{p,q,r^*}}{k(1-k^q)},$$

where $a := 1+q/r-q/p$.

Proof. We consider the case $p \neq \infty$. Differentiating $E_{p,q,r}(k)$ we have

$$\frac{dE_{p,q,r}}{dk} = \frac{q}{r}\int_0^{\pi_{p,q}/2} \frac{-k^{q-1}\sin_{p,q}^q\theta}{(1-k^q\sin_{p,q}^q\theta)^{1/r^*}}\,d\theta = \frac{q}{rk}(E_{p,q,r}-K_{p,q,r^*}).$$

Next, for $K_{p,q,r^*}(k)$

$$\frac{dK_{p,q,r^*}}{dk} = \frac{q}{r^*}\int_0^{\pi_{p,q}/2} \frac{k^{q-1}\sin_{p,q}^q\theta}{(1-k^q\sin_{p,q}^q\theta)^{1+1/r^*}}\,d\theta.$$

Here we see that

$$\frac{d}{d\theta}\left(\frac{-\cos_{p,q}^{p/r^*}\theta}{(1-k^q\sin_{p,q}^q\theta)^{1/r^*}}\right)=\frac{q(1-k^q)\sin_{p,q}^{q-1}\theta\cos_{p,q}^{1-p/r}\theta}{r^*(1-k^q\sin_{p,q}^q\theta)^{1+1/r^*}},$$
$$\lim_{\theta\to\pi_{p,q}/2}\cos_{p,q}^{p-1}\theta=\lim_{\theta\to\pi_{p,q}/2}(1-\sin_{p,q}^q\theta)^{1/p^*}=0;$$

so that we use integration by parts as

$$\begin{aligned}\frac{dK_{p,q,r^*}}{dk}&=\frac{k^{q-1}}{1-k^q}\int_0^{\pi_{p,q}/2}\frac{d}{d\theta}\left(\frac{-\cos_{p,q}^{p/r^*}\theta}{(1-k^q\sin_{p,q}^q\theta)^{1/r^*}}\right)\sin_{p,q}\theta\cos_{p,q}^{p/r-1}\theta\,d\theta\\&=\frac{k^{q-1}}{1-k^q}\left[\frac{-\sin_{p,q}\theta\cos_{p,q}^{p-1}\theta}{(1-k^q\sin_{p,q}^q\theta)^{1/r^*}}\right]_0^{\pi_{p,q}/2}\\&\quad+\frac{k^{q-1}}{1-k^q}\int_0^{\pi_{p,q}/2}\frac{\cos_{p,q}^{p/r^*}\theta}{(1-k^q\sin_{p,q}^q\theta)^{1/r^*}}\left(\cos_{p,q}^{p/r}\theta-\frac{(q/r-q/p)\sin_{p,q}^q\theta}{\cos_{p,q}^{p/r^*}\theta}\right)d\theta\\&=\frac{k^{q-1}}{1-k^q}\int_0^{\pi_{p,q}/2}\frac{\cos_{p,q}^p\theta-(q/r-q/p)\sin_{p,q}^q\theta}{(1-k^q\sin_{p,q}^q\theta)^{1/r^*}}\,d\theta\\&=\frac{k^{q-1}}{1-k^q}\int_0^{\pi_{p,q}/2}\frac{(1+q/r-q/p)(1-k^q\sin_{p,q}^q\theta)-(1+q/r-q/p-k^q)}{k^q(1-k^q\sin_{p,q}^q\theta)^{1/r^*}}\,d\theta\\&=\frac{(1+q/r-q/p)E_{p,q,r}-(1+q/r-q/p-k^q)K_{p,q,r^*}}{k(1-k^q)}.\end{aligned}$$

The case $p=\infty$ is proved similarly. This completes the proof. □

Proposition 5.2 now yields Theorem 5.1.

Proof of Theorem 5.1. Let $k':=(1-k^q)^{1/r}$, $E'_{p,r,q}(k):=E_{p,r,q}(k')$ and $K'_{p,r,q^*}(k):=K_{p,r,q^*}(k')$. As $dk'/dk=-(q/r)k^{q-1}/(k')^{r-1}$, Proposition 5.2 gives

$$\begin{aligned}\frac{dE_{p,q,r}}{dk}&=\frac{q(E_{p,q,r}-K_{p,q,r^*})}{rk},\\\frac{dK_{p,q,r^*}}{dk}&=\frac{aE_{p,q,r}-(a-k^q)K_{p,q,r^*}}{k(k')^r},\\\frac{dE'_{p,r,q}}{dk}&=\frac{k^{q-1}(-E'_{p,r,q}+K'_{p,r,q^*})}{(k')^r},\\\frac{dK'_{p,r,q^*}}{dk}&=\frac{q(-bE'_{p,r,q}+(b-(k')^r)K'_{p,r,q^*})}{rk(k')^r},\end{aligned}$$

where $a:=1+q/r-q/p$ and $b:=1+r/q-r/p$.

We denote the left-hand side of (5.1) by $L(k)$. A direct computation shows that

$$\begin{aligned}
&\frac{d}{dk}L(k)\\
&= \frac{dE_{p,q,r}}{dk}K'_{p,r,q^*} + E_{p,q,r}\frac{dK'_{p,r,q^*}}{dk}\\
&\quad + \frac{dK_{p,q,r^*}}{dk}E'_{p,r,q} + K_{p,q,r^*}\frac{E'_{p,r,q}}{dk} - \frac{dK_{p,q,r^*}}{dk}K'_{p,r,q^*} - K_{p,q,r^*}\frac{dK'_{p,r,q^*}}{dk}\\
&= \frac{dE_{p,q,r}}{dk}K'_{p,r,q^*} + (E_{p,q,r} - K_{p,q,r^*})\frac{dK'_{p,r,q^*}}{dk}\\
&\quad + \frac{dK_{p,q,r^*}}{dk}(E'_{p,r,q} - K'_{p,r,q^*}) + K_{p,q,r^*}\frac{E'_{p,r,q}}{dk}\\
&= \frac{q(E_{p,q,r} - K_{p,q,r^*})}{rk}\cdot K'_{p,r,q^*} + (E_{p,q,r} - K_{p,q,r^*})\cdot\frac{q(-bE'_{p,r,q} + (b-(k')^r)K'_{p,r,q^*})}{rk(k')^r}\\
&\quad + \frac{aE_{p,q,r} - (a-k^q)K_{p,q,r^*}}{k(k')^r}\cdot(E'_{p,r,q} - K'_{p,r,q^*}) + K_{p,q,r^*}\cdot\frac{k^{q-1}(-E'_{p,r,q} + K'_{p,r,q^*})}{(k')^r}\\
&= -\frac{bq}{rk(k')^r}(E_{p,q,r} - K_{p,q,r^*})(E'_{p,r,q} - K'_{p,r,q^*}) + \frac{a}{k(k')^r}(E_{p,q,r} - K_{p,q,r^*})(E'_{p,r,q} - K'_{p,r,q^*})\\
&= \frac{ar - bq}{rk(k')^r}(E_{p,q,r} - K_{p,q,r^*})(E'_{p,r,q} - K'_{p,r,q^*}).
\end{aligned}$$

Since $ar - bq = 0$, we see that $dL/dk = 0$. Thus $L(k)$ is a constant C.

We will evaluate C as follows. Since

$$\begin{aligned}
&|(K_{p,q,r^*} - E_{p,q,r})K'_{p,r,q^*}|\\
&\quad = \int_0^{\pi_{p,q}/2}\left(\frac{1}{(1-k^q\sin_{p,q}^q\theta)^{1/r^*}} - (1-k^q\sin_{p,q}^q\theta)^{1/r}\right)d\theta\\
&\qquad \times\int_0^{\pi_{p,r}/2}\frac{d\theta}{(1-(k')^r\sin_{p,r}^r\theta)^{1/q^*}}\\
&\quad = \int_0^{\pi_{p,q}/2}\frac{k^q\sin_{p,q}^q\theta}{(1-k^q\sin_{p,q}^q\theta)^{1/r^*}}\,d\theta\cdot\int_0^{\pi_{p,r}/2}\frac{d\theta}{(\cos_{p,r}^p\theta + k^q\sin_{p,r}^r\theta)^{1/q^*}}\\
&\quad \le k^qK_{p,q,r^*}(k)\cdot\frac{1}{k^{q-1}}\frac{\pi_{p,r}}{2}\\
&\quad = \frac{\pi_{p,r}}{2}kK_{p,q,r^*}(k),
\end{aligned}$$

we obtain $\lim_{k\to+0}(K_{p,q,r^*} - E_{p,q,r})K'_{p,r,q^*} = 0$. Therefore, from Proposition 5.1

$$C = \lim_{k\to+0}K_{p,q,r^*}E'_{p,r,q} = K_{p,q,r^*}(0)E_{p,r,q}(1) = \frac{\pi_{p,q}\pi_{s,r}}{4},$$

where $1/s = 1/p - 1/q$. Thus, we conclude the assertion. □

Finally, we will give a remark for Theorem 5.1. From the series expansion and the termwise integration, it is possible to express the generalized complete elliptic integrals by

Gaussian hypergeometric functions

$$K_{p,q,r}(k) = \frac{\pi_{p,q}}{2} F\left(\frac{1}{q}, \frac{1}{r}; \frac{1}{p^*} + \frac{1}{q}; k^q\right),$$
$$E_{p,q,r}(k) = \frac{\pi_{p,q}}{2} F\left(\frac{1}{q}, -\frac{1}{r}; \frac{1}{p^*} + \frac{1}{q}; k^q\right).$$

By these expressions and letting $1/p = 1/2 - b$, $1/q = 1/2 + a$, $1/r = 1/2 - c$ and $k^q = x$ in (5.1), we obtain *Elliott's identity* (see Elliott [24]; see also [2], [3, Theorem 3.2.8] and [25, (13) p. 85]):

$$\begin{aligned} F&\left({1/2+a, -1/2-c \atop a+b+1}; x\right) F\left({1/2-a, 1/2+c \atop b+c+1}; 1-x\right) \\ &+ F\left({1/2+a, 1/2-c \atop a+b+1}; x\right) F\left({-1/2-a, 1/2+c \atop b+c+1}; 1-x\right) \\ &- F\left({1/2+a, 1/2-c \atop a+b+1}; x\right) F\left({1/2-a, 1/2+c \atop b+c+1}; 1-x\right) \\ &= \frac{\Gamma(a+b+1)\Gamma(b+c+1)}{\Gamma(a+b+c+3/2)\Gamma(b+1/2)} \end{aligned} \tag{5.3}$$

for $|a|$, $|c| < 1/2$ and $b \in (-1/2, \infty)$, where Γ denotes the gamma function. Also, letting $1/p = 2 - c - a$ and $1/q = 1 - a$ in (5.2) of Corollary 5.1, we have the identity of [2, Corollary 3.13 (5)] for $a \in (0, 1)$ and $c \in (1 - a, \infty)$. A series of Vuorinen's works on Elliott's identity with his coauthors starting from [2] deals with the concavity/convexity properties of certain related functions to the left-hand side of (5.3).

6 Legendre's relation for the incomplete elliptic integrals

Legendre has also showed a relation as (4.3) for the *incomplete* elliptic integrals; see Cayley's monograph [15, p.136]. However, the proof is slightly complicated and we still have not generalized the relation to the generalized (incomplete) elliptic integrals. For our future work, we will give an elementary proof of Legendre's relation for the incomplete elliptic integrals.

Let Δ be the function of ϕ and k as $\Delta(\phi, k) := \sqrt{1 - k^2 \sin^2 \phi}$. Using Δ, we denote the incomplete elliptic integrals of the first kind F, the second kind E and the third kind Π by

$$F(\phi, k) := \int_0^\phi \frac{d\theta}{\Delta(\theta, k)},$$
$$E(\phi, k) := \int_0^\phi \Delta(\theta, k)\, d\theta,$$
$$\Pi(\phi, n, k) := \int_0^\phi \frac{d\theta}{(1 - n \sin^2 \theta)\Delta(\theta, k)}.$$

Moreover, $\Delta'(\psi, k) := \Delta(\psi, k')$, $F'(\psi, k) := F(\psi, k')$, $E'(\psi, k) := E(\psi, k')$ and $\Pi'(\psi, n, k) := \Pi(\psi, n, k')$.

Then, it is possible to obtain the following derivative formulas.

Lemma 6.1.

$$\begin{aligned}
\frac{d\Delta(\phi,k)}{dk} &= -\frac{k\sin^2\phi}{\Delta(\phi,k)},\\
\frac{dF(\phi,k)}{dk} &= \frac{E(\phi,k)-(k')^2F(\phi,k)}{k(k')^2} - \frac{k\sin\phi\cos\phi}{(k')^2\Delta(\phi,k)},\\
\frac{dE(\phi,k)}{dk} &= \frac{E(\phi,k)-F(\phi,k)}{k},\\
\frac{d\Pi(\phi,n,k)}{dk} &= \frac{k}{(k')^2(k^2-n)}\left(E(\phi,k)-(k')^2\Pi(\phi,n,k)-\frac{k^2\sin\phi\cos\phi}{\Delta(\phi,k)}\right),\\
\frac{d\Delta'(\psi,k)}{dk} &= \frac{k\sin^2\psi}{\Delta'(\psi,k)},\\
\frac{dF'(\psi,k)}{dk} &= -\frac{E'(\psi,k)-k^2F'(\psi,k)}{(k')^2k} + \frac{\sin\psi\cos\psi}{k\Delta'(\psi,k)},\\
\frac{dE'(\psi,k)}{dk} &= -\frac{k(E'(\psi,k)-F'(\psi,k))}{(k')^2},\\
\frac{d\Pi'(\psi,n,k)}{dk} &= -\frac{1}{k((k')^2-n)}\left(E'(\psi,k)-k^2\Pi'(\psi,n,k)-\frac{(k')^2\sin\psi\cos\psi}{\Delta'(\psi,k)}\right).
\end{aligned}$$

Proof. We give only the formula of the derivative of $\Pi(\phi,n,k)$.

$$\begin{aligned}
\frac{d\Pi}{dk} &= \int_0^\phi \frac{k\sin^2\theta}{(1-n\sin^2\theta)\Delta^3}\,d\theta\\
&= \frac{k}{n-k^2}\int_0^\phi \frac{(1-k^2\sin^2\theta)-(1-n\sin^2\theta)}{(1-n\sin^2\theta)\Delta^3}\,d\theta\\
&= \frac{k}{n-k^2}\left(\Pi-\int_0^\phi\frac{d\theta}{\Delta^3}\right).
\end{aligned}$$

Here,

$$\begin{aligned}
\int_0^\phi \frac{d\theta}{\Delta^3} &= \frac{1}{1-k^2}\int_0^\phi \frac{(1-k^2\sin^2\theta)^2-k^2(1-2\sin^2\theta+k^2\sin^4\theta)}{\Delta^3}\,d\theta\\
&= \frac{1}{(k')^2}\left(E-k^2\int_0^\phi\frac{1-2\sin^2\theta+k^2\sin^4\theta}{\Delta^3}\,d\theta\right),
\end{aligned}$$

and

$$\frac{d}{d\theta}\left(\frac{\sin\theta\cos\theta}{\Delta}\right) = \frac{1-2\sin^2\theta+k^2\sin^4\theta}{\Delta^3};$$

so that we have

$$\int_0^\phi\frac{d\theta}{\Delta^3} = \frac{1}{(k')^2}\left(E-\frac{k^2\sin\phi\cos\phi}{\Delta}\right).$$

Thus,

$$\frac{d\Pi}{dk} = \frac{k}{(k')^2(k^2-n)}\left(E-(k')^2\Pi-\frac{k^2\sin\phi\cos\phi}{\Delta}\right).$$

□

Now, let us define

$$
\begin{aligned}
L(\phi,\psi,k) &:= E(\phi,k)F'(\psi,k)+F(\phi,k)E'(\psi,k)-F(\phi,k)F'(\psi,k),\\
M(\phi,\psi,k) &:= \csc^2\psi\,\Pi(\phi,-\cot^2\psi,k)-F(\phi,k),\\
M'(\psi,\phi,k) &:= M(\psi,\phi,k')=\csc^2\phi\,\Pi'(\psi,-\cot^2\phi,k)-F'(\psi,k),\\
N(\phi,\psi,k) &:= \tan\phi\,\Delta(\phi,k)M'(\psi,\phi,k)+\tan\psi\,\Delta'(\psi,k)M(\phi,\psi,k).
\end{aligned}
$$

Then, it follows from Lemma 6.1 that

Lemma 6.2.

$$
\begin{aligned}
\frac{dL(\phi,\psi,k)}{dk} &= \frac{dE(\phi,k)}{dk}\frac{\sin\psi\cos\psi}{\Delta'(\psi,k)}+\frac{dE'(\psi,k)}{dk}\frac{\sin\phi\cos\phi}{\Delta(\phi,k)},\\
\frac{dM(\phi,\psi,k)}{dk} &= -\frac{1}{k\Delta'(\psi,k)^2}\left(\cos^2\psi E(\phi,k)-\Delta'(\psi,k)^2F(\phi,k)\right.\\
&\qquad\left.+k^2\Pi(\phi,-\cot^2\psi,k)-\frac{k^2\sin\phi\cos\phi\cos^2\psi}{\Delta(\phi,k)}\right),\\
\frac{dM'(\psi,\phi,k)}{dk} &= \frac{k}{(k')^2\Delta(\phi,k)^2}\left(\cos^2\phi E'(\psi,k)-\Delta(\phi,k)^2F'(\psi,k)\right.\\
&\qquad\left.+(k')^2\Pi'(\psi,-\cot^2\phi,k)-\frac{(k')^2\sin\psi\cos\psi\cos^2\phi}{\Delta'(\psi,k)}\right),\\
\frac{dN(\phi,\psi,k)}{dk} &= -\frac{dE(\phi,k)}{dk}\frac{\sin\psi\cos\psi}{\Delta'(\psi,k)}-\frac{dE'(\psi,k)}{dk}\frac{\sin\phi\cos\phi}{\Delta(\phi,k)}.
\end{aligned}
$$

Proof. It is sufficient to show the formulas of L, M and N.

$$
\begin{aligned}
\frac{dL}{dk} &= \frac{dE}{dk}F'+E\frac{dF'}{dk}+\frac{dF}{dk}E'+F\frac{dE'}{dk}-\frac{dF}{dk}F'-F\frac{dF'}{dk}\\
&= \frac{dE}{dk}F'+(E-F)\frac{dF'}{dk}+\frac{dF}{dk}(E'-F')+F\frac{dE'}{dk}\\
&= \frac{E-F}{k}F'+(E-F)\left(-\frac{E'-k^2F'}{(k')^2k}+\frac{\sin\psi\cos\psi}{k\Delta'}\right)\\
&\quad+\left(\frac{E-(k')^2F}{k(k')^2}-\frac{k\sin\phi\cos\phi}{(k')^2\Delta}\right)(E'-F')+F\left(-\frac{k(E'-F')}{(k')^2}\right)\\
&= (E-F)\left(\frac{F'}{k}-\frac{E'-k^2F'}{(k')^2k}+\frac{\sin\psi\cos\psi}{k\Delta'}\right)\\
&\quad+(E'-F')\left(\frac{E-(k')^2F}{k(k')^2}-\frac{k\sin\phi\cos\phi}{(k')^2\Delta}-\frac{kF}{(k')^2}\right)\\
&= (E-F)\left(\frac{F'-E'}{(k')^2k}+\frac{\sin\psi\cos\psi}{k\Delta'}\right)+(E'-F')\left(\frac{E-F}{k(k')^2}-\frac{k\sin\phi\cos\phi}{(k')^2\Delta}\right)\\
&= \frac{E-F}{k}\frac{\sin\psi\cos\psi}{\Delta'}+\frac{k(F'-E')}{(k')^2}\frac{\sin\phi\cos\phi}{\Delta}\\
&= \frac{dE}{dk}\frac{\sin\psi\cos\psi}{\Delta'}+\frac{dE'}{dk}\frac{\sin\phi\cos\phi}{\Delta}.
\end{aligned}
$$

$$
\begin{aligned}
\frac{dM}{dk} &= \csc^2\psi\frac{d\Pi}{dk} - \frac{dF}{dk}\\
&= \csc^2\psi\frac{k}{(k')^2(k^2+\cot^2\psi)}\left(E-(k')^2\Pi-\frac{k^2\sin\phi\cos\phi}{\Delta}\right)\\
&\quad -\left(\frac{E-(k')^2F}{k(k')^2}-\frac{k\sin\phi\cos\phi}{(k')^2\Delta}\right)\\
&= \left(\frac{k\csc^2\psi}{(k')^2(k^2+\cot^2\psi)}-\frac{1}{k(k')^2}\right)E+\frac{1}{k}F-\frac{k\csc^2\psi}{k^2+\cot^2\psi}\Pi\\
&\quad +\frac{k\sin\phi\cos\phi}{(k')^2\Delta}\left(-\frac{k^2\csc^2\psi}{k^2+\cot^2\psi}+1\right)\\
&= -\frac{\cos^2\psi}{k(\Delta')^2}E+\frac{1}{k}F-\frac{k}{(\Delta')^2}\Pi+\frac{k\sin\phi\cos\phi}{(k')^2\Delta}\frac{(k')^2\cos^2\psi}{(\Delta')^2}\\
&= -\frac{1}{k(\Delta')^2}\left(\cos^2\psi E-(\Delta')^2F+k^2\Pi-\frac{k^2\sin\phi\cos\phi\cos^2\psi}{\Delta}\right).
\end{aligned}
$$

$$
\begin{aligned}
\frac{dN}{dk} &= \tan\phi\left(\frac{d\Delta}{dk}M'+\Delta\frac{dM'}{dk}\right)+\tan\psi\left(\frac{d\Delta'}{dk}M+\Delta'\frac{dM}{dk}\right)\\
&= \tan\phi\left(-\frac{k\sin^2\phi}{\Delta}(\csc^2\phi\Pi'-F')\right.\\
&\quad\left.+\Delta\frac{k}{(k')^2\Delta^2}\left(\cos^2\phi E'-\Delta^2F'+(k')^2\Pi'-\frac{(k')^2\sin\psi\cos\psi\cos^2\phi}{\Delta'}\right)\right)\\
&\quad+\tan\psi\left(\frac{k\sin^2\psi}{\Delta'}(\csc^2\psi\Pi-F)\right.\\
&\quad\left.-\Delta'\frac{1}{k(\Delta')^2}\left(\cos^2\psi E+(\Delta')^2F-k^2\Pi+\frac{k^2\sin\phi\cos\phi\cos^2\psi}{\Delta}\right)\right)\\
&= \tan\phi\left(\frac{k\cos^2\phi}{(k')^2\Delta}E'-\frac{k\cos^2\phi}{(k')^2\Delta}F'-\frac{k\sin\psi\cos\psi\cos^2\phi}{\Delta\Delta'}\right)\\
&\quad+\tan\psi\left(-\frac{\cos^2\psi}{k\Delta'}E+\frac{\cos^2\psi}{k\Delta'}F+\frac{k\sin\phi\cos\phi\cos^2\psi}{\Delta\Delta'}\right)\\
&= \frac{k\sin\phi\cos\phi}{(k')^2\Delta}(E'-F')-\frac{\sin\psi\cos\psi}{k\Delta'}(E-F)\\
&= -\frac{dE'}{dk}\frac{\sin\phi\cos\phi}{\Delta}-\frac{dE}{dk}\frac{\sin\psi\cos\psi}{\Delta'}.
\end{aligned}
$$

□

Now, we are in a position to show the Legendre relation of the incomplete elliptic integrals.

Theorem 6.1 (Legendre). *Let* $\phi,\ \psi \in (0,\pi/2)$ *and* $k\in(0,1)$. *Then*

$$L(\phi,\psi,k)+N(\phi,\psi,k)=\frac{\pi}{2}.$$

Proof. It follows from Lemma 6.2 that

$$\frac{d}{dk}(L+N)=0,$$

which implies that $L+N=C$, a constant independent of k. Moreover, as $k\to+0$ we can show that

$$\begin{aligned}(E-F)F'&\to 0,\\ FE'+\tan\psi\Delta' M&\to\tan^{-1}\left(\frac{\tan\phi}{\sin\psi}\right),\\ \tan\phi\Delta M'&\to\tan^{-1}\left(\frac{\sin\psi}{\tan\phi}\right).\end{aligned}$$

Therefore,

$$C=\lim_{k\to+0}(L+N)=\tan^{-1}\left(\frac{\tan\phi}{\sin\psi}\right)+\tan^{-1}\left(\frac{\sin\psi}{\tan\phi}\right)=\frac{\pi}{2}.$$

□

As ϕ, $\psi\to\pi/2$, we have $L(\phi,\psi,k)\to L(k)$ and $N(\phi,\psi,k)\to 0$. Thus, we obtain Legendre's relation (4.3) as a corollary of Theorem 6.1.

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