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FURUTA INEQUALITY AND p-$wA(s,t)$ OPERATORS

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Dedicated to the memory of Professor Takayuki Furuta with deep gratitude

ABSTRACT. The aim of this paper is to introduce small history with Furata's inequality and relating class of p-$wA(s,t)$ operators.

1. INTRODUCTION

Let $B(\mathcal{H})$ be the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. In 1987, Furuta [5] proved the follwing inequality.

Theorem 1. [Furuta inequality]
Let $0 < p,q,r \in \mathbb{R}$ *and* $A, B \in B(\mathcal{H})$ *satisfy* $0 \le B \le A$. *If* $p + 2r \le (1+2r)q$ *and* $1 \le q$, *then* $B^{\frac{p+2r}{q}} \le (B^r A^p B^r)^{\frac{1}{q}}$ *and* $(A^r B^p A^r)^{\frac{1}{q}} \le A^{\frac{p+2r}{q}}$.

This is a good extension of Löwner-Heinz's inequality ([7] and [12]).

Theorem 2. [Löwner-Heinz's inequality]
Let $A, B \in B(\mathcal{H})$ *satisfy* $0 \le B \le A$ *and* $0 < p \le 1$. *Then* $B^p \le A^p$.

Recall that an operator T is said to be hyponormal if $T^*T \ge TT^*$. For $T \in B(\mathcal{H})$, set $|T| = (T^*T)^{\frac{1}{2}}$ as usual. By taking $U|T|x = Tx$ for $x \in \mathcal{H}$ and $Ux = 0$ for $x \in \ker |T|$, T has a unique polar decomposition $T = U|T|$ with $\ker U = \ker |T|$. Aluthge [1] defined Aluthge transformation $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, and studied interesting properties of p-hyponormal operators for $0 < p \le 1$.

Definition 3. $T \in B(\mathcal{H})$ is said to be p-hyponormal if $(T^*T)^p \ge (TT^*)^p$ where $p \in (0,1]$.

The class of p-hyponormal operators is a generalization of the class of hyponormal operators by Löwner-Heinz's inequality. Aluthge [1] proved follwing result.

Theorem 4. *Let T be p-hyponormal. If* $0 < p \le 1/2$, *then* $\tilde{T}$ *is* $(p+1/2)$-*hyponormal. If* $1/2 \le p \le 1$, *then* $\tilde{T}$ *is hyponormal.*

This is a epoc making result. Aluthge transformation is a strong tool of operator theory and many applications has been studied, for example, Putnam ineqality, Fuglede Putnam type theorem, Wyle type theorem. I think that generalization of class of operators may be good way to investigate non-normal operators. Furuta [6] and Yoshino [17] definded generalized transformation $T(s,t) = |T|^s U|T|^t$ with $0 < s,t$ and Yanagida [15], Ito [8], Yamazaki [9], Fujii, Jung, Lee, Nakamoto [4] studied $wA(s,t)$ operators.

Definition 5. T is said to be $wA(s,t)$ if

$$(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t} \tag{1.1}$$

and

$$|T|^{2s} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{s}{s+t}}. \tag{1.2}$$

Hence generalized Aluthge transformation $T(s,t)$ of $wA(s,t)$ operator T enjoys the following property.

Proposition 6. *Let T be $wA(s,t)$. Then*

$$|T(s,t)|^{\frac{2t}{s+t}} \geq |T|^{2t}$$

and

$$|T|^{2s} \geq |T(s,t)^*|^{\frac{2s}{s+t}}.$$

Hence

$$|T(s,t)|^{\frac{2r}{s+t}} \geq |T|^{2r} \geq |T(s,t)^*|^{\frac{2r}{s+t}}$$

for all $r \in (0, \min\{s,t\}]$.

Ito and Yamazaki [9] proved that (1.1) implies (1.2). This is a good result. This means that class of $wA(s,t)$ operators are coincides with $A(s,t)$ operators.

Definition 7. T is said to be $A(s,t)$ if

$$(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}. \tag{1.3}$$

Class $A(1,1)$ is said to be class A and class $A(1/2,1/2)$ is said to be w-hyponormal [4, 9, 15]. Prasad and Tanahshi [16] definded p-$wA(s,t)$ operator for $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$ as follows.

Definition 8. T is said to be p-$wA(s,t)$ if

$$(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{pt}{s+t}} \geq |T^*|^{2pt} \tag{1.4}$$

and

$$|T|^{2ps} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{ps}{s+t}}. \tag{1.5}$$

Hence p-$wA(s,t)$ operator is a generalization of $wA(s,t)$ operator by Löwner-Heinz's inequality. The aim of this paper is to prove several properties of p-$wA(s,t)$ operaor and show some open problems of p-$wA(s,t)$ operaor. Main results are proved in [16] and [2].

2. Results

At first, we show generalized Aluthge transformation $T(s,t)$ of p-$wA(s,t)$ operator T enjoys the following property [16].

Theorem 9. *Let T be p- $wA(s,t)$. Then*

$$|T(s,t)|^{\frac{2pt}{s+t}} \geq |T|^{2pt}$$

and

$$|T|^{2ps} \geq |T(s,t)^*|^{\frac{2ps}{s+t}}.$$

Hence

$$|T(s,t)|^{\frac{2pr}{s+t}} \geq |T|^{2pr} \geq |T(s,t)^*|^{\frac{2pr}{s+t}}$$

for all $r \in (0, \min\{s, t\}]$.

Proof.

$$\begin{aligned}
&(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{pt}{s+t}} \geq |T^*|^{2pt} \\
&\Longleftrightarrow \left(U|T|^tU^*|T|^{2s}U|T|^tU^*\right)^{\frac{pt}{s+t}} \geq U|T|^{2pt}U^* \\
&\Longleftrightarrow U\left(|T|^tU^*|T|^{2s}U|T|^t\right)^{\frac{pt}{s+t}}U^* \geq U|T|^{2pt}U^* (\text{ [8, Lemma 2.1]}) \\
&\Longleftrightarrow \left(|T|^tU^*|T|^{2s}U|T|^t\right)^{\frac{pt}{s+t}} \geq |T|^{2pt} (\text{ [8, lemma 2.1]}) \\
&\Longleftrightarrow |T(s,t)|^{\frac{2pt}{s+t}} \geq |T|^{2pt}.
\end{aligned}$$

Also,

$$\begin{aligned}
&\left(|T|^s|T^*|^{2t}|T|^s\right)^{\frac{ps}{s+t}} \leq |T|^{2ps} \\
&\Longleftrightarrow \left(|T|^sU|T|^{2t}U^*|T|^s\right)^{\frac{ps}{s+t}} \leq |T|^{2ps} \\
&\Longleftrightarrow |\{T(s,t)\}^*|^{\frac{2ps}{s+t}} \leq |T|^{2ps}.
\end{aligned}$$

□

Next we show class of p-$wA(s,t)$ operators are decreasing class of operators with $0 < p \leq 1$ and increasing with $0 < s, t \leq 1$. The proof is essentially due to C. Yang and J. Yuan ([19] Proposition 3.4).

Lemma 10. *If* T *is* p-$wA(s,t)$ *and* $0 < s \leq s_1, 0 < t \leq t_1, 0 < p_1 \leq p \leq 1$, *then* T *is* p_1-$wA(s_1,t_1)$.

Proof. Let T be p-$wA(s,t)$. Then

$$(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{tp}{s+t}} \geq |T^*|^{2tp} \tag{2.1}$$

and

$$|T|^{2sp} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{sp}{s+t}}. \tag{2.2}$$

We prove that T is p-$wA(s_1,t_1)$. Then T is p_1-$wA(s_1,t_1)$ by Lowner-Heinz's inequality.

Let $A_1 = (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{tp}{s+t}}$ and $B_1 = |T^*|^{2tp}$. Since (1) implies $A_1 \geq B_1$, we have

$$\left(B_1^{\frac{r_2}{2}}A_1^{p_2}B_1^{\frac{r_2}{2}}\right)^{\frac{1+r_2}{p_2+r_2}} \geq B_1^{1+r_2}$$

for any $r_2 > 0$ and $p_2 \geq 1$ by Furuta's inequality [5]. Let

$$\beta \geq t, p_2 = \frac{s+t}{tp} \geq 1, r_2 = \frac{\beta - t}{tp} \geq 0.$$

Then

$$\left(|T^*|^\beta|T|^{2s}|T^*|^\beta\right)^{\frac{tp+\beta-t}{s+\beta}} \geq |T^*|^{2tp+2\beta-2t}.$$

Hence we have

$$\left(|T^*|^\beta|T|^{2s}|T^*|^\beta\right)^{\frac{w}{s+\beta}} \geq |T^*|^{2w}$$

for any $0 < w \leq tp + \beta - t$.

Let

$$f_s(\beta) = \left(|T|^s|T^*|^{2\beta}|T|^s\right)^{\frac{s}{s+\beta}}$$

for $\beta \geq t$. Then

$$\begin{aligned}
f_s(\beta) &= \left\{\left(|T|^s|T^*|^{2\beta}|T|^s\right)^{\frac{s+\beta+w}{s+\beta}}\right\}^{\frac{s}{s+\beta+w}} \\
&= \left\{|T|^s|T^*|^{\beta}\left(|T^*|^{\beta}|T|^{2s}|T^*|^{\beta}\right)^{\frac{w}{s+\beta}}|T^*|^{\beta}|T|^s\right\}^{\frac{s}{s+\beta+w}} \\
&\geq \left\{|T|^s|T^*|^{\beta}|T^*|^{2w}|T^*|^{\beta}|T|^s\right\}^{\frac{s}{s+\beta+w}} \\
&= \left\{|T|^s|T^*|^{2(\beta+w)}|T|^s\right\}^{\frac{s}{s+\beta+w}} \\
&= f_s(\beta+w).
\end{aligned}$$

Hence $f_s(\beta)$ is decreasing for $\beta \geq t$.

Then, by (2.2),

$$\begin{aligned}
|T|^{2sp} &\geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{sp}{s+t}} \\
&= \{f_s(t)\}^p \\
&\geq \{f_s(t_1)\}^p = \left(|T|^s|T^*|^{2t_1}|T|^s\right)^{\frac{sp}{s+t_1}}.
\end{aligned}$$

Let $A_2 = |T|^{2sp}$ and $B_2 = \left(|T|^s|T^*|^{2t_1}|T|^s\right)^{\frac{sp}{s+t_1}}$. Then

$$A_2^{1+r_3} \geq \left(A_2^{\frac{r_3}{2}} B_2^{p_3} A_2^{\frac{r_3}{2}}\right)^{\frac{1+r_3}{p_3+r_3}}$$

for any $r_3 \geq 0$ and $p_3 \geq 1$ by Furuta's inequality [5]. Let

$$p_3 = \frac{s+t_1}{sp} \geq 1, r_3 = \frac{s_1 - s}{sp} \geq 0.$$

Then

$$|T|^{2sp+2s_1-2s} \geq \left(|T|^{s_1}|T^*|^{2t_1}|T|^{s_1}\right)^{\frac{sp+s_1-s}{s_1+t_1}}.$$

Since

$$sp + s_1 - s - s_1 p = (s_1 - s)(1-p) \geq 0,$$

we have

$$|T|^{2s_1p} \geq \left(|T|^{s_1}|T^*|^{2t_1}|T|^{s_1}\right)^{\frac{s_1p}{s_1+t_1}}.$$

Similarly, we have

$$(|T^*|^{t_1}|T|^{2s_1}|T^*|^{t_1})^{\frac{t_1p}{s_1+t_1}} \geq |T^*|^{2t_1p}.$$

Hence T is p-$wA(s_1, t_1)$.

□

The following result seems new, even for class $A(s,t)$ operators.

Theorem 11. *If T is p-$wA(s,t)$ and T is invertible, then T^{-1} is p-$wA(t,s)$.*

Proof. Let $T = U|T|$ the polar decomposition of T. Then

$$|T^{-1}|^2 = (T^{-1})^*T^{-1} = (T^*)^{-1}T^{-1} = (TT^*)^{-1} = |T^*|^{-2}.$$

Hence

$$|T^{-1}| = |T^*|.$$

Also,

$$|(T^{-1})^*|^2 = (T^{-1})(T^{-1})^* = T^{-1}(T^*)^{-1} = (T^*T)^{-1} = |T|^{-2}.$$

Hence

$$|(T^{-1})^*| = |T|^{-1}.$$

Then

$$\begin{aligned} &\{|(T^{-1})^*|^s|T^{-1}|^{2t}|(T^{-1})^*|^s\}^{\frac{sp}{s+t}} \\ &= (|T|^{-s}|T^*|^{-2t}|T|^{-s})^{\frac{sp}{s+t}} \\ &= (|T|^{s}|T^*|^{2t}|T|^{s})^{\frac{-ps}{s+t}} \\ &\geq |T|^{-2sp} = |(T^{-1})^*|^{2sp} \end{aligned}$$

and

$$\begin{aligned} &\{|T^{-1}|^t|(T^{-1})^*|^{2s}|T^{-1}|^t|\}^{\frac{tp}{s+t}} \\ &= \{|T^*|^{-t}|T|^{-2s}|T^*|^{-t}\}^{\frac{tp}{s+t}} \\ &= (|T^*|^{t}|T|^{2s}|T^*|^{t})^{\frac{-tp}{s+t}} \\ &\leq |T^*|^{-2tp} = |T^{-1}|^{2tp}. \end{aligned}$$

□

Corollary 12. *If T is $A(s,t)$ and T is invertible, then T^{-1} is $A(t,s)$.*

Let $0 < p \leq 1$ and $S, T \in B(\mathcal{H})$ be non zero operators. In [3], Duggal proved that tensor product $T \otimes S$ is p-hyponormal if and only if T and S are p-hyponormal. The passage of calss A operators is studied by Jeon and Duggal [10]. Tanahashi and Chō [13] proved that the tensor product $T \otimes S$ is of class $A(s,t)$ if and only if T and S are class $A(s,t)$. Now we will prove similar result for p-class $wA(s,t)$ operators by adopting the ideas in [13],[10].

Lemma 13. [11] *Let $T_1, T_2, S_1, S_2 \in B(\mathcal{H})$ be non negative operators. If $T_1 \neq 0$ and $S_1 \neq 0$, then the following conditions are equivalent.*

(1) $T_1 \otimes S_1 \leq T_2 \otimes S_2$.

(2) *There exists $c > 0$ such that $T_1 \leq cT_1$ and $S_1 \leq c^{-1}S_2$.*

Lemma 14. [13] *Let $T = U|T|$ and $S = V|S|$ be the polar decompositions of $T, S \in B(\mathcal{H})$. Then the following assertions hold.*

(1) $|T \otimes S| = |T| \otimes |S|$.

(2) $T \otimes S = (U \otimes V)(|T| \otimes |S|)$ *is the polar decomposition of $T \otimes S$.*

(3) $(T \otimes S)(s,t) = T(s,t) \otimes S(s,t)$ *for $s, t > 0$.*

Theorem 15. *Let $S, T \in B(\mathcal{H})$ be non zero operators. Then $T \otimes S$ is p-$wA(s,t)$ if and only if S, T are p-$wA(s,t)$.*

Proof. Let $S, T \in B(\mathcal{H})$ be non zero p-class $wA(s,t)$ operators and $S = V|S|, T = U|T|$ be polar decomposions of S, T. Then $|T(s,t)|^{\frac{2tp}{s+t}} \geq |T|^{2tp}$ and $|S(s,t)^*|^{\frac{2tp}{s+t}} \geq |S|^{2tp}$ by Theorem 9. By applying Lemma 14, we obtain

$$\begin{aligned}|(T \otimes S)(s,t)|^{\frac{2tp}{s+t}} &= |T(s,t) \otimes S(s,t)|^{\frac{2tp}{s+t}} \\ &= |T(s,t)|^{\frac{2tp}{s+t}} \otimes |S(s,t)|^{\frac{2tp}{s+t}} \geq |T|^{2tp} \otimes |S|^{2tp} = |T \otimes S|^{2tp}.\end{aligned}$$

Similarly, we have

$$|T \otimes S|^{2sp} \geq |\{(T \otimes S)(s,t)\}^*|^{\frac{2sp}{s+t}}.$$

Hence $T \otimes S$ is p-$wA(s,t)$.

Conversely, suppose that $T \otimes S$ is p-$wA(s,t)$. Then

$$\begin{aligned}|(T \otimes S)(s,t)|^{\frac{2tp}{s+t}} &= |T(s,t)|^{\frac{2tp}{s+t}} \otimes |S(s,t)|^{\frac{2tp}{s+t}} \\ &\geq |T|^{2tp} \otimes |S|^{2tp} = |T \otimes S|^{2tp}\end{aligned}$$

and

$$|T \otimes S|^{2sp} \geq |\{(T \otimes S)(s,t)\}^*|^{\frac{2sp}{s+t}}.$$

Hence there exists $c > 0$ such that

$$c|T(s,t)|^{\frac{2tp}{s+t}} \geq |T|^{2tp}$$

and

$$c^{-1}|S(s,t)|^{\frac{2tp}{s+t}} \geq |S|^{2tp}$$

by Lemma 13. Let x be a unit vector. Then

$$\begin{aligned}\||T|^{tp}x\|^2 = \langle |T|^{2tp}x, x\rangle &\leq \langle c|T(s,t)|^{\frac{2tp}{s+t}}x, x\rangle \\ &\leq c\||T(s,t)|^{\frac{tp}{s+t}}\|^2 = c\|T(s,t)\|^{\frac{2tp}{s+t}} \\ &= c\||T|^sU|T|^t\|^{\frac{2tp}{s+t}} \\ &\leq c\left(\||T|^s\| \cdot 1 \cdot \||T|^t\|\right)^{\frac{2tp}{s+t}} = c\||T|^{tp}\|^2.\end{aligned}$$

Hence $1 \leq c$. Similarly,

$$\begin{aligned}\||S|^{tp}x\|^2 = \langle |S|^{2tp}x, x\rangle &\leq \langle c^{-1}|S(s,t)|^{\frac{2tp}{s+t}}x, x\rangle \\ &\leq c^{-1}\||S(s,t)|^{\frac{tp}{s+t}}\|^2 = c^{-1}\|S(s,t)\|^{\frac{2tp}{s+t}} \\ &= c^{-1}\||S|^sV|S|^t\|^{\frac{2tp}{s+t}} \\ &\leq c^{-1}\left(\||S|^s\| \cdot 1 \cdot \||S|^t\|\right)^{\frac{2tp}{s+t}} = c^{-1}\||S|^{tp}\|^2.\end{aligned}$$

Hence $1 \leq c^{-1}$. Hence $c = 1$. This implies

$$|T(s,t)|^{\frac{2tp}{s+t}} \geq |T|^{2tp}$$

and

$$|S(s,t)|^{\frac{2tp}{s+t}} \geq |S|^{2tp}.$$

Similarly we have

$$|T|^{2sp} \geq |\{T(s,t)\}^*|^{\frac{2sp}{s+t}}$$

and

$$|S|^{2sp} \geq |\{S(s,t)\}^*|^{\frac{2sp}{s+t}}.$$

Thus T and S are p-$wA(s,t)$. $\square$

Corollary 16. *Let $S, T \in B(\mathcal{H})$ be non zero operators. Then $T \otimes S$ is p-$A(s,t)$ if and only if S, T are p-$A(s,t)$.*

Theorem 17. *Let $T \in B(\mathcal{H})$ be p-$wA(s,t)$ with $0 < s, t, s+t = 1$ and $0 < p \leq 1$. Let $\rho e^{i\theta} \in \mathbb{C}$ be an isolated point of $\sigma(T)$ and $0 < \rho$. Then the Riesz idempotent E for T with respect to $\rho e^{i\theta}$ is self-adjoint with*

$$\text{ran } E = \ker(T - \rho e^{i\theta}) = \ker\left((T - \rho e^{i\theta})^*\right).$$

and coincides with the Riesz idempotent $E(s,t)$ for $T(s,t)$ with respect to $\rho e^{i\theta}$.

Proof. Since $\sigma(T) = \sigma(T(s,t))$ by Lemma 6 of [14], $\rho e^{i\theta}$ is an isolated point of $\sigma(T(s,t))$. Since $T(s,t)$ is rp-hyponormal for all $r \in (0, \min\{s,t\}]$, $E(s,t)$ is self-adjoint and satisfies

$$\text{ran } E(s,t) = \ker\left(T(s,t) - \rho e^{i\theta}\right) = \ker(T - \rho e^{i\theta})$$

and

$$\rho e^{i\theta} \notin \sigma\left(T(s,t)|_{\text{ran } E(s,t)}\right).$$

Since $\ker(T - \rho e^{i\theta}) = \text{ran } E(s,t)$ reduces T, we have

$$T = \rho e^{i\theta} \oplus T' \quad \text{on} \quad \mathcal{H} = \text{ran } E(s,t) \oplus \text{ran } (1 - E(s,t)).$$

Then T' is also class p-$wA(s,t)$ and $T'(s,t) = T(s,t)|_{\text{ran } (1-E(s,t))}$. Hence $\rho e^{i\theta} \notin \sigma(T'(s,t)) = \sigma(T')$ by Lemma 6 of [14]. Hence $T' - \rho e^{i\theta}$ is invertible and $T - \rho e^{i\theta} = 0 \oplus (T' - \rho e^{i\theta})$. This implies $\ker(T - \rho e^{i\theta}) = \ker\left((T - \rho e^{i\theta})^*\right)$ and

$$E = \frac{1}{2\pi i}\int_\gamma (z - \rho e^{i\theta})^{-1} \oplus (z - T')^{-1} dz = 1 \oplus 0 = E(s,t)$$

where γ is a small circle containing $\rho e^{i\theta}$. $\square$

Theorem 18. *Let $T \in B(\mathcal{H})$ be p-$wA(s,t)$ with $0 < s, t, s+t \leq 1$ and $0 < p \leq 1$. Let $(T - \rho e^{i\theta})x_n \to 0$ for $x_n \in \mathcal{H}$ with $\|x_n\| = 1$ and $\rho e^{i\theta} \in \mathbb{C}, 0 < \rho$. Then $(|T| - \rho)x_n, (U - e^{i\theta})x_n, (U - e^{i\theta})^* x_n, (T - \rho e^{i\theta})^* x_n \to 0$.*

Proof. We may assume $s + t = 1$ by Lemma 10. Since

$$\left(T(s,t) - \rho e^{i\theta}\right)|T|^s x_n = |T|^s \left(T - \rho e^{i\theta}\right) x_n \to 0,$$

we have

$$\left(T(s,t) - \rho e^{i\theta}\right)^* |T|^s x_n \to 0,$$

because $T(s,t)$ is rp-hyponormal for all $r \in (0, \min\{s,t\}]$. Hence

$$\begin{aligned}
0 \leftarrow & \left(T(s,t) - \rho e^{i\theta}\right)^* \left(T(s,t) - \rho e^{i\theta}\right)|T|^s x_n \\
&= \left(|T(s,t)|^2 - \rho^2\right)|T|^s x_n \\
&\quad - \rho e^{-i\theta}\left(T(s,t) - \rho e^{i\theta}\right)|T|^s x_n - \rho\left(T(s,t) - \rho e^{i\theta}\right)^* |T|^s x_n.
\end{aligned}$$

This implies

$$\left(|T(s,t)|^2 - \rho^2\right)|T|^s x_n \to 0$$

and

$$\left(|T(s,t)|^{rp}-\rho^{rp}\right)|T|^{s}x_n \to 0.$$

Similarly, we have

$$\left(|T(s,t)^*|^{rp}-\rho^{rp}\right)|T|^{s}x_n \to 0.$$

Hence

$$\begin{aligned} 0 \leftarrow & \langle\left(|T(s,t)|^{rp}-\rho^{rp}\right)|T|^{s}x_n, |T|^{s}x_n\rangle \\ & \geq \langle\left(|T|^{rp}-\rho^{rp}\right)|T|^{s}x_n, |T|^{s}x_n\rangle \\ & \geq \langle\left(|T(s,t)^*|^{rp}-\rho^{rp}\right)|T|^{s}x_n, |T|^{s}x_n\rangle \to 0. \end{aligned}$$

This implies

$$\langle\left(|T|^{rp}-\rho^{rp}\right)|T|^{s}x_n, |T|^{s}x_n\rangle \to 0.$$

Since $\frac{r}{2} \in (0, \min\{s,t\}]$, we have

$$\langle\left(|T|^{\frac{rp}{2}}-\rho^{\frac{rp}{2}}\right)|T|^{s}x_n, |T|^{s}x_n\rangle \to 0$$

by the same argument. Then

$$\begin{aligned} & \|\left(|T|^{\frac{rp}{2}}-\rho^{\frac{rp}{2}}\right)|T|^{s}x_n\|^2 \\ & = \langle\left(|T|^{\frac{rp}{2}}-\rho^{\frac{rp}{2}}\right)^2|T|^{s}x_n, |T|^{s}x_n\rangle \\ & = \langle\left(|T|^{rp}-\rho^{rp}\right)|T|^{s}x_n, |T|^{s}x_n\rangle - 2\rho^{\frac{rp}{2}}\langle\left(|T|^{\frac{rp}{2}}-\rho^{\frac{rp}{2}}\right)|T|^{s}x_n, |T|^{s}x_n\rangle \\ & \to 0. \end{aligned}$$

Hence

$$\left(|T|^{\frac{rp}{2}}-\rho^{\frac{rp}{2}}\right)|T|^{s}x_n \to 0$$

and so

$$\left(|T|-\rho\right)|T|^{s}x_n \to 0.$$

Then

$$|T|(|T|-\rho)x_n = |T|^{1-s}(|T|-\rho)|T|^{s}x_n \to 0.$$

Since

$$\begin{aligned} 0 &= \lim\langle|T|(|T|-\rho)x_n, x_n\rangle \\ &= \lim\||T|x_n\|^2 - \rho\lim\langle|T|x_n, x_n\rangle \\ &= \lim\|Tx_n\|^2 - \rho\lim\langle|T|x_n, x_n\rangle \\ &= \rho^2 - \rho\lim\langle|T|x_n, x_n\rangle, \end{aligned}$$

we have

$$\langle|T|x_n, x_n\rangle \to \rho$$

and

$$\begin{aligned} \|(|T|-\rho)x_n\|^2 &= \||T|x_n\|^2 - 2\rho\langle|T|x_n, x_n\rangle + \rho^2 \\ &\to \rho^2 - 2\rho^2 + \rho^2 = 0. \end{aligned}$$

This implies

$$(|T|-\rho)x_n \to 0.$$

Since

$$0 \leftarrow (T-\rho e^{i\theta})x_n = U\left(|T|-\rho\right)x_n + \rho(U-e^{i\theta})x_n$$

and $0 < \rho$, we have

$$(U - e^{i\theta})x_n \to 0.$$

Also,

$$\begin{aligned}\|(U - e^{i\theta})^* x_n\|^2 &= \|U^* x_n\|^2 - \langle U^* x_n, e^{-i\theta} x_n\rangle - \langle e^{-i\theta} x_n, U^* x_n\rangle + 1 \\ &\leq 1 - e^{i\theta}\langle x_n, U x_n\rangle - e^{-i\theta}\langle U x_n, x_n\rangle + 1 \\ &\leq -e^{i\theta}\langle x_n, (U - e^{i\theta})x_n\rangle - e^{-i\theta}\langle (U - e^{i\theta})x_n, x_n\rangle \to 0.\end{aligned}$$

Hence

$$(U - e^{i\theta})^* x_n \to 0$$

and

$$(T - \rho e^{i\theta})^* x_n = |T|(U - e^{i\theta})^* x_n + e^{-i\theta}(|T| - \rho)x_n \to 0.$$

□

Open problems

It is known that class A operator satisfies Putnam type inequality. However it is not known that Putnam type inequality holds for p-$wA(s,t)$ operators. It seems a difficult problem. We note some open problems for p-$wA(s,t)$ operators.

(1) M. Ito and T. Yamazaki [9] proved that $A(s,t)$ implies $wA(s,t)$. However it is not known whether p-class $A(s,t)$ implies p-$wA(s,t)$ for $0 < p < 1$ or not.

(2) It is known that if T is class $A(s,t)$ and $\mathcal{M} \subset \mathcal{H}$ is a T-invariant subspace, then $T|_{\mathcal{M}}$ is class $A(s,t)$. However it is not known whether this property holds for p-$wA(s,t)$ operator T.

(3) It is known that class A operator T is normaloid. But it is not known that p-$wA(s,t)$ operator T is normaloid or not.

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