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HEAT EQUATIONS BY FOURIER AND POISSON

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ABSTRACT.
We discuss the historical wave theories including the fluid dynamics, the heat theory, in particular, that of Fourier and Poisson. We think, from the heat theories, we have had the many mathematically fruitful productions of derivatives from the heat theories, such as the trigonometric series, the eigenvalue problems and the rapidly decreasing function.

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1. INTRODICTION

[1,2,3] Fourier explains the motion of the heat in the interior of solid. The difference is that determines its increment of the temperature during an instant :

$$Kdydz\ d\Big(\frac{dv}{dx}\Big)dt + Kdxdz\ d\Big(\frac{dv}{dy}\Big)dt + Kdxdy\ d\Big(\frac{dv}{dz}\Big)dt \quad \Rightarrow \quad Kdxdydz\Big(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\Big)dt$$

$$(d)_{F2.5} \qquad \frac{du}{dt} = \frac{K}{C.D}\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\right) \tag{1}$$

where, K internal conductibility, C capacity, D density of the substance. ([4, pp.120-2]). We think that Fourier's deductive method is very diffuse style and simpler than Poisson's inductive method described over 10 pages in original [18], we show his point below in § 3.3.

1.1. Poisson's paradigm and singularity.

Poisson publishes the last books consist of three elements : [15, 16, 17, 18]. ([16, 17] are the same title and are divided into two volumes.) These are his paradigm of the mathematical physics through all his academic life, entitled a study of mathematical phisics. (*Un Traité de Physique Mathématique.*) In the rivalry to Euler, Lagrange, Laplace, Fourier, Navier, et al.,

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[1]Translation from Latin/French/German into English mine, except for Boltzmann.

[2]To establish a time line of these contributor, we list for easy reference the year of their birth and death: Newton (1643-1727), Euler(1707-83), d'Alembert(1717-83), Lagrange(1736-1813), Laplace(1749-1827), Fourier(1768-1830), Poisson(1781-1840), Cauchy(1789-1857), Dirichlet(1805-59), Riemann(1826-66), Boltzmann(1844-1906), Hilbert(1862-1943), Schrödinger(1887-1961).

[3]We use (⇓) means our remark not original, when we want to avoid the confusions between our opinion and sic. (⇐) means our translation in citing the origin.

we think, he struggles to make his paradigm. On the other hand, as its proofs, there are some singular but important sugestions such as :

- rigorous sum instead of integral, (cf. §1.2)
- critics to easy applying the rule comes from real to transcendental function, (cf. fig.1)
- conjecture on the defect of the proof in the eternity of exact differential,
- contribution to the fluid dynamics, especially, to the Navier-Stokes equations, (cf. [9, pp. 261-271], Table 2, and 3)
- deduction of another heat equation from the basically molecular analysis. (cf. § 3 and § 3.3)

1.2. A comment on continuum by Duhamel.

Duhamel 1829 [2] points out the theory of continuum from the viewpoint of scientific history, citing from the Poisson's paper in the argument with Navier on the nonsense of Navier's null action in nature.

> (⇐) Up to now, the reserchers have considered the corps of the nature as continue, it makes illusion to this regards, however, partly because this hypothesis simplify the calcul, and partly because they think that it gives a sufficient approximation. Mr. Poisson think that this hypothesis isn't never admissible, and justify his opinion with following considerations.

> In this state, the distance which separate the molecules must be such that this condition were replaced, in having regard to their mutual attraction and the caloric repulsion which we take also among the molecular actions. However the corps is hard or something solid, the force which opposes the separation of their parties is zero or doesn't exist in the state of which we discuss. It doesn't begin the existence that when we seek to effectuate this separation, and when we change only a few distance of the molecules. Namely, *if we explain this force with a integral, it gets to as its value being zero in the natural state of corps, it will be so even if after the variation of the molecular distances, so that, the corps will opposite any resistance to the separatiopn of its parties ; this is what will be nonsense.* It results from here, that the sum which explain the total action of a series of disjoint molecules can't convert the sum instead of the definite integral ; this is what holds in the nature of the *function of distances* which represent the action of each molecule. The molecular force, of which we will find the expression in the §1 of this Memoire, is calculated according to this principle, and reduced at least in the simplest form of which it were susceptible. [2, pp.98-99] (trans. and italics mine.)

2. Confusions and unify on continuum theory

The physico-mathematicians are must construct at first the physical structure, then allpies the mathematical concept on it. The former is necessary to fit with the actual phenomena. Arago 1829 [?] seeks to separate these items to Navier 1829 [10] in the current of dispute with Poisson and Arago. This is comes from the word what-Navier-called *l'une sur l'autre*, he fails to explain exactly it, and since then, his theories and the equations are neglected up to the top of the 20th century. We consider that the confusions and unify are as follows :

- Poisson and Fourier discuss on the handling of the De Gua's theorem into the transcendental equations. Without clear explanation, Fourier passed away in 1830. cf. (fig.1)

- On the attraction and replusion of molecule, Navier depends on Fourier's principle of heat molecule. The then hysico-mathematicians had little evaluated Navier until the top of the 20th century. For formulation of heat motion in the fluid, Fourier cites not Navier's fluid equations, but Euler's fluid equations.
- The hydrodynamists like Navier, Poisson, Cauchy are propose the wave equations in the elasticity, and the last two hydrodynamists proposes the total equations in unity on the continuum.
- On the formulation of heat motion in the fluid, Fourier had submitted this paper, however, until his death, he has not published it, in which he seems to aim the unity of hydro- and thermodynamics, however, he has given up it.

3. The heat and fluid theories in the 19th century

Poisson [13] traces Fourier's work of heat theory, from the another point of view. Poisson emphasizes, in the head paragraph of his paper, that although he totally takes the different approaches to formulate the heat differential equations or to solove the various problems or to deduce the solutions from them, the results by Poisson are coincident with Fourier's.

> I will take care of, through this Memoire, to cite the principle result which Mr. Fourier have obtained before me ; and I dare to say at first, in all the particular problems which we have taken the one and the another for examples, and which being naturally indicated in this material, the formulae of my Memoire coincides with that this piece includes. However, *just only that there is common between our two oeuvres* ; because, it were to formulate the differential equations of the motion of the heat, or it were to solve them and deduce the definitive solution of each problem, *I am using the entirely different methods from that Mr. Fourier is tracing.* [13, pp.1-2] (trans. and italics mine.)

3.1. The deduction of heat equations by Prévost.

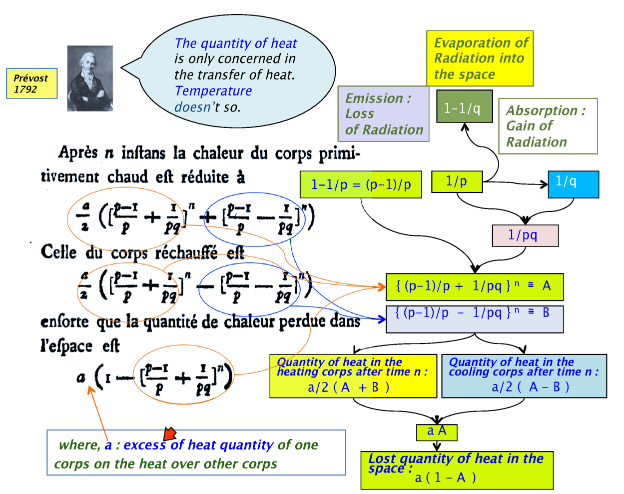

fig.4 Heat communication theory by Prévost

3.2. The deduction of heat equations by Fourier.

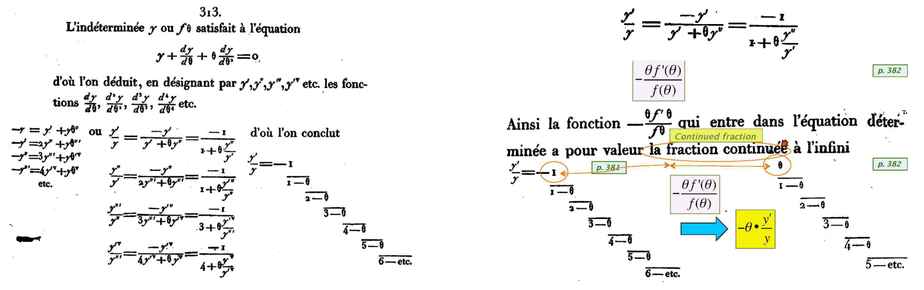

fig.6 Continued fraction in Fourier 1822

Fourier shows the article 313 as follows :

> the indeterminate y viz. $f(\theta)$ satisfies width the equation : $y + \frac{dy}{d\theta} + \theta\frac{d^2y}{d\theta^2} = 0$, from where, we deduce, in designating with y', y'', y''', y^{IV} etc. the function $\frac{dy}{d\theta}$, $\frac{d^2y}{d\theta^2}$, $\frac{d^3y}{d\theta^3}$, $\frac{d^4y}{d\theta^4}$ etc.. Hence, the function $-\frac{\theta f'(\theta)}{f(\theta)}$ which exists in the determined equation has for value the continued fraction to the infinite. [4, pp.381-382].

In fact, in 1807, Fourier has used the continued fraction to explain the last temperatures and the curve which represents them as follows :

> we have remarked precedingly what is the form of the curve which the ordinates y are $f(\theta)$, namely,
>
> $$1 - \theta + \frac{\theta^2}{1^2.2^2} - \frac{\theta^3}{1^2.2^2.3^2} + \frac{\theta^4}{1^2.2^2.3^2.4^2} - \&.$$
>
> and which θ is the abcissa. If we suppose $\theta = \frac{\epsilon^2}{2^2}$ and if we take ϵ for new abcissa and for ordinate the same quantity y viz. $f(\theta)$, which turns F, namely
>
> $$1 - \frac{\epsilon^2}{2^2} + \frac{\epsilon^4}{2^2.4^2} - \frac{\epsilon^6}{2^2.4^2.6^2} + \&.$$
>
> it will be easy to know the form of the new curve which y and ϵ are the coordinates. It will need to conserve the ordinates with the preceding curves and reduce the abscissae, in addition to that it holds the equation $\theta = \frac{\epsilon^2}{2^2}$. To explain with ϵ the function $-\theta\frac{f'(\theta)}{f(\theta)}$ which exists in the determined equation we put $f(\theta) = F(\epsilon)$, from where we put $\frac{d\theta f'(\theta)}{d\epsilon} = F'(\epsilon)$. It holds hence $-\theta\frac{f'(\theta)}{f(\theta)} = -\frac{\theta F'(\epsilon)}{\frac{d\theta}{d\epsilon}F(\epsilon)}$. The equation $\theta = \frac{\epsilon^2}{2^2}$ gives $\frac{d\theta}{d\epsilon} = \frac{2\epsilon}{2^2}$. However, $-\theta\frac{f'(\theta)}{f(\theta)} = -\epsilon\frac{F'(\epsilon)}{2F(\epsilon)}$. Hence the determined equation $\frac{hR}{2K} = -2\theta\frac{f'(\theta)}{f(\theta)}$ turns into $\frac{hR}{2K} = -\epsilon\frac{F'(\epsilon)}{F(\epsilon)}$. [6, pp.371-2] (trans mine.) cf. fig. 2 and ??.

3.3. The deduction of heat equations by Poisson.

Poisson deduces his heat equations of the motion in interior of solid corps or liquid with the function R, which depend on the distance between the two molecules.

(§44.) There is always the heat in motion in all the corps, even when of all their points is invariable,

- were each point would have a particular temperature,
- were its would have all a same temperature.

However, the expression *motion of the heat* is taken here, in the another sense ; it signifies the variation of temperature which holds from an instant to the other in a corps which is heated or is cooled ; and the velocity of this motion, in each point of the corps, is the primary differential coefficient of the temperature with respect to the time.

I will call A the corps solid or liquid, homogeneous or heterogeneous, in which we are going to consider the motion of the heat. Let

- M a certain point of A,
- and m a particle of this corps, of insensible magnitude (no. 7),
- and take the point M.

At the end of a certain time t,

- designate with $x,\ y,\ z$, the three rectangular coordinates of M,
- with v the volume of m,
- and with ρ its density,

so that we have $m = v\rho$. Let also, at the same instant, u the temperature and $\mho$ [4] the velocity of motion of the heat which responds to the point M.

The quantity u will be a function of $t,\ x,\ y,\ z$, dependent on an equation in the partial differences with respect to these four variables, which it is the problem to form. If A is a corps solid, and which we make neglect its small dilations, positive or negative, products with the variations of u relative to time, the coordinates $x,\ y,\ z$, according to independent of t, and we will have simply, $\mho = \frac{du}{dt}$.

- If in contrast, we have regard to small displacement of the point M caused from these dilations,
- or also, if A is a fluid in which the integrality of temperature, or all other cause, hold to the motions of its molecules,

then the coordinates $x,\ y,\ z$, will be the function of t ; and then we will have with the known rules of the differentiation of functions made of functions,[5]

$$(1)_{PS4} \qquad \mho = \frac{du}{dt} + \frac{dx}{dt}\frac{du}{dx} + \frac{dy}{dt}\frac{du}{dy} + \frac{dz}{dt}\frac{du}{dz} \; ; \qquad (2)$$

where, expression in which $\frac{dx}{dt},\ \frac{dy}{dt},\ \frac{dz}{dt}$, will be the components of the velocities at the point M, parallel to the axes $x,\ y,\ z$.

quad The unknown u will be the only that it will need to determine, for recognition completely of the calorific state of the corps A at a certain instant. Suppose that we divide this corps into two parts B and B', with a certain surface, traced in its interior. Let ω an element of this surface (no. 9) containing the point M, there will be continuously, crosswise ω, a *flux of heat* sensible to that of the radiating heat which holds crosswise the element of the surface of A, and that I will represent with $\Gamma\ \omega\ dt$ during the instant dt, of manner which this product, positive or negative, were the excess of the heat which traverses the ω in passing from B in B', during this instant, on that which traverse in the same time, in passing from B' into the B. The coefficient Γ, or the flux of heat relative to the units of time and of surface. will depend on the material and of the temperature of A at the point M, and of the direction of ω ; it will be important to determine it, in function of $t,\ x,\ y,\ z$, for each direction given with ω. Hence, u and Γ will be the two unknown of the problem with which we will have to occupy in this chapter. When the corps A is obey to the influence of foci constants of heat all its parts arrive generally, after a certain time, to the variable temperatures of a point to another, however, independent of the

[4]($\Downarrow$) We use $\mho$, because, in origin, Poisson uses the vertical type of $\propto$ like the opened shape in upper of the numerical letter 8, however, this exact type isn't in our LaTex font system.

[5]($\Downarrow$) sic. The function is repeated.

time. In this stationary state of A, the velocity $\mho$ is zero in all the points ; however, the flux of heat Γ exists still, and merely its value is independent of t, like that of u.

(§45.) Let M' a second point of A very near to M, and m' a particle of A of insensible magnitude, like m which will contain M'. At the end of time t, we call x', y', z', the coordinates of M' in relating to same axes with x, y, z, and designate with u' the temperature of m' ; also let r the distance MM'.

According to the general hypothesis on which the mathematical theory of the heat (no. 7) is based, there will be a continuous exchange of heat between m and m'. I will represent with δ the augmentation of heat which will result then for m during the instant dt, namely, the excess positive or negative, during this instant,

- of the heat emitted from m' and absorbed with m,
- over the heat emitted from m and absorbed with m'.

It will be able to suppose this excess proportional to product $m\ m'\ dt$, or to $v\ v'\ \rho\ \rho' dt$, in calling v' and ρ' the volume and the density of m', so that we would have $m' = v'\ \rho'$, as we have already $m = v\ \rho$. It will be zero in the case of $u' = u$, and same sign with the difference $u' - u$, when it won't be zero ; in the vacuum, it will come in the reverse ratio of the square of r ; and generally its value will be the form

$$(2)_{PS4} \qquad \delta = \frac{v\ v'}{r^2}\ R\ (u' - u)\ dt, \tag{3}$$

where, in designating with R a positive coefficient, in which we contain the factor $\rho\ \rho'$, which will decrease very rapidly for the values increasing with r, which will be also able to depend on materials and the temperatures of m and m', and will vary with the direction of MM', when the absorption of the heat won't be the same in all direction around of M. [6]

In the supposition the most general, R will be hence a function of r, u, u', and the coordinates of M and M' ; so that we will have

$$R = \Phi\ (r,\ u,\ u',\ x,\ y,\ z,\ x',\ y',\ z').$$

However, if we call δ' the dimension of heat of m' during the instant dt, causing the exchange between m and m', we will have evidently $\delta' = -\delta$; in addition, the value of δ' will come to be deduced from that of δ with the permutation of quantities relative to the one of the points M and M', and the analogous quantities which respond to the other ; in consequence, it will need that the function Φ were symmetric with respect to u and u', x and x', y and y', z and z'.

The corps A being a solid or a liquid, this function Φ will vary very rapidly with r and will be insensible or zero, as long as r will have arrived at a very small grade. I will designate this limit with l, so that this function Φ were zero, as long as we will have $r > l$ or merely $r = l$. This segment l will be hence very small, however, of the sensible grade and measurable (no. 41), and in consequence, extremely greater with relation to the dimensions of m and m'.

(§46.) The total augmentation of heat of m during the instant dt will be the sum of values of δ, extended to all the point M' of which the distance at the point M is smaller than l. I will indicate a such sum with the characteristic Σ. The factor $v\ dt$ being common to all the value of δ, their sum will be

$$v\ dt \sum \frac{R}{r^2}\ (u' - u)\ v'. \tag{4}$$

However, during the instance dt, the temperature of m augments with $\mho\ dt$; if hence, we call c its specific heat, $c\ v\ \mho\ dt$ will be also its augmentation of heat during this instant ; hence in

[6](⇓) The expression (3) is combined Newton's law : $\frac{v\ v'}{r^2}$ with Prévost's theory : $R\ (u' - u)$.

suppressing the common factor $v\ dt$, we will have

$$(3)_{PS4} \qquad c\,\mho = \sum \frac{R}{r^2}\,(u' - u)\,v', \tag{5}$$

for the equation of motion of the heat equally applicable to a corps solid and to a liquid, in substituting the convenient expression with $\mho$. The sum $\sum$ contained in this equation, doesn't depend in effect, merely on the calorific state of m and of the parties surrounding with A, which exists at the end of the time t, and in any manner of change which would be able to hold the next instant ; so that it wouldn't be necessary to the heat, like the mathematicians [7] have considered, a particular equation for the motion of the heat in the liquids [8], distinct from one which responds to corps solids heterogeneous, and which had been given since long ago. The value of a sum $\sum$ relative to the parties of insensible grade, such that the preceding, can be explained with a series of which the primary term is an integral taken between the same limits which this sum, and of which the other preceding terms following the dimensions of these particles, raised to the increasing power. These dimensions being insensible with hypothesis, it is followed that the series is, in general, extremely *convergent*, and may be reduced to its primary term. Hence, in designating with $d\,v'$ the differential element of the volume of A, which responds to the point M', we will have, without appreciative error,

$$\sum \frac{R}{r^2}\,(u' - u)\,v' = \int \frac{R}{r^2}\,(u' - u)\,dv'\ ;$$

The integral is extending to all the elements dv', of which the distance r at the point M is smaller than l.

In effect, I remarked in other occasions which the reduction of a sum to an integral is no more permitted in a certain case which is presented, for instance, in the calculation of molecular forces ; however, for that this exception should hold, it needs that the function of which we are going to sum the values, varies very rapidly and change the sign between the limits of this sum ; hence, here the coefficient R vary well in effect very rapidly with the variable r, however, without never change of sign ; and for this reason, the exception of which it is important isn't to be afraid. In all the calculations of quantities of heat which result of exchange between the parties of a corps, of insensible grade, we will be able to decompose immediately its volume in elements infinitely small, and replace the sum with the integrals, as if this corps being would be formed of a material, contained and not of the disjoint molecules, separated with the pores or vacant space.

(§47.) Of the point M as center and a radius equal to the linear unit, we describe a spherical surface ; were ds the differential element of this surface, to which gets, the radius of which the direction is that of MM', we will have $dv' = r^2\ dr\ ds$; and according to the value of the sum $\sum$, the equation (5) will turn out

$$(4)_{PS4} \qquad c\,\frac{du}{dt} = \iint R\,(u' - u)\,dr\ ds\ ; \tag{6}$$

We put here, for abridgement, $\frac{du}{dt}$, instead of $\mho$; however, we will remember that this differential coefficient needs to be taken with relation to t and to all this that depend ; so that it needs to replace $\frac{du}{dt}$ with the formula (2), when the coordinates $x,\ y,\ z,$ of the point M will vary with the time.

The limit relative to r of the integral contains in this equation (6) won't be the same, according to the distance of the point M to the surface of A will surpass l or will be shorter than this small segment. In this chapter we will suppose that this were the primary case which holds ;

[7](⇓) F. geometricians. Now, it means mathematician.

[8](⇓) Poisson may cite as the mathematician Fourier [5].

the integral relative to r will come to be hence taken from $r = 0$ to $r = l$, in all the direction around M ; we will be able hence to describe the equation (6) under the form

$$(5)_{PS4} \qquad c\,\frac{du}{dt} = \int_0^l \Big[\int R\,(u' - u)\ ds\Big]\ dr\ ; \tag{7}$$

where, the integral in respecting to ds will come to be extended to all the element ds from the spherical surface, and with the reduction in series, we will obtain easily the approximate value.

(§48.) For these things, I designate with $\alpha,\ \beta,\ \gamma$, the angles which the segment MM' makes with the parallels to the axes $x,\ y,\ z$, traced through the point M. Because of $MM' = r$, then it will result

$$x' - x = r\ \cos\ \alpha, \qquad y' - y = r\ \cos\ \beta, \qquad z' - z = r\ \cos\ \gamma\ ;$$

and, according to the theory of Taylor, we will have

$$\begin{aligned} u' - u = \ & \frac{du}{dx}\,r\ \cos\ \alpha + \frac{du}{dy}\,r\ \cos\ \beta + \frac{du}{dz}\,r\ \cos\ \gamma \\ + \ & \frac{1}{2}\frac{d^2u}{dx^2}\,r^2\ \cos^2\ \alpha + \frac{1}{2}\frac{d^2u}{dy^2}\,r^2\ \cos^2\ \beta + \frac{1}{2}\frac{d^2u}{dz^2}\,r^2\ \cos^2\ \gamma \\ + \ & \frac{d^2u}{dx\ dy}\,r^2\ \cos\ \alpha\ \cos\ \beta + \frac{d^2u}{dx\ dz}\,r^2\ \cos\ \alpha\ \cos\ \gamma + \frac{d^2u}{dy\ dz}\,r^2\ \cos\ \beta\ \cos\ \gamma \\ \cdots\ . \end{aligned}$$

If we develop similarly R in accordance with the power and the products of $u' - u,\ x' - x,\ y' - y,\ z' - z$, we will have also

$$R = V + \Big(\frac{dR}{du'}\Big)\ (u' - u) + \Big(\frac{dR}{dx'}\Big)\ (x' - x) + \Big(\frac{dR}{dy'}\Big)\ (y' - y) + \Big(\frac{dR}{dz'}\Big)\ (z' - z) + \cdots\ ;$$

where, the parentheses indicating here that it needs to put $u' = u$, $x' = x$, $y' = y$, $z' = z$ according to the differentiation which supposes r invariable, and V designating this which comes at the same time from the function Φ of the (no. 45), so that we have

$$V = \Phi\ (r,\ u,\ u,\ x,\ y,\ z,\ x,\ y,\ z). \tag{8}$$

By means of these developments of R and of $u' - u$, this one of product $\int R\ (u' - u)$ will be composed of terms of this form

$$H\ r^n\ \cos^i\ \alpha\ \cos^{i'}\ \beta\ \cos^{i''}\ \gamma\ ;$$

where, H designating a coefficient independent of $\alpha,\ \beta,\ \gamma$, and the exponential $i,\ i',\ i''$, being the number entire and positive which won't be zeros all the three to the times, and of which the exponent n is the sum $i + i' + i''$. Hence in having regard to the limits of the integral relative to ds, we will have

$$\int \cos^i\ \alpha\ \cos^{i'}\ \beta\ \cos^{i''}\ \gamma\ ds = 0,$$

here all times which the one of the three numbers $i,\ i',\ i''$, will be odd ; for then this integral will be composed of the elements which will be equal two by two and the contrary sign. When any of number $i,\ i',\ i''$, won't be odd, the integral won't be zero ; the ordinary rules give the exact values, whatever these three number were ; and with this manner, we will have

$$(6)_{PS4} \qquad R\ (u' - u) = H_2\ r^2 + H_4\ r^4 + H_6\ r^6 + \text{etc.}\ ; \tag{9}$$

where, $H_2,\ H_4,\ H_6$, etc., being the differential function of known form, in any of which the partial differences [9] of u will be taken in respect to $x,\ y,\ z$, and are raised to the order marked with its inferior index.

[9](⇓) id. This means the partial differentials.

For a temperature u which would vary very rapidly, so that it should have the values very different in the extent of interior radiation, the coefficients H_2, H_4, H_6, etc., would form a series very rapidly increasing, by reason of partial differences [10] of u on which they depend. The series (9) would cease hence to be converged, though the smallness of r^2 ; however, this case doesn't hold in a point M sufficiently separated, as we suppose it, of the surface of A ; and we will be able, in consequence, to regard the series (9) as extremely convergent.

In stopping at its nth term, the equation in the partial differences [11] of the motion of the heat will be the order $2n$; however, its complete integral will include certain parties which will vary very rapidly, and that we be will be able to suppress for this reason, in the value of u, as a layperson to the question ; this one which will reduce always this value at the same degree of generality, whatever its degree of approximation were, dependent on the terms of the series (9) which we will have conserved. This is here what we see successively, on a particular example, in which we will show also the influence which can have the sensible extent of the interior radiation on the value of u. However, to reduce the general equation of the motion of the heat to the simplest form, namely, to the form of an equation in the partial differences [12] of second order, also which we make ordinarily, we restrict the approximation to the primary term of the series (9) ; this here returns to consider as insensible the extent of the radiation in the interior of corps solid and of liquid.

(§49.) (General equation of the motion of heat) [13]

In this hypothesis, we will stop the development of R at the terms dependent on the square of r exclusively. By reason of the system of R in respect to u and u', x and x', y and y', z and z', and of this one which V represents, we have evidently

$$\left(\frac{dR}{du'}\right)=\frac{1}{2}\frac{dV}{du},\quad \left(\frac{dR}{dx'}\right)=\frac{1}{2}\frac{dV}{dx},\quad \left(\frac{dR}{dy'}\right)=\frac{1}{2}\frac{dV}{dy},\quad \left(\frac{dR}{dz'}\right)=\frac{1}{2}\frac{dV}{dz}\ ;$$

then, it will result hence

$$R=V+\frac{1}{2}\frac{dV}{du}(u'-u)+\frac{1}{2}\frac{dV}{dx}(x'-x)+\frac{1}{2}\frac{dV}{dy}(y'-y)+\frac{1}{2}\frac{dV}{dz}(z'-z)\ ;$$

and of this value jointed to that of $u'-u$, we will conclude

$$\begin{aligned} H_2 &= \frac{1}{2}\left[V\frac{d^2u}{dx^2}+\left(\frac{dV}{du}\frac{du}{dx}+\frac{dV}{dx}\right)\frac{du}{dx}\right]\int\cos^2\alpha\ ds+\frac{1}{2}\left[V\frac{d^2u}{dy^2}+\left(\frac{dV}{du}\frac{du}{dy}+\frac{dV}{dy}\right)\frac{du}{dy}\right]\int\cos^2\beta\ ds \\ &+ \frac{1}{2}\left[V\frac{d^2u}{dz^2}+\left(\frac{dV}{du}\frac{du}{dz}+\frac{dV}{dz}\right)\frac{du}{dz}\right]\int\cos^2\gamma\ ds, \end{aligned}$$

or more simply

$$\begin{aligned} H_2 &= \frac{1}{2}\left[V\frac{d^2u}{dx^2}+\frac{dV}{dx}\frac{du}{dx}\right]\int\cos^2\alpha\ ds+\frac{1}{2}\left[V\frac{d^2u}{dy^2}+\frac{dV}{dy}\frac{du}{dy}\right]\int\cos^2\beta\ ds \\ &+ \frac{1}{2}\left[V\frac{d^2u}{dz^2}+\frac{dV}{dz}\frac{du}{dz}\right]\int\cos^2\gamma\ ds\ ; \end{aligned}$$

the partial differences [14] of V with respect to x, y, z, being taken in considering u as a function of these three coordinates, and without varying r.

[10](⇓) id.

[11](⇓) id.

[12](⇓) id.

[13](⇓) This article is the most frequently referred from other article, such as 52, 58, 64, 68, 70, **76**, 85, 89, 117, 119, 120, 137, **162**. (These are the article numbers, referred to the no. 49, and in the bold numbers, the another equations are expressed.)

[14](⇓) id.

We have additionally

$$\int \cos^2 \alpha \; ds = \int \cos^2 \beta \; ds = \int \cos^2 \gamma \; ds.$$

Moreover, if we call ψ the angle which makes the plane of the segment MM' and of a parallel to the axis of x traced through the point M, with a fixed plane traced through this parallel, we will have $ds = \sin \; \alpha \; d\alpha \; d\psi$; and the integral relative to ds will come to be extended to all the spherical surface, to which this element belongs, then it will result

$$\int \cos^2 \alpha \; ds = \int_0^{\pi} \cos^2 \alpha \; \sin \; \alpha \; d\alpha \int_0^{2\pi} d\psi = \frac{4\pi}{3}.$$

[15] Hence, in reducing the value of $\int R \; (u' - u)$ at the primary term $H_2 \; r^2$ of the series (9), the equation (7) will come to be

$$\begin{aligned} c\frac{du}{dt} &= \frac{2\pi}{3}\Big(\frac{d^2u}{dx^2}\int_0^l V\; r^2 \; dr + \frac{du}{dx}\int_0^l \frac{dV}{dx}\; r^2 \; dr\Big) + \frac{2\pi}{3}\Big(\frac{d^2u}{dy^2}\int_0^l V\; r^2 \; dr + \frac{du}{dy}\int_0^l \frac{dV}{dy}\; r^2 \; dr\Big) \\ &+ \frac{2\pi}{3}\Big(\frac{d^2u}{dz^2}\int_0^l V\; r^2 \; dr + \frac{du}{dz}\int_0^l \frac{dV}{dz}\; r^2 \; dr\Big). \qquad (10) \end{aligned}$$

The function V being zero for all the value of r longer than l, we will be able to now extend the integral relative to r beyond this limit, and if we want to be until $r = \infty$. If we put also

$$\frac{2\pi}{3}\int_0^{\infty} Vr^2 \; dr \equiv k, \qquad (11)$$

where, k will be a function of $u, \; x, \; y, \; z$, and we will have

$$\frac{2\pi}{3}\int_0^{\infty}\frac{dV}{dx}\; r^2 \; dr = \frac{dk}{dx}, \quad \frac{2\pi}{3}\int_0^{\infty}\frac{dV}{dy}\; r^2 \; dr = \frac{dk}{dy}, \quad \frac{2\pi}{3}\int_0^{\infty}\frac{dV}{dz}\; r^2 \; dr = \frac{dk}{dz} \; ;$$

in consequence, the general equation of the motion of the heat will come to be finally [16]

$$(7)_{PS4} \qquad c\frac{du}{dt} = \frac{d.k\frac{du}{dx}}{dx} + \frac{d.k\frac{du}{dy}}{dy} + \frac{d.k\frac{du}{dz}}{dz}. \qquad (12)$$

When all the point of A gets to a stationary state, we will have $\frac{du}{dt} = 0$, and then it will result

$$\frac{d.k\frac{du}{dx}}{dx} + \frac{d.k\frac{du}{dy}}{dy} + \frac{d.k\frac{du}{dz}}{dz} = 0,$$

for the equation relative to this stationary state.

(§50.) The equation (12) coincides with that which I found in years ago for the case of a heterogeneous corps[17], however, in never supposing hence that the quantity k depended on the temperature u.

If A is a corps homogeneous, [18]

- k will depend only on u,

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[15](⇓) According to [11, p.41, no.277],

$$\int \cos^m x \; \sin x \; dx = -\frac{\cos^{m+1} x}{m+1}.$$

[16](⇓) The expression (10) is reduced into

$$c\frac{du}{dt} = \Big(\frac{d^2u}{dx^2}k + \frac{du}{dx}\frac{dk}{dx}\Big) + \Big(\frac{d^2u}{dy^2}k + \frac{du}{dy}\frac{dk}{dy}\Big) + \Big(\frac{d^2u}{dz^2}k + \frac{du}{dz}\frac{dk}{dz}\Big) \quad \Rightarrow \quad (12).$$

[17]sic. *Journal de l'École Polytechnique*, 19ᵉ cahier, page 87. (⇓) Poisson [13], [17, p. 677].

[18](⇓) We regret, in the last report [9, p.168], that we made an incorrect statement in this condition as 'heterogeneous' on the (13). Correctly, 'homogeneous'.

- and the equation (12) will be changed as follows : [19]

$$(8)_{PS4} \qquad c\frac{du}{dt} = k\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) + \frac{dk}{du}\left(\frac{du^2}{dx^2} + \frac{du^2}{dy^2} + \frac{du^2}{dz^2}\right). \qquad (13)$$

In supposing that this quantity k were independent of u, we could have the equation

$$(9)_{PS4} \qquad c\frac{du}{dt} = k\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right), \qquad (14)$$

[20] which we give it ordinarily, and which is reduced, in the case of the stationary state, to an equation independent of two quantities c and k, viz.,

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = 0. \qquad (15)$$

[21] Poisson puts also the another heat equations :

$$(1)_{PS11} \qquad \frac{du}{dt} = a^2\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right), \qquad \frac{k}{c} = a^2, \qquad (16)$$

where, u is the heat, k and c are the conductibility and the specific heat of the material.

4. Conclusions

Fourier doesn't show the pricise deduction of the heat equation (1), while Poisson takes 9 pages to descrive it from §44 to §50. The diffefrence between Fourier and Poisson is the common kernel function of molecular distance, which Poisson considers and manipulates in both fluid motion and heat motion. Additionally, we concludes as follows :

1. We consider our problem as the totality among the definite integral, the trigonometric series, etc., for Poisson's objection to Fourier is relating the universal and fundamental problem of analytics, as we show Poisson's analytical/mathematical thought or sight in the Chapter ??, etc. In fact, Poisson's work-span covers them.
2. Fourier doesn't show the pricise deduction of the heat equation (1), while Poisson takes 9 pages to descrive it from §44 to §50. The diffefrence between Fourier and Poisson is the common kernel function of molecular distance, which Poisson manipulates in both fluid motion and heat motion. Poisson descreminate the homogeneous and heterogeneous corps, the former corresponds to the equation (13) and the latter to the equation (14), which we get it ordinarily and it equals to Fourier's (1), if we have no problem of the coefficient, or, we put $\frac{k}{c} \equiv a^2$.
3. Both aim the mathematical physics, however, Fourier's mathematical bases are algebraic, while Poisson's one are analytical one adapting for wide phenomina of nature. We think, Poisson's method comes from the fluid dynamics and the wave theory in which he introduces an origin of the Navier-Stokes equations and the wave equations. We see Poisson's deduction of the heat equation based on the hypothesis of molecular emission and absorption of heat owing to the Newton's law and Prvost's law.
4. Owing to the arrival of continuum, we are able to discuss the solution of the problem on the continuous space of mathematics. As Duhamel [2] says, at first, Poisson performs it with the concept of mathematically infinite continuity. This allows us to discuss, without depending on the microscopic-description, by the vectorially description, like Saint-Venant, Stokes.

[19]($\Downarrow$) The second term of the right hand-side of the equation (13) is for $k = k(u)$, in sic.

[20]($\Downarrow$) The equation (14) means $c\frac{du}{dt} = k\Delta u$, where Δ meaning the Laplacian.

[21]($\Downarrow$) This function u satisfying the equation (15) is called harmonic function. Poisson doesn't mention the harmonic function, however, Poincaré [19, p.237] calls it so.

5. Although the confusion of knowledges on continuum, the unity in the mathematics are gained, however, the applicabilities of the unite or general equations are then not yet defined, which comes from the misunderstandings interphysico-mathematics, such as the identity of fluid and elasticity, or, fluid and heat.
6. About the describability of the trigonometric series of an arbitrary function, nobody succeeds in it including Fourier, himself.

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[4] J.-B.-J. Fourier, *Théorie analytique de la chaleur.*, Paris, 1822. → http://gallica.bnf.fr/ark:/12148/bpt6k1045508v (Remark. This is the original book version of BnF : *Bibliothèque Nationale France*, namely, it seems the original book of [3] edited by G.Darboux. After p.32, there are 4 pages of pp.23-26, however, p.33ff are followed after p.26. Total pages are 639 plus the 2 pages of figures, while the book [3] is consisted of 563 pages.)

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