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SURJECTIVE ISOMETRIES ON $C^1[0,1]$ WITH RESPECT TO SEVERAL NORMS

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ABSTRACT. Let $C^1[0,1]$ be a complex linear space of all continuously differentiable complex valued functions on the unit interval $[0,1]$. We give a characterization of surjective, not necessarily linear, isometries on $C^1[0,1]$ with respect to the following norms: $\|f\|_\Sigma = \|f\|_\infty + \|f'\|_\infty$, $\|f\|_C = \sup\{|f(t)|+|f'(t)| : t \in [0,1]\}$ and $\|f\|_\sigma = |f(0)|+\|f'\|_\infty$ for $f \in C^1[0,1]$, respectively.

1. INTRODUCTION

Let M and N be real or complex normed linear spaces with norms $\|\cdot\|_M$ and $\|\cdot\|_N$, respectively. We say that a mapping $T\colon M \to N$ is an *isometry* if and only if

$$\|T(a) - T(b)\|_N = \|a - b\|_M \qquad (a, b \in M).$$

It should be emphasized that we never assume linearity of isometries unless otherwise stated. Let X be a compact Hausdorff space and $C(X)$ the Banach space of all continuous complex valued functions on X with the supremum norm $\|\cdot\|_\infty$. Denote by $C_{\mathbb{R}}(X)$ the real Banach space of all continuous real valued functions on X. Banach [1, Theorem 3 in Chapter XI] proved that if $T\colon C_{\mathbb{R}}(X) \to C_{\mathbb{R}}(Y)$ is a surjective isometry and if X and Y are compact metric spaces, then there exist a continuous function $u\colon Y \to \{\pm 1\}$ and a homeomorphism $\varphi\colon Y \to X$ such that $T(f)(y) = T(0)(y) + u(y)f(\varphi(y))$ for all $f \in C_{\mathbb{R}}(X)$ and $y \in Y$. Stone [18, Theorem 83] generalized the result by Banach for compact Hausdorff spaces X and Y. On the other hand, the so-called Banach-Stone theorem states that if $T\colon C(X) \to C(Y)$ is a surjective *complex linear* isometry, then there exist a continuous function $u\colon Y \to \mathbb{C}$ with $|u(y)| = 1$ for $y \in Y$ and a homeomorphism $\varphi\colon Y \to X$ such that $T(f)(y) = u(y)f(\varphi(y))$ for all $f \in C(X)$ and $y \in Y$.

Let $C^1[0,1]$ be the Banach space of all continuously differentiable complex valued functions on the unit interval $[0,1]$ with the norm $\|f\|_C = \sup\{|f(t)| + |f'(t)| : t \in [0,1]\}$ for $f \in C^1[0,1]$. Cambern [4, Theorem 1.5] gave a characterization for surjective complex linear isometries from $C^1[0,1]$ onto itself; to be more explicit, if $T\colon C^1[0,1] \to C^1[0,1]$ is a surjective *complex linear* isometry, then there exists $c \in \mathbb{C}$ with $|c| = 1$ such that $T(f)(t) = cf(t)$ for all $f \in C^1[0,1]$ and $t \in [0,1]$, or $T(f)(t) = cf(1-t)$ for all $f \in C^1[0,1]$ and $t \in [0,1]$. The result by Cambern has been extended in various directions; Pathak [16, Theorem 2.5] described surjective complex linear isometries between the Banach space of

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all n times continuously differentiable functions. Rao and Roy [17, Theorem 4.1] considered surjective complex linear isometries on $C^1[0,1]$ with the norm $\|f\|_\infty + \|f'\|_\infty$ for $f \in C^1[0,1]$. Jarosz and Pathak [9, Theorem 3] gave a scheme to verify that surjective complex linear isometries are given by homeomorphisms. Botelho and Jamison [2, Theorem 3.5] investigated surjective complex linear isometries between $C^1([0,1], E)$, where E denotes a finite dimensional Hilbert space. We refer the reader to [6, 7] for a survey of the study of isometries on various function spaces.

The purpose of this paper is to describe surjective isometries on $C^1[0,1]$ without assuming linearity of maps. In fact, the following is the main theorem of this paper, which extends the result by Rao and Roy [17, Theorem 4.1]:

2. MAIN RESULTS

Theorem 2.1. *Let $T\colon C^1[0,1] \to C^1[0,1]$ be a surjective isometry, which need not be linear, with respect to the norm $\|f\|_\Sigma = \|f\|_\infty + \|f'\|_\infty$. Then there exists a constant $c \in \mathbb{C}$ with $|c| = 1$ such that*

$$\begin{aligned}
&T(f)(t) = T(0)(t) + cf(t) && (\forall f \in C^1[0,1],\ \forall t \in [0,1]), \quad \textit{or}\\
&T(f)(t) = T(0)(t) + cf(1-t) && (\forall f \in C^1[0,1],\ \forall t \in [0,1]), \quad \textit{or}\\
&T(f)(t) = T(0)(t) + \overline{cf(t)} && (\forall f \in C^1[0,1],\ \forall t \in [0,1]), \quad \textit{or}\\
&T(f)(t) = T(0)(t) + \overline{cf(1-t)} && (\forall f \in C^1[0,1],\ \forall t \in [0,1]),
\end{aligned}$$

where $\overline{\cdot}$ denotes the complex conjugate.

Conversely, each of the above maps is a surjective isometry on $C^1[0,1]$ with respect to $\|\cdot\|_\Sigma$, where $T(0)$ is an arbitrary element of $C^1[0,1]$.

The following result is a special case of [2, Theorem 3.5] by Botelho and Jamison; in fact, they consider surjective linear isometries on $C^1([0,1], H)$ with respect to the norm $\sup\{\|f(t)\|_H + \|f'(t)\|_H : t \in [0,1]\}$, where H denotes a finite dimensional Hilbert space. We can identify $C^1[0,1]$ with $C^1([0,1], \mathbb{R}^2)$. If T_0 is a surjective real linear isometry on $C^1[0,1]$, then we may regard T_0 as a surjective linear isometry on $C^1([0,1], \mathbb{R}^2)$. Thus, T_0 is characterized by [2, Theorem 3.5]. On the other hand, we can prove the following result as a corollary to Theorem 2.1.

Corollary 2.2. *Let $T\colon C^1[0,1] \to C^1[0,1]$ be a surjective isometry, which need not be linear, with respect to the norm $\|f\|_C = \sup\{|f(t)| + |f'(t)| : t \in [0,1]\}$. Then there exists a constant $c \in \mathbb{C}$ with $|c| = 1$ such that*

$$\begin{aligned}
&T(f)(t) = T(0)(t) + cf(t) && (\forall f \in C^1[0,1],\ \forall t \in [0,1]), \quad \textit{or}\\
&T(f)(t) = T(0)(t) + cf(1-t) && (\forall f \in C^1[0,1],\ \forall t \in [0,1]), \quad \textit{or}\\
&T(f)(t) = T(0)(t) + \overline{cf(t)} && (\forall f \in C^1[0,1],\ \forall t \in [0,1]), \quad \textit{or}\\
&T(f)(t) = T(0)(t) + \overline{cf(1-t)} && (\forall f \in C^1[0,1],\ \forall t \in [0,1]).
\end{aligned}$$

Conversely, each of the above maps is a surjective isometry on $C^1[0,1]$ with respect to $\|\cdot\|_C$, where $T(0)$ is an arbitrary element of $C^1[0,1]$.

Theorem 2.3. *Let $T\colon C^1[0,1] \to C^1[0,1]$ be a surjective isometry, which need not be linear, with respect to the norm $\|f\|_\sigma = |f(0)| + \|f'\|_\infty$. Then there exist a constant $c \in \mathbb{C}$ with $|c| = 1$, a continuous unimodular function $\beta\colon [0,1] \to \mathbb{C}$ and a homeomorphism $\rho\colon [0,1] \to [0,1]$ such that*

$$T_0(f)(t) = cf(0) + \int_0^t \beta(s) f'(\rho(s))\, ds \qquad (\forall f \in C^1[0,1],\ \forall t \in [0,1]), \quad \textit{or}$$

$$T_0(f)(t) = c\overline{f(0)} + \int_0^t \beta(s) f'(\rho(s))\, ds \qquad (\forall f \in C^1[0,1],\ \forall t \in [0,1]), \quad \textit{or}$$

$$T_0(f)(t) = cf(0) + \int_0^t \beta(s) \overline{f'(\rho(s))}\, ds \qquad (\forall f \in C^1[0,1],\ \forall t \in [0,1]), \quad \textit{or}$$

$$T_0(f)(t) = c\overline{f(0)} + \int_0^t \beta(s) \overline{f'(\rho(s))}\, ds \qquad (\forall f \in C^1[0,1],\ \forall t \in [0,1]),$$

where $T_0(f)(t) = T(f)(t) - T(0)(t)$.

Conversely, each of the above maps is a surjective isometry on $C^1[0,1]$ with respect to $\|\cdot\|_\sigma$, where $T(0)$ is an arbitrary element of $C^1[0,1]$.

A key of proofs of the main results is a significant result related to isometries proven by Mazur and Ulam. The Mazur-Ulam theorem [13] states that if T is a surjective isometry between normed linear spaces, then $T - T(0)$ is real linear; consequently $T - T(0)$ is a surjective, *real linear* isometry. Väisälä [19] gave a simple proof of the Mazur-Ulam theorem. Theorem 2.1 states that surjective real linear isometry $T - T(0)$ on $C^1[0,1]$ is the same as complex linear one up to the complex conjugate; similar results were proven for function algebras [5, 8, 14] and for function spaces under additional assumptions [12]. On the other hand, real linear isometries are quite different from complex linear ones in general; such an elementary example is given in [12, Example 6.2]. A characterization is obtained in [15] in order that surjective real linear isometries on function spaces with respect to the supremum norm be of the canonical form, that is, a combination of weighted composition operators and the complex conjugate. Surjective, non-canonical isometries are investigated in [10].

Let $C^1[0,1]$ be the Banach space of all continuously differentiable complex valued functions on the unit interval $[0,1]$ with respect to the following norms:

$$\|f\|_\Sigma = \|f\|_\infty + \|f'\|_\infty, \qquad \|f\|_\sigma = |f(0)| + \|f'\|_\infty \qquad \text{and}$$

$$\|f\|_C = \sup\{|f(t)| + |f'(t)| : t \in [0,1]\}$$

for $f \in C^1[0,1]$, where $\|\cdot\|_\infty$ denotes the supremum norm on $[0,1]$. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle in the complex plane $\mathbb{C}$, and set $X_\Sigma = [0,1] \times [0,1] \times \mathbb{T}$,

$$X_\sigma = \{(r,s,z) \in X_\Sigma : r = 0\} \quad \text{and} \quad X_c = \{(r,s,z) \in X_\Sigma : r = s\}$$

with the product topology. Define

$$\tilde{f}(r,s,z) = f(r) + zf'(s) \tag{1}$$

for $f \in C^1[0,1]$ and $(r,s,z) \in X_\Sigma$; thus $\tilde{f}(r,s,z) = f(0) + zf'(s)$ if $(r,s,z) \in X_\sigma$, and $\tilde{f}(r,s,z) = f(s)+zf'(s)$ if $(r,s,z) \in X_c$. The function $\tilde{f}$ is continuous on X_Σ. Let $C(K)$ be the Banach space of all continuous complex valued functions on a compact Hausdorff space K with respect to the supremum norm $\|\cdot\|_\infty$. We define $A_\Sigma = \{\tilde{f} \in C(X_\Sigma) : f \in C^1[0,1]\}$, $A_\sigma = A_\Sigma|_{X_\sigma}$ and $A_C = A_\Sigma|_{X_c}$. Let $(A,X) \in \{(A_\Sigma, X_\Sigma), (A_\sigma, X_\sigma), (A_C, X_c)\}$. Then A is a normed linear subspace of $C(X)$. Let $\mathbf{1} \in C^1[0,1]$ be the constant function such that $\mathbf{1}(t) = 1$ for all $t \in [0,1]$. By (1), we see that A has constant function $\tilde{\mathbf{1}}$. Notice that A separates points of X in the sense that for each pair of distinct points $x_1, x_2 \in X$ there exists $\tilde{f} \in A$ such that $\tilde{f}(x_1) \neq \tilde{f}(x_2)$. The correspondence $f \mapsto \tilde{f}$ is a complex linear isometry from $(C^1[0,1], \|\cdot\|)$ onto $(A, \|\cdot\|_\infty)$; where, $\|f\| = \|f\|_\Sigma$ if $A = A_\Sigma$, $\|f\| = \|f\|_\sigma$ if $A = A_\sigma$ and $\|f\| = \|f\|_C$ if $A = A_C$. Note that $\widetilde{if} = i\tilde{f}$ for $f \in C^1[0,1]$. We denote by A^* the complex dual space of $(A, \|\cdot\|_\infty)$. Let $\delta_x \colon A \to \mathbb{C}$ be the point evaluation defined as $\delta_x(\tilde{f}) = \tilde{f}(x)$ for $\tilde{f} \in A$ and $x \in X$. We see that the set of all extreme points of the unit ball of A^* is $\{\lambda\delta_x : x \in X, \lambda \in \mathbb{T}\}$.

Let $T\colon C^1[0,1] \to C^1[0,1]$ be a surjective isometry. Define a mapping $T_0\colon C^1[0,1] \to C^1[0,1]$ as $T_0 = T - T(0)$. By the Mazur-Ulam theorem, T_0 is a surjective, *real linear* isometry from $C^1[0,1]$ onto itself. We define $S\colon A \to A$ as

$$S(\tilde{f}) = \widetilde{T_0(f)} \qquad (\tilde{f} \in A). \tag{2}$$

$$\begin{array}{ccc} C^1[0,1] & \xrightarrow{T_0} & C^1[0,1] \\ \Big\downarrow{\scriptstyle \sim} & & \Big\downarrow{\scriptstyle \sim} \\ A & \xrightarrow[S]{} & A \end{array}$$

Since $f \mapsto \tilde{f}$ is a surjective isometry from $C^1[0,1]$ onto A, it is a bijection, and thus S is well defined. As $f \mapsto \tilde{f}$ is a surjective complex linear isometry, S is a surjective real linear isometry on A. We define a mapping $S_*\colon A^* \to A^*$ as

$$S_*(\eta)(\tilde{f}) = \operatorname{Re}\eta(S(\tilde{f})) - i\operatorname{Re}\eta(S(i\tilde{f})) \tag{3}$$

for $\eta \in A^*$ and $\tilde{f} \in A$. It is routine to check that the mapping S_* is a surjective real linear isometry with respect to the operator norm on A^* (cf. [15, Proposition 1]).

Proof of Theorem 2.1, Corollary 2.2 and Theorem 2.3 are given in [11]. In fact, Kawamura, Koshimizu and the author of this paper generalize these results.

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