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Introduction to the Theory of Quasi-orthogonal Integrals

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Abstract

We give a brief introduction to the theory of continuous quasi-orthogonal decomposition, which is one of the major ingredients of the proof of the Bieberbach conjecture by L. de Branges. However, since it seems that the original text including this theory is no longer available, we refer mainly to Vasyunin-Nikol'skiĭ [5]. Furthermore, there is a slight difference between our article and [5]. Our approach to main results is based on the formulation of Ando [1].

1 Integral operators

Definition 1.1. *Let $\mathcal{H}$ be a separable Hilbert space. An $\mathcal{H}$-valued function f on the closed interval $[a,b]$ is said to be L^2-Bochner integrable if there exists a sequence of $\mathcal{H}$-valued step functions $\{f_n\}_{n\geq 1}$ satisfying the following (i) and (ii):*

(i) $\|f_n(s)-f(s)\|_{\mathcal{H}} \to 0\ (n\to\infty)$ *a.e.,*

(ii) $\displaystyle\int_a^b \|f(s)\|_{\mathcal{H}}^2\, ds < \infty.$

The set of all L^2-Bochner integrable functions is denoted by $L^2(\mathcal{H})$.

Let $\mathcal{G}$ be another separable Hilbert space. We consider families of operators $T_s : \mathcal{H} \to \mathcal{G}$ $(a \leq s \leq b)$ satisfying the following two conditions:

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(i) $M := \sup_{a \le s \le b} \|T_s\|$ is finite, that is, $\{T_s\}_{a \le s \le b}$ is uniformly bounded,

(ii) T_s^* is continuous in the strong sense with respect to the variable s.

Since $T_s^* y$ belongs to $L^2(\mathcal{H})$ for any y in $\mathcal{G}$ and

$$|\int_a^b \langle f(s), T_s^* y\rangle_{\mathcal{H}}\, ds| \le (b-a)^{1/2} M \|f\|_{L^2(\mathcal{H})} \|y\|_{\mathcal{G}} \quad (y \in \mathcal{G}),$$

the conjugate linear functional

$$\varphi : y \mapsto \int_a^b \langle f(s), T_s^* y\rangle_{\mathcal{H}}\, ds \quad (y \in \mathcal{G})$$

is bounded and

$$\|\varphi\| \le (b-a)^{1/2} M \|f\|_{L^2(\mathcal{H})}.$$

Hence, by the Riesz representation theorem, there exists an element z in $\mathcal{G}$ such that

$$\langle z, y\rangle_{\mathcal{G}} = \int_a^b \langle f(s), T_s^* y\rangle_{\mathcal{H}}\, ds$$

and

$$\|z\|_{\mathcal{G}} = \|\varphi\| \le (b-a)^{1/2} M \|f\|_{L^2(\mathcal{H})}. \tag{1.1}$$

This z will be denoted by

$$\int_a^b T_s f(s)\, ds.$$

Then we have the following identity:

$$\langle \int_a^b T_s f(s)\, ds, y\rangle_{\mathcal{G}} = \int_a^b \langle T_s f(s), y\rangle_{\mathcal{G}}\, ds \quad (y \in \mathcal{G}).$$

Further, we set

$$\mathbb{T} : L^2(\mathcal{H}) \to \mathcal{G}, \quad \mathbb{T} f = \int_a^b T_s f(s)\, ds.$$

Then the inequality (1.1) means that $\mathbb{T}$ is bounded.

2 Shmuly'an's theorem

Let S be a bounded linear operator from $\mathcal{H}$ to $\mathcal{G}$. Then we endow the pull-back norm $\|Sx\|_{\mathcal{M}(S)} = \|P_{(\ker S)^\perp} x\|_{\mathcal{H}}$ on the range of S. Then $\mathcal{M}(S) = (\operatorname{ran} S, \|\cdot\|_{\mathcal{M}(S)})$ is a Hilbert space, and which is called the de Branges-Rovnyak space induced by S.

Further, if S is contractive, then $\mathcal{H}(S) = \mathcal{M}(\sqrt{I_\mathcal{G} - SS^*})$ is called the de Branges-Rovnyak complement of $\mathcal{M}(S)$ (for details, see Ando [1], Sarason [3], Seto [4] or Vasyunin-Nikol'skiĭ [5]).

We need a theorem due to Shmuly'an (see Corollary 2 in Fillmore-Williams [2]) for our proof of the main theorem.

Theorem 2.1 (Shmuly'an). *Let S be a bounded linear operator from $\mathcal{H}$ to $\mathcal{G}$. Then, for any u in $\mathcal{G}$, u belongs to* $\operatorname{ran} S$ *if and only if*

$$\gamma := \sup_{S^*y \neq 0} \frac{|\langle y, u\rangle_\mathcal{G}|}{\|S^*y\|_\mathcal{H}} < \infty.$$

Further, then $\|u\|_{\mathcal{M}(S)} = \gamma$.

Proof. This proof is taken from Ando [1]. First, we suppose that $u = Sx$. Then we have that

$$|\langle y, u\rangle_\mathcal{G}| = |\langle y, Sx\rangle_\mathcal{G}| = |\langle S^*y, x\rangle_\mathcal{H}| \leq \|S^*y\|_\mathcal{H}\|x\|_\mathcal{H}$$

for any y in $\mathcal{G}$. Hence we have that $\gamma \leq \|x\|_\mathcal{H}$. Therefore γ is finite. Next, conversely, we suppose that γ is finite. Then, setting

$$\varphi : S^*y \mapsto \langle y, u\rangle_\mathcal{G} \quad (S^*y \neq 0),$$

φ is well defined as a linear functional on $\operatorname{ran} S^*$. Indeed, if $S^*y = 0$, then, for any $z \in (\ker S^*)^\perp$ and any $\varepsilon > 0$, we have that

$$0 \leq |\langle y + \varepsilon z, u\rangle_\mathcal{G}| \leq \gamma\|S^*(y + \varepsilon z)\|_\mathcal{H} = \gamma\varepsilon\|S^*z\|_\mathcal{H}.$$

It follows from this that $\langle y, u\rangle_\mathcal{G} = 0$. Further, by the assumption that γ is finite, φ can be extended as a bounded linear functional on $\overline{\operatorname{ran}} S^*$. Then, by the Riesz representation theorem, there exists a vector x in $\overline{\operatorname{ran}} S^*$ such that

$$\langle y, u\rangle_\mathcal{G} = \varphi(S^*y) = \langle S^*y, x\rangle_\mathcal{H}$$

for any y in $\mathcal{G}$ and $\gamma = \|x\|_\mathcal{H}$. Therefore we have that $u = Sx$. Lastly, norm identity $\|u\|_{\mathcal{M}(S)} = \gamma$ follows from the property of the above x. Indeed, $\gamma = \|x\|_\mathcal{H} = \|P_{(\ker S)^\perp}x\|_\mathcal{H} = \|u\|_{\mathcal{M}(S)}$. □

3 Integral representation

Let's recall that $\mathbb{T}f$ is defined by the identity

$$\langle \int_a^b T_s f(s)\ ds, y\rangle_{\mathcal{G}} = \int_a^b \langle f(s), T_s^* y\rangle_{\mathcal{H}}\ ds \quad (y \in \mathcal{G}).$$

Since T_s^* is continuous in the strong sense, $\langle f(s), T_s^* y\rangle_{\mathcal{H}}$ is measurable for any y in $\mathcal{G}$, that is, so is $T_s f(s)$ in the weak sense. Further, it is well known that the weak measurability implies the strong measurability in separable cases. Hence $T_s f(s)$ is measurable in the strong sense. Moreover, we note that

$$\int_a^b T_s T_s^*\ ds$$

converges to a non-negative self-adjoint operator acting on $\mathcal{G}$, because T_s^* is continuous in the strong sense with respect to the variable s.

The next theorem is fundamental in this article.

Theorem 3.1.

$$\mathcal{M}(\mathbb{T}) = \mathcal{M}((\int_a^b T_s T_s^*\ ds)^{1/2}). \tag{3.1}$$

Particularly, for any u in $\mathcal{M}((\int_a^b T_s T_s^*\ ds)^{1/2})$, there exists some f in $L^2(\mathcal{H})$ such that

$$u = \int_a^b T_s f(s)\ ds$$

and

$$\| \int_a^b T_s f(s)\ ds\|^2_{\mathcal{M}((\int_a^b T_s T_s^*\ ds)^{1/2})} \leq \int_a^b \|T_s f(s)\|^2_{\mathcal{M}(T_s)}\ ds.$$

In this sense, we may write

$$\mathcal{M}((\int_a^b T_s T_s^*\ ds)^{1/2}) = \int_a^b \mathcal{M}(T_s)\ ds.$$

Proof. This proof is based on that of Theorem 3.6 in Ando [1], where sums of two de Branges-Rovnyak spaces are discussed[1]. We set

$$\mathbb{S} = \left(\int_a^b T_s T_s^*\ ds\right)^{1/2}.$$

We divide our proof into three steps.

[1]See also lines 7–8 in p. 260 of Fillmore-Williams [2].

(Step 1) We shall show that $\mathcal{M}(\mathbb{S}) \hookrightarrow \mathcal{M}(\mathbb{T})$ (which means that $\mathcal{M}(\mathbb{S})$ is embedded into $\mathcal{M}(\mathbb{T})$ contractively). If $u = \mathbb{S}x$, then, by Theorem 2.1 and

$$\|\mathbb{S}^* y\|_{\mathcal{G}}^2 = \|\mathbb{S}y\|_{\mathcal{G}}^2 = \int_a^b \|T_s^* y\|_{\mathcal{H}}^2 \, ds \quad (y \in \mathcal{G}),$$

we have that

$$\|u\|_{\mathcal{M}(\mathbb{S})}^2 = \sup_{\mathbb{S}^* y \neq 0} \frac{|\langle y, u \rangle_{\mathcal{G}}|^2}{\|\mathbb{S}^* y\|_{\mathcal{G}}^2} = \sup_{\mathbb{S}^* y \neq 0} \frac{|\langle y, u \rangle_{\mathcal{G}}|^2}{\int_a^b \|T_s^* y\|_{\mathcal{H}}^2 \, ds}$$

is finite. Hence

$$\varphi : f \mapsto \langle y, u \rangle_{\mathcal{G}} \quad (f(s) = T_s^* y)$$

defines a bounded linear functional on the closure of

$$\{f \in L^2(\mathcal{H}) : \exists y \in \mathcal{G} \text{ s.t. } f(s) = T_s^* y\}.$$

Then, it follows from the Riesz representation theorem that there exists a function g in $L^2(\mathcal{H})$ such that

$$\langle y, u \rangle_{\mathcal{G}} = \langle f, g \rangle_{L^2(\mathcal{H})} \quad \text{and} \quad \|u\|_{\mathcal{M}(\mathbb{S})}^2 = \|g\|_{L^2(\mathcal{H})}^2.$$

Further, we have that

$$\begin{aligned} \langle y, u \rangle_{\mathcal{G}} &= \langle f, g \rangle_{L^2(\mathcal{H})} \\ &= \int_a^b \langle T_s^* y, g(s) \rangle_{\mathcal{H}} \, ds \\ &= \int_a^b \langle y, T_s g(s) \rangle_{\mathcal{G}} \, ds \\ &= \langle y, \int_a^b T_s g(s) \, ds \rangle_{\mathcal{G}}. \end{aligned}$$

Hence, we have that

$$u = \int_a^b T_s g(s) \, ds = \mathbb{T} g.$$

Therefore u belongs to $\mathcal{M}(\mathbb{T})$ and

$$\|u\|_{\mathcal{M}(\mathbb{T})}^2 \leq \|g\|_{L^2(\mathcal{H})}^2 = \|u\|_{\mathcal{M}(\mathbb{S})}^2.$$

Thus we concludes that $\mathcal{M}(\mathbb{S}) \hookrightarrow \mathcal{M}(\mathbb{T})$.

(Step 2) We shall show that $\mathcal{M}(\mathbb{T}) \hookrightarrow \mathcal{M}(\mathbb{S})$. Let u be a vector in $\mathcal{M}(\mathbb{T})$. Then we may assume that $u = \mathbb{T}f$ where f is taken from $(\ker \mathbb{T})^{\perp}$. It follows from this assumption that $\|u\|^2_{\mathcal{M}(\mathbb{T})} = \|f\|^2_{L^2(\mathcal{H})}$. Further, for any y in $\mathcal{G}$, we have that

$$\begin{aligned}\langle y, u\rangle_{\mathcal{G}} &= \langle y, \int_a^b T_s f(s)\, ds\rangle_{\mathcal{G}} \\ &= \int_a^b \langle y, T_s f(s)\rangle_{\mathcal{G}}\, ds \\ &= \int_a^b \langle T_s^* y, f(s)\rangle_{\mathcal{H}}\, ds.\end{aligned}$$

Hence we have that

$$\begin{aligned}|\langle y, u\rangle|^2_{\mathcal{G}} &\le \left(\int_a^b \|T_s^* y\|^2_{\mathcal{H}}\, ds\right)\left(\int_a^b \|f(s)\|^2_{\mathcal{H}}\, ds\right) \\ &= \|g\|^2_{L^2(\mathcal{H})}\|u\|^2_{\mathcal{M}(\mathbb{T})},\end{aligned}$$

where we set $g(s) = T_s^* b$. This inequality means that the linear functional

$$\varphi : g \mapsto \langle y, u\rangle_{\mathcal{G}} \quad (g(s) = T_s^* y)$$

is bounded and $\|\varphi\| \le \|u\|_{\mathcal{M}(\mathbb{T})}$. Further we have that

$$\sup_{\mathbb{S}^* y \neq 0} \frac{|\langle y, u\rangle_{\mathcal{G}}|^2}{\|\mathbb{S}^* y\|^2_{\mathcal{G}}} = \sup_{\mathbb{S}^* y \neq 0} \frac{|\langle y, u\rangle_{\mathcal{G}}|^2}{\int_a^b \|T_s^* y\|^2_{\mathcal{H}}\, ds} = \|\varphi\|^2 \le \|u\|^2_{\mathcal{M}(\mathbb{T})}.$$

Therefore, by Theorem 2.1, u belongs to $\operatorname{ran} \mathbb{S}$ and $\|u\|_{\mathcal{M}(\mathbb{S})} \le \|u\|_{\mathcal{M}(\mathbb{T})}$.

(Step 3) Step 1 and Step 2 conclude the identity (3.1). Finally, we shall show norm inequalities. Setting

$$\begin{aligned}&\mathbb{T}_1 : L^2(\mathcal{H}) \to L^2(\mathcal{G}), \quad f \mapsto T_\cdot f(\cdot) \\ &\mathbb{T}_2 : L^2(\mathcal{G}) \to \mathcal{G}, \quad g \mapsto \int_a^b g(s)\, ds,\end{aligned}$$

we have that $\mathbb{T} = \mathbb{T}_2\mathbb{T}_1$, and it is easy to see that $\mathbb{T}_1$ and $\mathbb{T}_2$ are bounded. We also note that

$$(\ker \mathbb{T}_1)^{\perp} = \{f \in L^2(\mathcal{H}) : f(s) \in (\ker T_s)^{\perp}\}.$$

Then we have that

$$
\begin{aligned}
\| \int_a^b T_s f(s)\, ds \|^2_{\mathcal{M}((\int_a^b T_s T_s^* \, ds)^{1/2})} &= \|\mathbb{T} f\|^2_{\mathcal{M}(\mathbb{T})} \\
&= \|\mathbb{T}_2 \mathbb{T}_1 f\|^2_{\mathcal{M}(\mathbb{T}_2\mathbb{T}_1)} \\
&= \|P_{(\ker \mathbb{T}_2\mathbb{T}_1)^\perp} f\|^2_{L^2(\mathcal{H})} \\
&\le \|P_{(\ker \mathbb{T}_1)^\perp} f\|^2_{L^2(\mathcal{H})} \\
&= \int_a^b \|(P_{(\ker \mathbb{T}_1)^\perp} f)(s)\|^2_{\mathcal{H}} \, ds \\
&= \int_a^b \|P_{(\ker T_s)^\perp} f(s)\|^2_{\mathcal{H}} \, ds \\
&= \int_a^b \|T_s f(s)\|^2_{\mathcal{M}(T_s)} \, ds.
\end{aligned}
$$

This concludes the proof. □

4 Evolution families

Let $\{T_s\}_{a \le s \le b}$ be a family of contractive linear operators acting on a Hilbert space $\mathcal{H}$. We suppose that there exists a two-parameter family of contractions $\{T_{rs}\}_{a \le r \le s \le b}$ satisfying the following (i), (ii), (iii) and (iv):

(i) $T_s = T_r T_{rs}$ $(r \le s)$,

(ii) $T_{rt} = T_{rs} T_{st}$ $(r \le s \le t)$,

(iii) $T_{ss} = I_{\mathcal{H}}$,

(iv) T_{rs} is sufficiently smooth to do calculus.

Then $\{T_{rs}\}_{a \le r \le s \le b}$ is called an evolution family.

Lemma 4.1. *We set*

$$
\Omega(s) = \frac{\partial T_{rs}}{\partial r}\Big|_{r=s} = \lim_{r \to s} \frac{T_{rs} - I_{\mathcal{H}}}{r - s}.
$$

Then the following formulas hold.

(i) $\dfrac{\partial T_{rs}}{\partial r} = \Omega(r) T_{rs}$,

(ii) $\dfrac{\partial T_{rs}}{\partial s} = -T_{rs} \Omega(s)$,

(iii) $\dfrac{\partial}{\partial s}(-T_{rs}T_{rs}^*) = T_{rs}(2\operatorname{Re}\Omega(s))T_{rs}^*.$

Proof.

$$\frac{\partial T_{rs}}{\partial r} = \frac{\partial T_{ts}}{\partial t}|_{t=r} = (\frac{\partial T_{tr}}{\partial t}|_{t=r})T_{rs} = \Omega(r)T_{rs}.$$

Thus we have (i). Next,

$$(\frac{\partial T_{rt}}{\partial s})|_{s=t} = (\frac{\partial}{\partial s}T_{rs}T_{st})|_{s=t} = (\frac{\partial T_{rs}}{\partial s}T_{st} + T_{rs}\frac{\partial T_{st}}{\partial s})|_{s=t}$$

and $\partial T_{rt}/\partial s = 0$ imply (ii). It is easy to see that (iii) follows from (ii). □

By (iii) of Lemma 4.1, we have that

$$2\operatorname{Re}\Omega(s) = \{\frac{\partial}{\partial s}(-T_{rs}T_{rs}^*)\}|_{r=s}.$$

Further, since T_{ts} is contractive, for $r \le t \le s$, we have that

$$\|T_{rs}^*x\|_{\mathcal{H}}^2 = \|T_{ts}^*T_{rt}^*x\|_{\mathcal{H}}^2 \le \|T_{rt}^*x\|_{\mathcal{H}}^2.$$

Hence $\langle T_{rs}T_{rs}^*x, x\rangle_{\mathcal{H}}$ is a decreasing function with respect to the variable s. Therefore we have that

$$\operatorname{Re}\Omega(s) \ge O.$$

Now, we shall consider operator $\Delta(s)$ defined by

$$\Delta(s)\Delta(s)^* = 2\operatorname{Re}\Omega(s),$$

where $\Delta(s)$ is assumed to be continuous and uniformly bounded.

Theorem 4.1. *For $r \le t$, $\mathcal{H}(T_{rt})$ has the following integral representation:*

$$\mathcal{H}(T_{rt}) = \int_r^t \mathcal{M}(T_{rs}\Delta(s))\,ds$$

in the sense of Theorem 3.1. Particularly, for any $f \in L^2(\mathcal{H})$,

$$\|\int_r^t T_{rs}\Delta(s)f(s)\,ds\|_{\mathcal{H}(T_{rt})}^2 \le \int_r^t \|\Delta(s)f(s)\|_{\mathcal{M}(\Delta(s))}^2\,ds \le \int_r^t \|f(s)\|_{\mathcal{H}}^2\,ds.$$

Proof. By the definition of $\Delta(s)$ and (iii) of Lemma 4.1, we have that

$$\begin{aligned}\int_r^t T_{rs}\Delta(s)(T_{rs}\Delta(s))^*\,ds &= \int_r^t T_{rs}(2\operatorname{Re}\Omega(s))T_{rs}^*\,ds \\ &= \int_r^t \frac{\partial}{\partial s}(-T_{rs}T_{rs}^*)\,ds \\ &= I_{\mathcal{H}} - T_{rt}T_{rt}^*.\end{aligned}$$

By Theorem 3.1, this implies the integral representation of $\mathcal{H}(T_{rs})$. Further,

$$\begin{aligned}\|\int_r^t T_{rs}\Delta(s)f(s)\,ds\|^2_{\mathcal{H}(T_{rt})} &\leq \int_r^s \|T_{rs}\Delta(s)f(s)\|^2_{\mathcal{M}(T_{rs}\Delta(s))}\,ds \\ &= \int_r^t \|P_{(\ker T_{rs}\Delta(s))^\perp}f(s)\|^2_{\mathcal{H}}\,ds \\ &\leq \int_r^t \|P_{(\ker \Delta(s))^\perp}f(s)\|^2_{\mathcal{H}}\,ds \\ &= \int_r^t \|\Delta(s)f(s)\|^2_{\mathcal{M}(\Delta(s))}\,ds.\end{aligned}$$

Thus we obtain norm inequalities. □

5 Weighted quasi-orthogonal integrals

Let $\sigma(s)$ be a positive-operator valued differentiable function on $[a, b]$. Further, we assume that $\sigma(s)$ is invertible and $\sigma(s)^{-1}$ is uniformly bounded on $\mathcal{H}^2$. We shall consider the one-parameter family of Hilbert spaces $\{\mathcal{H}_s : \mathcal{H}_s = \mathcal{M}(\sigma^{-1/2}(s))\ (a \leq s \leq b)\}$ associated with σ.

Lemma 5.1. *Let $\{T_{rs}\}_{a\leq r\leq s\leq b}$ be an evolution family acting on a Hilbert space $\mathcal{H}$. Then T_{rs} is contractive as an operator from $\mathcal{H}_s$ to $\mathcal{H}_r$ if and only if*

$$\Lambda(s) = \sigma'(s) + 2\,\mathrm{Re}(\sigma(s)\Omega(s)) \geq O. \tag{5.1}$$

Proof. Since

$$\langle x, y\rangle_{\mathcal{H}_s} = \langle \sigma(s)^{1/2}x, \sigma^{1/2}(s)y\rangle_{\mathcal{H}} = \langle \sigma(s)x, y\rangle_{\mathcal{H}},$$

it is easy to see that T_{rs} is contractive as an operator from $\mathcal{H}_s$ to $\mathcal{H}_r$ if and only if

$$T^*_{rs}\sigma(r)T_{rs} \leq \sigma(s). \tag{5.2}$$

We shall show that (5.1) and (5.2) are mutually equivalent. First, we suppose (5.1). Then we have that

$$\frac{\partial}{\partial s}(T^*_{st}\sigma(s)T_{st}) = T^*_{st}\Lambda(s)T_{st} \geq O$$

by Lemma 4.1. Hence

$$\sigma(t) - T^*_{rt}\sigma(r)T_{rt} = \int_r^t \frac{\partial}{\partial s}(T^*_{st}\sigma(s)T_{st})\,ds \geq O.$$

[2] If not, then we will deal with $\sigma(s) + \varepsilon I_{\mathcal{H}}$ $(\varepsilon > 0)$.

Thus we obtain (5.2). Next, conversely, we suppose (5.2). Then, we have that

$$\begin{aligned}
&\frac{\langle(\sigma(s) - T_{rs}^*\sigma(r)T_{rs})x, x\rangle_{\mathcal{H}}}{s-r} \\
&= \frac{\langle(\sigma(s) - \sigma(r))x, x\rangle_{\mathcal{H}} - \langle\sigma(r)(T_{rs} - I)x, x\rangle_{\mathcal{H}} - \langle(T_{rs}^* - I)\sigma(r)T_{rs}x, x\rangle_{\mathcal{H}}}{s-r} \\
&\to \langle(\sigma'(r) + 2\operatorname{Re}\sigma(r)\Omega(r))x, x\rangle_{\mathcal{H}}
\end{aligned}$$

as s tends to r by Lemma 4.1. Thus we obtain (5.1). □

Let $\{T_{rs} \in \mathcal{L}(\mathcal{H}_s, \mathcal{H}_r) : a \le r \le s \le b\}$ be a contractive evolution family in the sense of Lemma 5.1. We set

$$\sigma(s) = \tau(s)^*\tau(s),$$

where we assume that $\tau(s)$ is differentiable. Then we define $\widetilde{T}_{rs}$ as follows:

$$\begin{array}{ccc}
\mathcal{H}_s & \xrightarrow{T_{rs}} & \mathcal{H}_r \\
{\scriptstyle \tau(s)}\downarrow & & \downarrow{\scriptstyle \tau(r)} \\
\mathcal{H} & \xrightarrow[\widetilde{T}_{rs}]{} & \mathcal{H},
\end{array} \tag{5.3}$$

that is, we set

$$\widetilde{T}_{rs} = \tau(r)T_{rs}\tau(s)^{-1}.$$

Then, trivially, $\{\widetilde{T}_{rs}\}_{0\le r\le s\le 1}$ has the evolution property:

$$\widetilde{T}_{rs}\widetilde{T}_{st} = \widetilde{T}_{rt} \quad (a \le r \le s \le t \le b).$$

Further, $\tau(s) : \mathcal{H}_s \to \mathcal{H}$ is a unitary operator. Indeed, let u be any vector in $\mathcal{H}$. Then we have that

$$\begin{aligned}
\|\tau(s)\sigma^{-1/2}(s)u\|_{\mathcal{H}}^2 &= \langle\sigma(s)^{-1/2}\tau(s)^*\tau(s)\sigma(s)^{-1/2}u, u\rangle_{\mathcal{H}} \\
&= \|u\|_{\mathcal{H}}^2 \\
&= \|\sigma(s)^{-1/2}u\|_{\mathcal{M}(\sigma(s)^{-1/2})}^2 \\
&= \|\sigma(s)^{-1/2}u\|_{\mathcal{H}_s}^2.
\end{aligned}$$

The adjoint operator of $\tau(s) : \mathcal{H}_s \to \mathcal{H}$ will be denoted by $\tau(s)^\sharp$ (needless to say, $\tau(s)^*$ denotes the adjoint operator of $\tau(s) : \mathcal{H} \to \mathcal{H}$). The adjoint operator of $T_{rs} : \mathcal{H}_s \to \mathcal{H}_r$ is also denoted by $T_{rs}^\sharp$. Then it is trivial that

$$\widetilde{T}_{rs} = \tau(r)T_{rs}\tau(s)^\sharp \quad \text{and} \quad (\widetilde{T}_{rs})^* = \tau(s)T_{rs}^\sharp\tau(r)^\sharp.$$

Definition 5.1. *$\mathcal{H}^{\sigma(r)}_{\sigma(s)}(T_{rs})$ denotes the de Branges-Rovnyak complement induced by $T_{rs} : \mathcal{H}_s \to \mathcal{H}_r$, that is, we set*

$$\mathcal{H}^{\sigma(r)}_{\sigma(s)}(T_{rs}) = \mathcal{M}\left(\sqrt{I_{\mathcal{H}_r} - T_{rs}T^{\sharp}_{rs}}\right).$$

Let $\widetilde{\Omega}$ and $\widetilde{\Delta}$ be operators corresponding to the evolution family $\{\widetilde{T}_{rs}\}_{a\le r\le s\le b}$.

Theorem 5.1. *Let $\{T_{rs} \in \mathcal{L}(\mathcal{H}_s, \mathcal{H}_r) : a \le r \le s \le b\}$ be a contractive evolution family in the sense of Lemma 5.1. We set $\Gamma = \tau^{-1}\widetilde{\Delta}$. Then $\mathcal{H}^{\sigma(t)}_{\sigma(r)}(T_{rt})$ $(a \le r \le t \le b)$ has the following integral representation:*

$$\mathcal{H}^{\sigma(t)}_{\sigma(r)}(T_{rt}) = \int_r^t \mathcal{M}(T_{rs}\Gamma(s))\, ds$$

in the sense of Theorem 3.1. Particularly, for any $f \in L^2(\mathcal{H})$,

$$\left\| \int_r^t T_{rt}\Gamma(s)f(s)\, ds\right\|^2_{\mathcal{H}^{\sigma(t)}_{\sigma(r)}(T_{rt})} \le \int_r^t \|\Gamma(s)f(s)\|^2_{\mathcal{M}(\Gamma(s))}\, ds \le \int_r^t \|f(s)\|^2_{\mathcal{H}}\, ds.$$

Proof. Since

$$\begin{aligned}
\int_r^t T_{rs}\Gamma(s)\Gamma(s)^{\sharp}T^{\sharp}_{rs}\, ds &= \int_r^t T_{rs}\tau(s)^{\sharp}\widetilde{\Delta}(s)\widetilde{\Delta}(s)^*\tau(s)T^{\sharp}_{rs}\, ds \\
&= \int_r^t \tau(r)^{\sharp}\widetilde{T}_{rs}\widetilde{\Delta}(s)\widetilde{\Delta}(s)^*\widetilde{T}^*_{rs}\tau(r)\, ds \\
&= \tau(r)^{\sharp}\Big(\int_r^t \widetilde{T}_{rs}(2\operatorname{Re}\widetilde{\Omega}(s))\widetilde{T}^*_{rs}\, ds\Big)\tau(r) \\
&= \tau(r)^{\sharp}\Big(\int_r^t \frac{\partial}{\partial s}(-\widetilde{T}_{rs}\widetilde{T}^*_{rs})\, ds\Big)\tau(r) \\
&= \tau(r)^{\sharp}(I_{\mathcal{H}} - \widetilde{T}_{rt}\widetilde{T}^*_{rt})\tau(r) \\
&= I_{\mathcal{H}_r} - T_{rt}T^{\sharp}_{rt}
\end{aligned}$$

by (iii) of Lemma 4.1, applying Theorem 3.1 to $T_{rs}\Gamma(s)$, we obtain the integral representation of $\mathcal{H}^{\sigma(t)}_{\sigma(r)}(T_{rt})$. Further,

$$\begin{aligned}
\left\| \int_r^t T_{rs}\Gamma(s)f(s)\, ds\right\|^2_{\mathcal{H}^{\sigma(t)}_{\sigma(r)}(T_{rt})} &\le \int_r^t \|T_{rs}\Gamma(s)f(s)\|^2_{\mathcal{M}(T_{rs}\Gamma(s))}\, ds \\
&= \int_r^t \|P_{(\ker T_{rs}\Gamma(s))^\perp} f(s)\|^2_{\mathcal{H}}\, ds \\
&\le \int_r^t \|P_{(\ker \Gamma(s))^\perp} f(s)\|^2_{\mathcal{H}}\, ds \\
&= \int_r^t \|\Gamma(s)f(s)\|^2_{\mathcal{M}(\Gamma(s))}\, ds.
\end{aligned}$$

Thus we obtain norm inequalities. ∎

6 Isometric representation

We shall consider the following differential equation:

$$x'(r) = \Omega(r)x(r) - g(r), \tag{6.1}$$

where $g(r) = \Gamma(r)f(r)$ for some fixed f in $L^2(\mathcal{H})$. Then the solution of (6.1) is written as follows:

$$x(r) = T_{rb}x(b) + \int_r^b T_{rs}g(s)\, ds. \tag{6.2}$$

Indeed, since

$$\begin{aligned}
&\frac{1}{h}\left(\int_{r+h}^b T_{r+h,s}g(s)\, ds - \int_r^b T_{rs}g(s)\, ds\right) \\
&= \int_r^b \frac{T_{r+h,s} - T_{rs}}{h} g(s)\, ds - \frac{1}{h}\int_r^{r+h} T_{r+h,s}g(s)\, ds \\
&\to \int_r^b \Omega(r)T_{rs}g(s)\, ds - g(r) \quad (h \to 0) \\
&= \Omega(r)(x(r) - T_{rb}x(b)) - g(r).
\end{aligned}$$

Hence we have that

$$\begin{aligned}
x'(r) &= \Omega(r)T_{rb}x(b) + \Omega(r)(x(r) - T_{rb}x(b)) - g(r) \\
&= \Omega(r)x(r) - g(r).
\end{aligned}$$

Here, we should note that (6.2) gives a de Branges-Rovnyak decomposition of $x(r)$ with respect to $T_{rb} : \mathcal{H}_b \to \mathcal{H}_r$. Hence we have that

$$\begin{aligned}
\|x(r)\|^2_{\mathcal{H}_r} &\le \|T_{rb}x(b)\|^2_{\mathcal{M}(T_{rb})} + \left\|\int_r^b T_{rs}g(s)\, ds\right\|^2_{\mathcal{H}^{\sigma(b)}_{\sigma(r)}(T_{rb})} \\
&\le \|x(b)\|^2_{\mathcal{H}_b} + \int_r^b \|g(s)\|^2_{\mathcal{M}(\Gamma(s))}\, ds
\end{aligned}$$

by Theorem 5.1. Therefore we have the inequality

$$\|x(r)\|^2_{\mathcal{H}_r} - \|x(b)\|^2_{\mathcal{H}_b} \le \int_r^b \|g(s)\|^2_{\mathcal{M}(\Gamma(s))}\, ds. \tag{6.3}$$

Lemma 6.1. $\Lambda = \sigma\Gamma\Gamma^*\sigma$ *as operators acting on* $\mathcal{H}$.

Proof. Differentiating the operator identity $\tau(r)T_{rs} = \widetilde{T}_{rs}\tau(s)$ on $\mathcal{H}$ with respect to r, we have that

$$\tau'(r)T_{rs} + \tau(r)\Omega(r)T_{rs} = \widetilde{\Omega}(r)\widetilde{T}_{rs}\tau(s).$$

Further, putting $r = s$, we have that

$$\tau'(s) + \tau(s)\Omega(s) = \widetilde{\Omega}(s)\tau(s).$$

It follows from the above identity that

$$\begin{aligned}\tau^*\widetilde{\Delta}\widetilde{\Delta}^*\tau &= \tau^*\widetilde{\Omega}\tau + \tau^*\widetilde{\Omega}^*\tau \\ &= \tau^*(\tau' + \tau\Omega) + (\tau' + \tau\Omega)^*\tau \\ &= \sigma' + \sigma\Omega + \Omega^*\sigma \\ &= \Lambda.\end{aligned}$$

Further, since $\Gamma = \tau^{-1}\widetilde{\Delta}$, we have that $\sigma\Gamma = \tau^*\widetilde{\Delta}$. This concludes the proof. □

Theorem 6.1. *Let $x(r)$ be the solution of (6.1). Then (6.3) is an equality if and only if one of the following (i) and (ii) is satisfied:*

(i) $(\sigma(s)x(s))' + \Omega(s)^*\sigma(s)x(s) = 0,$

(ii) $\sigma(s)x(s) = T^*_{rs}\sigma(r)x(r).$

Proof. Let $x(r)$ be the solution of (6.1). We set

$$F(r) = \|x(r)\|^2_{\mathcal{H}_r} - \|x(b)\|^2_{\mathcal{H}_b} - \int_r^b \|g(s)\|^2_{\mathcal{M}(\Gamma(s))}\, ds,$$

and choose h from $L^2(\mathcal{H})$ satisfying $\Gamma(s)h(s) = g(s)$ and $\|h(s)\|_{\mathcal{H}} = \|g(s)\|_{\mathcal{M}(\Gamma(s))}$[3]. Then, by (6.1) and Lemma 6.1, we have that

$$\begin{aligned}F'(r) &= \langle\sigma'(r)x(r), x(r)\rangle_{\mathcal{H}} + 2\,\mathrm{Re}\langle\sigma(r)x'(r), x(r)\rangle_{\mathcal{H}} + \|g(r)\|^2_{\mathcal{M}(\Gamma(r))} \\ &= \langle\sigma'(r)x(r), x(r)\rangle_{\mathcal{H}} + 2\,\mathrm{Re}\langle\sigma(r)\{\Omega(r)x(r) - g(r)\}, x(r)\rangle_{\mathcal{H}} + \|g(r)\|^2_{\mathcal{M}(\Gamma(r))} \\ &= \langle\Lambda(r)x(r), x(r)\rangle_{\mathcal{H}} - 2\,\mathrm{Re}\langle\sigma(r)\Gamma(r)h(r), x(r)\rangle_{\mathcal{H}} + \|h(r)\|^2_{\mathcal{H}} \\ &= \|\Gamma(r)^*\sigma(r)x(r) - h(r)\|^2_{\mathcal{H}}.\end{aligned}$$

Hence $F(r) \equiv 0$ if and only if $h = \Gamma^*\sigma x$, because $F(b) = 0$.

Suppose that $h = \Gamma^*\sigma x$. Then, by Lemma 6.1, we have that $\sigma g = \Lambda x$, and which implies (i) by (5.1) and (6.1). Conversely, we suppose (i). Then it follows from (5.1) and (6.1) that $\Lambda x = \sigma x$. Hence we have that $\sigma\Gamma(\Gamma^*\sigma x - h) = 0$ by Lemma 6.1, that is, $\Gamma^*\sigma x - h$ belongs to $\ker\Gamma$. Since h and $\Gamma^*\sigma x$ belong to $(\ker\Gamma)^\perp$, we have that $h = \Gamma^*\sigma x$.

Finally, it is easy to see that (i) and (ii) are mutually equivalent by Lemma 4.1. □

[3]Since $\mathbf{\Gamma} : f \mapsto \Gamma(\cdot)f(\cdot)$ is a bounded linear operator acting on $L^2(\mathcal{H})$, we may take h from $(\ker\mathbf{\Gamma})^\perp$.

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