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Pointwise multipliers on Musielak-Orlicz and Musielak-Orlicz-Morrey spaces

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1 Introduction

This report is an announcement of [17] and [18].

Let (Ω, μ) be a complete σ-finite measure space. We denote by $L^{0}(\Omega)$ the set of all measurable functions from Ω to $\mathbb{R}$ or $\mathbb{C}$. Let E_{1} and E_{2} be subspaces of $L^{0}(\Omega)$. We say that a function $g\in L^{0}(\Omega)$ is a pointwise multiplier from E_{1} to E_{2}, if the pointwise multiplication fg is in E_{2} for any $f\in E_{1}$. We denote by $\mathrm{PWM}(E_{1}, E_{2})$ the set of all pointwise multipliers from E_{1} to E_{2}. We abbreviate $\mathrm{PWM}(E, E)$ to $\mathrm{PWM}(E)$.

For $p\in(0, \infty]$, we denote by $L^{p}(\Omega)$ the usual Lebesgue spaces. It is well known as Hölder's inequality that

$$\|fg\|_{L^{p_{2}}(\Omega)}\leq\|f\|_{L^{p_{1}}(\Omega)}\|g\|_{L^{p_{3}}(\Omega)},$$

for $1/p_{2}=1/p_{1}+1/p_{3}$ with $p_{i}\in(0, \infty]$, $i=1,2,3$. This shows that

$$\mathrm{PWM}(L^{p_{1}}(\Omega), L^{p_{2}}(\Omega))\supset L^{p_{3}}(\Omega).$$

Conversely, we can show the reverse inclusion by using the uniform boundedness theorem or the closed graph theorem. That is,

$$\mathrm{PWM}(L^{p_{1}}(\Omega), L^{p_{2}}(\Omega))=L^{p_{3}}(\Omega). \tag{1.1}$$

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This equality was extended to Orlicz spaces by [7, 8]. In this report we extend the above equality to Musielak-Orlicz spaces and Musielak-Orlicz-Morrey spaces.

Recall that, for a normed or quasi-normed space $E \subset L^0(\Omega)$, we say that E has the lattice (ideal) property if the following holds:

$$f \in E,\ h \in L^0(\Omega),\ |h(x)| \le |f(x)| \text{ a.e.} \quad \Rightarrow \quad h \in E,\ \|h\|_E \le \|f\|_E.$$

It is known that, if E has the lattice property and is complete, then

$$\mathrm{PWM}(E) = L^\infty(\Omega) \quad \text{and} \quad \|g\|_{\mathrm{Op}} = \|g\|_{L^\infty(\Omega)},$$

where $\|g\|_{\mathrm{Op}}$ is the operator norm of $g \in \mathrm{PWM}(E)$. In this report we consider pointwise multipliers from a Musielak-Orlicz-Morrey space to another Musielak-Orlicz-Morrey space.

For the introduction, first we show the proof of (1.1). To do this we first show the following lemma.

Lemma 1.1.

$$g \in L^{p_3}(\Omega) \quad \Rightarrow \quad \|g\|_{\mathrm{Op}} = \|g\|_{L^{p_3}(\Omega)}. \tag{1.2}$$

Proof. Let $g \in L^{p_3}(\Omega)$. Then, by Hölder's inequality, g is a bounded operator from $L^{p_1}(\Omega)$ to $L^{p_2}(\Omega)$ and

$$\|g\|_{\mathrm{Op}} \le \|g\|_{L^{p_3}(\Omega)}.$$

Let $f = |g|^{p_3/p_1}$. Then $f \in L^{p_1}(\Omega)$ and $\|f\|_{L^{p_1}(\Omega)} = \|g\|_{L^{p_3}(\Omega)}^{p_3/p_1}$. Moreover, $fg \in L^{p_2}(\Omega)$, $\|fg\|_{L^{p_2}(\Omega)} = \|g\|_{L^{p_3}(\Omega)}^{p_3/p_2}$, and

$$\|f\|_{L^{p_1}(\Omega)}\|g\|_{L^{p_3}(\Omega)} = \|fg\|_{L^{p_2}(\Omega)},$$

since

$$\frac{p_3}{p_1} + 1 = p_3\left(\frac{1}{p_1} + \frac{1}{p_3}\right) = \frac{p_3}{p_2}.$$

This shows that (1.2). □

To prove (1.1) we need to show

$$\mathrm{PWM}(L^{p_1}(\Omega), L^{p_2}(\Omega)) \subset L^{p_3}(\Omega). \tag{1.3}$$

Proof of (1.3). Let $g \in \mathrm{PWM}(L^{p_1}(\Omega), L^{p_2}(\Omega))$. Take a sequence of finitely simple functions $g_j \geq 0$ such that $g_j \nearrow |g|$ a.e. Then, for any $f \in L^{p_1}(\Omega)$, we have

$$\|fg_j\|_{L^{p_2}(\Omega)} \leq \|fg\|_{L^{p_2}(\Omega)}.$$

By the uniform boundedness theorem and Lemma 1.1 we have

$$\sup_j \|g_j\|_{\mathrm{Op}} < \infty \quad \text{and} \quad \sup_j \|g_j\|_{L^{p_3}(\Omega)} < \infty.$$

Therefore, $g \in L^{p_3}(\Omega)$. □

Another proof of (1.3). Let $g \in \mathrm{PWM}(L^{p_1}(\Omega), L^{p_2}(\Omega))$. Then g is a closed operator from $L^{p_1}(\Omega)$ to $L^{p_2}(\Omega)$. Actually, if

$$f_j \to f \text{ in } L^{p_1}(\Omega) \quad \text{and} \quad f_j g \to h \text{ in } L^{p_2}(\Omega),$$

then we can take its subsequence $f_{j(k)}$ such that

$$f_{j(k)} \to f \text{ a.e.} \quad \text{and} \quad f_{j(k)} g \to h \text{ a.e.}$$

This shows that $h = fg$ a.e. That is, g is a closed operator.

By the closed graph theorem g is a bounded operator. Take a sequence of finitely simple functions $g_j \geq 0$ such that $g_j \nearrow |g|$ a.e. Then $g_j \in \mathrm{PWM}(L^{p_1}(\Omega), L^{p_2}(\Omega)) \cap L^{p_3}(\Omega)$ and then, by Lemma 1.1 we have

$$\|g_j\|_{L^{p_3}(\Omega)} = \|g_j\|_{\mathrm{Op}} \leq \|g\|_{\mathrm{Op}},$$

for all j. Therefore, $g \in L^{p_3}(\Omega)$. □

2 Orlicz and Musielak-Orlicz spaces

Let $\bar{\Phi}$ be the set of all functions $\Phi : [0, \infty] \to [0, \infty]$ such that

$$\lim_{t \to +0} \Phi(t) = \Phi(0) = 0 \quad \text{and} \quad \lim_{t \to \infty} \Phi(t) = \Phi(\infty) = \infty.$$

Let

$$a(\Phi) = \sup\{t \geq 0 : \Phi(t) = 0\}, \quad b(\Phi) = \inf\{t \geq 0 : \Phi(t) = \infty\}.$$

Definition 2.1. A function $\Phi \in \bar{\varPhi}$ is called a Young function (or sometimes also called an Orlicz function) if Φ is nondecreasing on $[0,\infty)$ and convex on $[0, b(\Phi))$, and

$$\lim_{t \to b(\Phi)-0} \Phi(t) = \Phi(b(\Phi)) \ (\le \infty).$$

Any Young function is neither identically zero nor identically infinity on $(0,\infty)$. We denote by $\varPhi_Y$ the set of all Young functions.

We define three subsets $\mathcal{Y}^{(i)}$ $(i = 1,2,3)$ of Young functions as

$$\begin{aligned}
\mathcal{Y}^{(1)} &= \{\Phi \in \varPhi_Y : b(\Phi) = \infty\}, \\
\mathcal{Y}^{(2)} &= \{\Phi \in \varPhi_Y : b(\Phi) < \infty, \ \Phi(b(\Phi)) = \infty\}, \\
\mathcal{Y}^{(3)} &= \{\Phi \in \varPhi_Y : b(\Phi) < \infty, \ \Phi(b(\Phi)) < \infty\}.
\end{aligned}$$

See Figure 1.

Definition 2.2 (Orlicz space). For a function $\Phi \in \varPhi_Y$, let

$$L^\Phi(\Omega) = \left\{ f \in L^0(\Omega) : \int_\Omega \Phi(k|f(x)|)\, d\mu(x) < \infty \text{ for some } k > 0 \right\},$$

$$\|f\|_{L^\Phi(\Omega)} = \inf\left\{ \lambda > 0 : \int_\Omega \Phi\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \le 1 \right\}.$$

For example

$$\Phi(t) = t^p \ (\in \mathcal{Y}^{(1)}) \quad \Rightarrow \quad L^\Phi(\Omega) = L^p(\Omega),$$

$$\Phi(t) = \begin{cases} 0 & (0 \le t \le 1) \\ \infty & (t > 1) \end{cases} \ (\in \mathcal{Y}^{(3)}) \quad \Rightarrow \quad L^\Phi(\Omega) = L^\infty(\Omega).$$

To show

$$\mathrm{PWM}(L^{\Phi_1}(\Omega), L^{\Phi_2}(\Omega)) = L^{\Phi_3}(\Omega),$$

we need generalized Hölder's inequality

$$\|fg\|_{L^{\Phi_2}(\Omega)} \le C\|f\|_{L^{\Phi_1}(\Omega)}\|g\|_{L^{\Phi_3}(\Omega)}$$

and

$$\|g\|_{\mathrm{Op}} \sim \|g\|_{L^{\Phi_3}(\Omega)} \quad \text{for } g \in L^{\Phi_3}(\Omega). \tag{2.1}$$

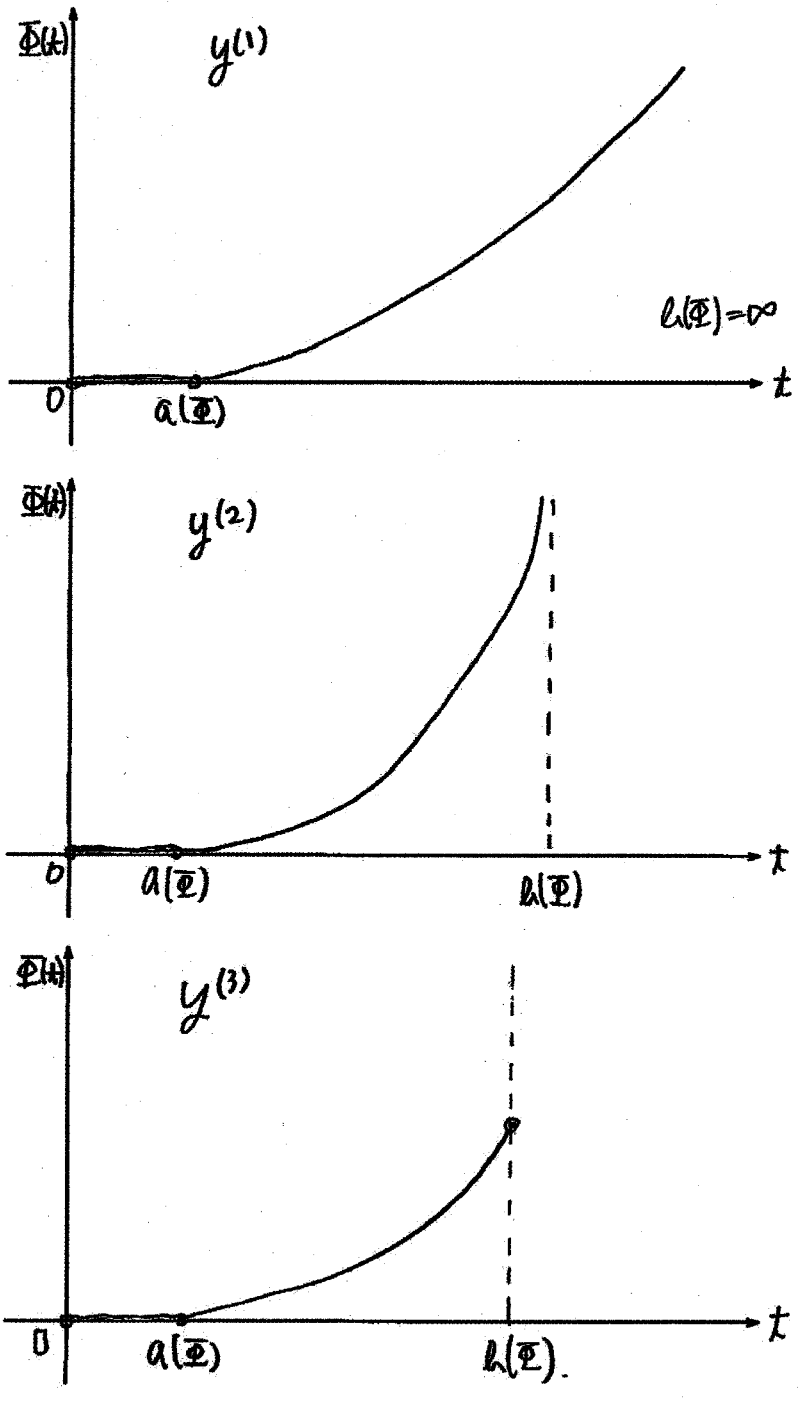

Figure 1: Three types of Young functions

If we prove

$$\int_\Omega \Phi_3\left(\frac{|g(x)|}{\|g\|_{L^{\Phi_3}(\Omega)}}\right) d\mu(x) = 1 \quad \text{for all } g \in L^{\Phi_3}(\Omega) \text{ with } g \not\equiv 0,$$

then we get (2.1). However, this holds if and only if $\Phi_3 \in \Delta_2$, which is strong restriction. So we prove it for all finitely simple functions $g \not\equiv 0$. To do this we need $\Phi_3 \in \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}$.

Definition 2.3. Let $\varPhi_Y^v$ be the set of all $\Phi : \Omega \times [0,\infty] \to [0,\infty]$ such that $\Phi(x,\cdot)$ is a Young function for every $x \in \Omega$, and that $\Phi(\cdot,t)$ is measurable on Ω for every $t \in [0,\infty]$. Assume also that, for any subset $A \subset \Omega$ with finite measure, there exists $t \in (0,\infty)$ such that $\Phi(\cdot,t)\chi_A$ is integrable.

Definition 2.4. (i) Let $\varPhi_{GY}$ be the set of all $\Phi \in \bar{\varPhi}$ such that $\Phi((\cdot)^{1/\ell})$ is in $\varPhi_Y$ for some $\ell \in (0,1]$.

(ii) Let $\varPhi_{GY}^v$ be the set of all $\Phi : \Omega \times [0,\infty] \to [0,\infty]$ such that $\Phi(\cdot,(\cdot)^{1/\ell})$ is in $\varPhi_Y^v$ for some $\ell \in (0,1]$.

For example, let $\Phi(x,t) = t^{p(x)}$.

$$\begin{aligned} p_- \geq 1 \quad &\Rightarrow \quad \Phi \in \varPhi_Y^v, \\ p_- > 0 \quad &\Rightarrow \quad \Phi \in \varPhi_{GY}^v. \end{aligned}$$

For $\Phi, \Psi \in \bar{\varPhi}$, we write $\Phi \approx \Psi$ if there exists a positive constant C such that

$$\Phi(C^{-1}t) \leq \Psi(t) \leq \Phi(Ct) \quad \text{for all } t \in (0,\infty).$$

For $\Phi, \Psi : \Omega \times [0,\infty] \to [0,\infty]$, we also write $\Phi \approx \Psi$ if there exists a positive constant C such that

$$\Phi(x, C^{-1}t) \leq \Psi(x,t) \leq \Phi(x, Ct) \quad \text{for all } (x,t) \in \Omega \times (0,\infty).$$

Lemma 2.1. *Let $\Phi \in \varPhi_{GY}^v$. For a subset $A \subset \Omega$ with $0 < \mu(A) < \infty$, let $\Phi^A(t) = \int_A \Phi(x,t)\,d\mu(x)$. Then $\Phi^A \in \varPhi_{GY}$.*

Remark 2.1. (i) $\forall \Phi \in \mathcal{Y}^{(3)}\ \exists \Psi \in \mathcal{Y}^{(2)}$ s.t. $\Phi \approx \Psi$.
(ii) $\exists \Phi \in \varPhi_Y^v$ with $\Phi(x,\cdot) \in \mathcal{Y}^{(1)}$ for each x, but $\Phi^A \in \mathcal{Y}^{(3)}$. Actually, let $\Omega = (0,1) \subset \mathbb{R}$ with the Lebesgue measure and take Young functions $\Phi(x,\cdot) \in \mathcal{Y}^{(1)}$ for all $x \in \Omega$ such that $\Phi(x,1) = 1$ and $\Phi(x,1+x) = 2/x$. Then $\Phi^\Omega \in \mathcal{Y}^{(3)}$.

Definition 2.5. Let $\bar{\varPhi}_Y$, $\bar{\varPhi}_Y^v$, $\bar{\varPhi}_{GY}$ and $\bar{\varPhi}_{GY}^v$ be the sets of all $\Phi \in \bar{\varPhi}$ such that $\Phi \approx \Psi$ for some Ψ in $\varPhi_Y$, $\varPhi_Y^v$, $\varPhi_{GY}$ and $\varPhi_{GY}^v$, respectively.

Definition 2.6. For a function $\Phi \in \bar{\varPhi}_{GY}^v$, let

$$L^\Phi(\Omega) = \left\{ f \in L^0(\Omega) : \int_\Omega \Phi(x, k|f(x)|)\, d\mu(x) < \infty \text{ for some } k > 0 \right\},$$

$$\|f\|_{L^\Phi} = \inf\left\{ \lambda > 0 : \int_\Omega \Phi\left(x, \frac{|f(x)|}{\lambda}\right) d\mu(x) \le 1 \right\}.$$

If $\Phi \approx \Psi$, then $L^\Phi(\Omega) = L^\Psi(\Omega)$ with equivalent quasi-norms.

Example 2.1. Let $p = p(\cdot)$ be a variable exponent, that is, it is a measurable function defined on Ω valued in $(0,\infty]$, and let $\Phi(x,t) = t^{p(x)}$. In this case we denote $L^\Phi(\Omega)$ by $L^{p(\cdot)}(\Omega)$.

Example 2.2. Let w be a weight function, that is, it is a measurable function defined on Ω valued in $(0,\infty)$ a.e., and $\int_A w(x)\, d\mu(x) < \infty$ for any $A \subset \Omega$ with finite measure. Let p be a variable exponent, and let

$$\Phi(x,t) = t^{p(x)} w(x).$$

In this case we denote $L^\Phi(\Omega)$ by $L_w^{p(\cdot)}(\Omega)$.

Example 2.3. Let p be a variable exponent, and let

$$\Phi(x,t) = \begin{cases} 1/\exp(1/t^{p(x)}), & t \in [0,1], \\ \exp(t^{p(x)}), & t \in (1,\infty]. \end{cases}$$

In this case we denote $L^\Phi(\Omega)$ by $\exp(L^{p(\cdot)})(\Omega)$.

Next we recall the generalized inverse of Young function Φ in the sense of O'Neil [20, Definition 1.2]. For a Young function Φ and $u \in [0,\infty]$, let

$$\Phi^{-1}(u) = \inf\{t \ge 0 : \Phi(t) > u\}, \tag{2.2}$$

where $\inf\emptyset = \infty$. For $\Phi \in \bar{\Phi}^v_{GY}$, we define also its generalized inverse with respect to t by (2.2) for each x and denote it by Φ^{-1}. That is,

$$\Phi^{-1}(x,u) = \inf\{t \geq 0 : \Phi(x,t) > u\}, \quad (x,u) \in \Omega \times [0,\infty]. \tag{2.3}$$

Theorem 2.2. *Let $\Phi_i \in \bar{\Phi}^v_{GY}$, $i = 1,2,3$. Assume that there exists a constant $C > 0$ such that*

$$\frac{1}{C}\Phi_2^{-1}(x,t) \leq \Phi_1^{-1}(x,t)\Phi_3^{-1}(x,t) \leq C\Phi_2^{-1}(x,t) \qquad \text{for} \quad (x,t) \in \Omega \times (0,\infty). \tag{2.4}$$

Assume also that there exists $\Psi_3 \in \Phi^v_{GY}$ such that

$$\Phi_3 \approx \Psi_3 \quad \text{and} \quad \Psi_3^A((\cdot)^{1/\ell}) \in \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}, \tag{2.5}$$

for some $\ell \in (0,1]$ and for any $A \subset \Omega$ with $0 < \mu(A) < \infty$, where $\Psi_3^A(t) = \int_A \Psi_3(x,t)\,d\mu(x)$. Then

$$\mathrm{PWM}(L^{\Phi_1}(\Omega), L^{\Phi_2}(\Omega)) = L^{\Phi_3}(\Omega),$$

$$\|g\|_{\mathrm{Op}} \sim \|g\|_{L^{\Phi_3}(\Omega)}.$$

Let p_i be variable exponents, w_i be weight functions, $i = 1,2,3$, and

$$\Omega_\infty = \{x \in \Omega : p_3(x) = \infty\}.$$

Assume that $\inf_{x\in\Omega} p_i(x) > 0$, $i = 1,2,3$, and $\sup_{x\in\Omega\setminus\Omega_\infty} p_3(x) < \infty$.

Example 2.4. Let

$$\frac{1}{p_1(x)} + \frac{1}{p_3(x)} = \frac{1}{p_2(x)}.$$

Then

$$\mathrm{PWM}(L^{p_1(\cdot)}(\Omega), L^{p_2(\cdot)}(\Omega)) = L^{p_3(\cdot)}(\Omega),$$

$$\mathrm{PWM}(\exp(L^{p_1(\cdot)})(\Omega), \exp(L^{p_2(\cdot)})(\Omega)) = \exp(L^{p_3(\cdot)})(\Omega).$$

Example 2.5. Let

$$\frac{1}{p_1(x)} + \frac{1}{p_3(x)} = \frac{1}{p_2(x)}, \quad w_1(x)^{1/p_1(x)}w_3(x)^{1/p_3(x)} = w_2(x)^{1/p_2(x)}.$$

Then

$$\mathrm{PWM}(L^{p_1(\cdot)}_{w_1}(\Omega), L^{p_2(\cdot)}_{w_2}(\Omega)) = L^{p_3(\cdot)}_{w_3}(\Omega).$$

3 Musielak-Orlicz-Morrey spaces

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and μ the Lebesgue measure. For a function $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ and a ball $B = B(x,r)$, we write $\phi(B) = \phi(x,r)$.

Definition 3.1 (Musielak-Orlicz-Morrey space). For $\Phi \in \bar{\Phi}^v_{GY}$, $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ and a ball B, let

$$\|f\|_{\Phi,\phi,B} = \inf\left\{\lambda > 0 : \frac{1}{\phi(B)\mu(B)}\int_B \Phi\left(x, \frac{|f(x)|}{\lambda}\right) d\mu(x) \le 1\right\},$$

and let

$$L^{(\Phi,\phi)}(\mathbb{R}^n) = \left\{f \in L^0(\mathbb{R}^n) : \|f\|_{L^{(\Phi,\phi)}(\mathbb{R}^n)} < \infty\right\},$$
$$\|f\|_{L^{(\Phi,\phi)}(\mathbb{R}^n)} = \sup_B \|f\|_{\Phi,\phi,B},$$

where the supremum is taken over all balls B.

If $\phi(B) = 1/\mu(B)$, then $L^{(\Phi,\phi)}(\mathbb{R}^n) = L^\Phi(\mathbb{R}^n)$.

For functions $\theta, \kappa : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, we write $\theta \sim \kappa$ if there exists a positive constant C such that

$$\frac{1}{C} \le \frac{\theta(x,r)}{\kappa(x,r)} \le C \quad \text{for all } (x,r) \in \mathbb{R}^n \times (0,\infty).$$

If $\Phi \approx \Psi$ and $\phi \sim \psi$, then $L^{(\Phi,\phi)}(\mathbb{R}^n) = L^{(\Psi,\psi)}(\mathbb{R}^n)$ with equivalent quasi-norms.

Definition 3.2. A function $\theta : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ is *almost increasing* (*almost decreasing*) *with respect to the order by ball inclusion* if there exists a positive constant C such that

$$\theta(B_1) \le C\theta(B_2) \quad (\theta(B_1) \ge C\theta(B_2)) \qquad \text{for all balls } B_1 \text{ and } B_2 \text{ with } B_1 \subset B_2.$$

Definition 3.3. Let $\mathcal{G}^v$ be the set of all $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ such that ϕ is almost decreasing with respect to the order by ball inclusion and $\phi(B)\mu(B)$ is almost increasing with respect to the order by ball inclusion.

Theorem 3.1. *Let* $\Phi_i \in \bar{\Phi}^v_{GY}$ *and* $\phi_i \in \mathcal{G}^v$, $i = 1, 2, 3$. *Assume that there exists a positive constant* C *such that*

$$\begin{aligned} C^{-1}\Phi_2^{-1}(x, t\phi_2(x,r)) &\le \Phi_1^{-1}(x, t\phi_1(x,r))\Phi_3^{-1}(x, t\phi_3(x,r)) \\ &\le C\Phi_2^{-1}(x, t\phi_2(x,r)), \quad \text{for all } x \in \mathbb{R}^n \text{ and } r, t \in (0, \infty), \end{aligned}$$

and that ϕ_3/ϕ_1 *is almost increasing with respect to the order by ball inclusion. Assume also one of the following:*

(i) Φ_3 *satisfies the* Δ_2 *condition, that is,* $\Phi_3(x, 2t) \le \exists C_{\Phi_3}\Phi_3(x, t)$.

(ii) $\displaystyle\lim_{r\to\infty}\inf_{x\in\mathbb{R}^n} \phi_3(x,r)\mu(B(x,r)) = \infty$, $\phi_3(x,r)$ *is continuous with respect to* x *and* r, *and, for all balls* B,

  (a) $\exists \Psi_B \in \mathcal{Y}^{(1)}$ *s.t.* $\displaystyle\sup_{x\in B}\Phi_3(x,t) \le \Psi_B(t)$ *for all* t, *and,*

  (b) $\displaystyle\lim_{r\to+0}\inf_{x\in B}\phi_3(x,r) = \infty$.

Then

$$\mathrm{PWM}(L^{(\Phi_1,\phi_1)}(\mathbb{R}^n), L^{(\Phi_2,\phi_2)}(\mathbb{R}^n)) = L^{(\Phi_3,\phi_3)}(\mathbb{R}^n),$$

$$\|g\|_{\mathrm{Op}} \sim \|g\|_{L^{(\Phi_3,\phi_3)}(\mathbb{R}^n)}.$$

Corollary 3.2. *Let* $p_i(\cdot)$ *be variable exponents with* $0 < (p_i)_- \le (p_i)_+ \le \infty$, w_i *be weights and* $\phi_i \in \mathcal{G}^v$, $i = 1, 2, 3$. *Assume that*

$$1/p_1(x) + 1/p_3(x) = 1/p_2(x),$$

that there exists a positive constant C *such that*

$$\begin{aligned} &C^{-1}(\phi_2(x,r)/w_2(x))^{1/p_2(x)} \\ &\quad\le (\phi_1(x,r)/w_1(x))^{1/p_1(x)}(\phi_3(x,r)/w_3(x))^{1/p_3(x)} \\ &\quad\quad\le C\,(\phi_2(x,r)/w_2(x))^{1/p_2(x)}, \\ &\qquad\qquad\qquad \text{for all } x \in \mathbb{R}^n \text{ and } r \in (0, \infty), \end{aligned}$$

and that ϕ_3/ϕ_1 *is almost increasing with respect to the order by ball inclusion. If* $(p_3)_+ < \infty$, *then*

$$\mathrm{PWM}(L^{(p_1,\phi_1)}_{w_1}(\mathbb{R}^n), L^{(p_2,\phi_2)}_{w_2}(\mathbb{R}^n)) = L^{(p_3,\phi_3)}_{w_3}(\mathbb{R}^n),$$

$$\|g\|_{\mathrm{Op}} \sim \|g\|_{L^{(p_3,\phi_3)}_{w_3}(\mathbb{R}^n)}.$$

Corollary 3.3. *Let $p_i(\cdot)$ and $\lambda_i(\cdot)$ be variable exponents with $0 < (p_i)_- \le (p_i)_+ \le \infty$ and $-n \le (\lambda_i)_- \le (\lambda_i)_+ < 0$, w_i be weights, $i = 1, 2, 3$. Let λ^* be a constant with $-n \le \lambda^* < 0$, and let*

$$\phi_i(x,r) = \begin{cases} r^{\lambda_i(x)}, & r \le 1/e, \\ r^{\lambda^*}, & r > 1/e. \end{cases}$$

Assume that $(p_3)_+ < \infty$, that $\lambda_i(\cdot)$, $i = 1, 2, 3$, are log-Hölder continuous, and that

$$\begin{cases} \dfrac{1}{p_1(x)} + \dfrac{1}{p_3(x)} = \dfrac{1}{p_2(x)}, \quad \dfrac{\lambda_1(x)}{p_1(x)} + \dfrac{\lambda_3(x)}{p_3(x)} = \dfrac{\lambda_2(x)}{p_2(x)}, \\ w_1(x)^{1/p_1(x)} w_3(x)^{1/p_3(x)} = w_2(x)^{1/p_2(x)}, \\ \lambda_3(x) \ge \lambda_1(x), \qquad \text{for all } x \in \mathbb{R}^n. \end{cases}$$

Then

$$\mathrm{PWM}(L^{(p_1,\phi_1)}_{w_1}(\mathbb{R}^n), L^{(p_2,\phi_2)}_{w_2}(\mathbb{R}^n)) = L^{(p_3,\phi_3)}_{w_3}(\mathbb{R}^n),$$

$$\|g\|_{\mathrm{Op}} \sim \|g\|_{L^{(p_3,\phi_3)}_{w_3}(\mathbb{R}^n)}.$$

Corollary 3.4. *Let $p_i(\cdot)$ be variable exponents with $0 < (p_i)_- \le (p_i)_+ \le \infty$, and let*

$$\Phi_i(x,t) = \begin{cases} 1/\exp(1/t^{p_i(x)}), & t \in [0,1], \\ \exp(t^{p_i(x)}), & t \in (1,\infty], \end{cases} \qquad i = 1, 2, 3.$$

Let λ be a constant with $-1 < \lambda < 0$, and let $\phi(B) = \mu(B)^\lambda$. Assume that $(p_3)_+ < \infty$ and that $1/p_1(x) + 1/p_3(x) = 1/p_2(x)$. Then

$$\mathrm{PWM}(L^{(\Phi_1,\phi)}(\mathbb{R}^n), L^{(\Phi_2,\phi)}(\mathbb{R}^n)) = L^{(\Phi_3,\phi)}(\mathbb{R}^n),$$

$$\|g\|_{\mathrm{Op}} \sim \|g\|_{L^{(\Phi_3,\phi)}(\mathbb{R}^n)}.$$

The results in this section can be extended to Musielak-Orlicz-Morrey spaces defined on spaces of homogeneous type or metric measure spaces with non-doubling measure.

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