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Perturbations of C^1 norms and associated isometry groups

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Abstract

This note announces a recent result on isometries of C^1-function spaces over compact Riemannian manifolds [7]. We characterize isometries (with respect to certain norms inducing the C^1-topology) of the C^1-function spaces over compact Riemannian manifolds as generalized weighted composition operators, under some regularity assumption. We also apply the characterization to study continuous deformations of isometry groups induced by perturbations of norms on the function spaces.

Let M be a compact Riemannian manifold with a Riemannian metric g and let $C^1(M,\mathbb{R}^d)$ be the space of all C^1-maps of M to $\mathbb{R}^d$ with the C^1-topology. For $p\in[1,\infty]$ and for a submanifold K of M, we define a norm $\|\cdot\|_{<M,K;p>}$ on $C^1(M,\mathbb{R}^d)$ by

$$\|f\|_{<M,K;p>}=(\|f|K\|_\infty^p+\|Df\|_\infty^p)^{1/p}$$

for $f\in C^1(M,\mathbb{R}^d)$, where Df denotes the derivative of f. Here the norm $\|Df\|_\infty$ is defined as follows. Take a point $q\in M$ and take a local chart $(x^1,\ldots x^n)$ at q and let $g_{ij}=g(\partial_i,\partial_j),\partial_i=\frac{\partial}{\partial x^i}$. The inverse matrix of (g_{ij}) is denoted by (g^{ij}): $(g^{ij})=(g_{ij})^{-1}$. Let

$$\|D_qf\|=\left(\sum_{i,j=1}^n g^{ij}\partial_i f\cdot\partial_j f\right)^{1/2}$$

where "$\cdot$" denotes the standard inner product on $\mathbb{R}^d$. It can be shown that $\|D_qf\|$ does not depend on the local chart and let $\|Df\|_\infty=\sup_{q\in M}\|D_qf\|$.

When $\dim M=1$ or $K=M$, we essentially have characterized surjective linear isometries $T:(C^1(M,\mathbb{R}^d),\|\cdot\|_{<M,K;p>})\to(C^1(M,\mathbb{R}^d),\|\cdot\|_{<M,K;p>})$ as generalized weighted composition operators [8], [5], [6]. As a continuation of the research the following theorem is proved in [7]. An operator $T: C^1(M,\mathbb{R}^d)\to C^1(M,\mathbb{R}^d)$ is said to be C^k-preserving, if Tf is a C^k-map for each C^k-map $f\in C^1(M,\mathbb{R}^d)$. Also T is said to preserve the constant maps if Tc is a constant map whenever $c:M\to\mathbb{R}^d$ is a constant map.

Theorem 1 *[7] Let M be compact connected Riemannian manifold with* $\dim M > 1$ *and let K and L be connected submanifolds of M. For $p \in (1, \infty]$, let $\|\cdot\|_{<M,K;p>}$ and $\|\cdot\|_{<M,L;p>}$ be the norms defined by the above and let $T : (C^1(M, \mathbb{R}^d), \|\cdot\|_{<M,K;p>}) \to (C^1(M, \mathbb{R}^d), \|\cdot\|_{<M,L;p>})$ be a surjective linear isometry. Assume that T and T^{-1} are C^3-preserving and preserve the constant maps. Then K and L are homeomorphic and we have the following.*

(1) Assume that $\dim K = \dim L > 0$. *Then there exist a Riemannian isometry $\varphi : M \to M$ and a linear isometry $U : \mathbb{R}^d \to \mathbb{R}^d$ such that*

(1.1) $\varphi(L) = K$ and

(1.2) $Tf(x) = U(f(\varphi(x)))$ for each $x \in M$ and for each $f \in C^1(M, \mathbb{R}^d)$.

(2) Assume that $K = \{a\}, L = \{b\}$. There exist a Riemannian isometry $\varphi : M \to M$ and linear isometries $U, V : \mathbb{R}^d \to \mathbb{R}^d$ such that

(2.1) $Tf(b) = U(f(a))$ and

(2.2) $Tf(x) = V(f(\varphi(x))) + \{U(f(a)) - V(f(\varphi(b)))\}$ for each $x \in M$ and for each $f \in C^1(M, \mathbb{R}^d)$.

For $p = 1$, we can obtain a similar result when $\dim M = 1$ by applying the argument due to Botelho and Jamison [1] which relies on the Borsuk-Ulam theorem. It should be mentioned that detailed study for $M = [0, 1]$ and $p = 1$ has been carried out by [1], [2], [3], [10], [11], [13] etc. It is not known to the author whether the same conclusion as Theorem 1 holds for the case $\dim M > 1$ and $p = 1$.

Let K and L be two positive-dimensional connected submanifolds of a compact Riemannian manifold M of dimension at least 2 and assume that $(C^1(M, \mathbb{R}^d), \|\cdot\|_{<M,K;p>})$ and $(C^1(M, \mathbb{R}^d), \|\cdot\|_{<M,L;p>})$ are isometric. It follows from the above theorem that (M, K) and (M, L) are isometric manifold pairs. In this sense the isometry type of the function space $(C^1(M, \mathbb{R}^d), \|\cdot\|_{<M,K;p>})$ not only determines the isometry type of the ambient manifold M but also the *embedding type* of the submanifold K up to isometry.

We may apply the above theorem along the line of [6] to study perturbations of norms on $C^1(M, \mathbb{R}^d)$ and deformations of associated isometry groups. To be more precise we give the following definitions. For a norm $\|\cdot\|$ on $C^1(M, \mathbb{R}^d)$, the group of all linear $\|\cdot\|$-isometries is denoted by $\mathcal{U}(\|\cdot\|)$.

Definition 2 *Let M be a compact connected Riemannian manifold and $d \geq 1$.*

(1) Let $\mathcal{N}(M, \mathbb{R}^d)$ be the space of all norms on $C^1(M, \mathbb{R}^d)$ which induce the C^1-topology. The space $\mathcal{N}(M, \mathbb{R}^d)$ is endowed with the coarsest topology such that the map $e_f : C^1(M, \mathbb{R}^d) \to \mathbb{R}$ defined by

$$e_f(\|\cdot\|) = \|f\|, \quad \|\cdot\| \in \mathcal{N}(M, \mathbb{R}^d)$$

is continuous for each $f \in C^1(M, \mathbb{R}^d)$.

(2) Let $\mathcal{B}(M, \mathbb{R}^d)$ be the space of all linear operators on $C^1(M, \mathbb{R}^d)$ which are continuous with respect to the C^1-topology. The space $\mathcal{B}(M, \mathbb{R}^d)$ is endowed with the coarsest topology such that the map $E_f : \mathcal{B}(M, \mathbb{R}^d) \to C^1(M, \mathbb{R}^d)$ given by

$$E_f(T) = Tf, \quad T \in \mathcal{B}(M, \mathbb{R}^d)$$

is continuous for each $f \in C^1(M, \mathbb{R}^d)$.

(3) We define "the bundle of isometries" as:

$$\begin{aligned} \mathcal{U}(M, \mathbb{R}^d) &= \{(\|\cdot\|, T) \in \mathcal{N}(M, \mathbb{R}^d) \times \mathcal{B}(M, \mathbb{R}^d) \mid T \in \mathcal{U}(\|\cdot\|)\} \\ &\subset \mathcal{N}(M, \mathbb{R}^d) \times \mathcal{B}(M, \mathbb{R}^d) \end{aligned}$$

with the projection $\Pi : \mathcal{U}(M, \mathbb{R}^d) \to \mathcal{N}(M, \mathbb{R}^d)$ given by $\Pi(\|\cdot\|, T) = \|\cdot\|$. We have $\Pi^{-1}(\|\cdot\|) = \mathcal{U}(\|\cdot\|)$.

Let $\nu : [0,1] \to \mathcal{N}(M, \mathbb{R}^d)$ be a continuous path and take an isometry $T \in \mathcal{U}(\nu(0))$. Motivated by covering space/fiber bundle theory we study the existence/uniqueness of a continuous path $\tau : [0,1] \to \mathcal{U}(M, \mathbb{R}^d)$, called a lift of ν starting with T, such that $P \circ \tau = \nu$ and $\tau(0) = T$. Theorem 1 reduces the problem to the existence/uniqueness of appropriate paths in the isometry group $\mathrm{Isom}(M)$ and in the orthogonal group $O(\mathbb{R}^d)$.

Let $\mathcal{K}_M$ be the space of all connected submanifolds of M with the Hausdorff metric. Motivated by [10] and [8], we consider a path $\alpha : [0,1] \to \mathcal{K}_M$ such that

($\star$) $\alpha(0)$ is a singleton, and $\dim \alpha(t) > 0$ and $\alpha(t) \neq M$ for each $t \in (0,1)$.

Fix $p \in [1, \infty]$ and let

$$\nu_\alpha(t) = \| \cdot \|_{<M,\alpha(t);p>}, \ t \in [0,1].$$

This defines a continuous path $\nu_\alpha : [0,1] \to \mathcal{N}(M, \mathbb{R}^d)$.

Convention. Taking account of the additional condition of C^3-preservation and constant-maps-preservation which is assumed on the isometry $T : (C^1(M, \mathbb{R}^d), \| \cdot \|_{<M,K;p>}) \to ((C^1(M, \mathbb{R}^d), \| \cdot \|_{<M,K;p>})$ of Theorem 1, we assume in the sequel that every lift $\tau : [0,1] \to \mathcal{U}(M, \mathbb{R}^d)$ satisfies the additional condition:

($\flat$) for each $t \in [0,1]$, the isometries $\tau(t)$ and $\tau(t)^{-1}$ are C^3-preserving and preserve the constant maps.

Let $\alpha : [0,1] \to \mathcal{K}_M$ be a continuous path satisfying the condition ($\star$) and assume that a continuous path $\tau : [0,1] \to \mathcal{U}(M, \mathbb{R}^d)$ is a lift of of ν_α. By Theorem 1 and the above convention, there exist isometries $\{\varphi_t \mid 0 \leq t \leq 1\} \subset \mathrm{Isom}(M)$ and linear isometries $\{U_t \mid 0 \leq t \leq 1\} \cup \{V\} \subset O(\mathbb{R}^d)$ such that

$$\begin{aligned} \tau(0)f &= V \circ f \circ \varphi_0 + (U_0(f(a)) - V(f(\varphi_0(a))) \quad (*) \\ \tau(t)f &= U_t \circ f \circ \varphi_t \quad t \in (0,1) \quad (**) \end{aligned}$$

for each $f \in C^1(M, \mathbb{R}^d)$, where $\varphi_t(\alpha(t)) = \alpha(t)$ for each $t \in (0,1]$. We can show from the continuity of τ that $U_0 = V$ and $\varphi_0(a) = a$, and thus $\tau(0)f = U_0 \circ f \circ \varphi_0$ for $f \in C^1(M, \mathbb{R}^d)$. Hence in order to obtain a lift of ν_α, the initial isometry $T \in \mathcal{U}(\nu_\alpha(0))$ must be of the form ($**$).

The validity of the converse depends on manifolds M, paths $\alpha : [0,1] \to \mathcal{K}_M$ and the initial isometry $T \in \mathcal{U}(\nu_\alpha(0))$. Here we study the lifting problem when M is the standard sphere, the flat torus and a manifold with finite isometry group $\mathrm{Isom}(M)$ (e.g. hyperbolic surfaces). For an isometry $\varphi \in \mathrm{Isom}(M)$ and for a linear isometry $U \in O(\mathbb{R}^d)$, $C_{\varphi,U}$ stands for the weighted composition operator given by

$$C_{\varphi,U} f = U \circ f \circ \varphi, \quad f \in C^1(M, \mathbb{R}^d).$$

The isometry $\varphi \in \mathrm{Isom}(M)$ and $U \in O(\mathbb{R}^d)$ above are called the symbol and the weight of the operator $C_{\varphi,U}$.

In what follows, $p > 1$ and $d \geq 1$ are fixed. To simplify terminology we say that a path $\alpha : [0,1] \to \mathcal{K}_M$ satisfying the condition ($\star$) is a *path of disks* if $\alpha(t)$ is homeomorphic to a disk of dimension $\dim M$ for each $t \in (0,1)$. Also we say that two lifts $\tau_1, \tau_2 : [0,1] \to \mathcal{U}(M, \mathbb{R}^d)$ *have different symbols* if $\tau_1(t)$ and $\tau_2(t)$ have different symbols for some t.

Let $n \geq 1$ and let $S^n = \{(x_i)_{1 \leq i \leq n+1} \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\}$ be the standard sphere. It is known that $\mathrm{Isom}(S^n) \cong O(n+1)$. A similar result to the next proposition for $n = 1$ has been proved in [6] (cf.[7]).

Proposition 3 *Let $n \geq 2$ and take a point $a \in S^n$.*

(1) There exist two paths of disks $\alpha, \beta : [0,1] \to \mathcal{K}_{S^n}$ with $\alpha(0) = \beta(0) = \{a\}$ such that

(1.1) for each $T = C_{\varphi,U}$ with $\varphi(a) = a$ there exist infinitely many lifts of ν_α starting with T such that they have mutually distinct symbols, and

(1.2) for each $T = C_{\varphi,U}$ with $\varphi(a) = a$ and φ is not isotopic to id_{S^n}, there exist no lifts of ν_β starting with T.

(2) There exist $(n+1)$ paths of disks $\alpha_1, \dots \alpha_{n+1}$ such that $\alpha_i(0) = \{a\}, i = 1, \dots n+1$, such that

(2.1) for each $i = 1, \dots, n+1$ and for each $T = C_{\varphi,U}$ with $\varphi(a) = a$, there exist infinitely many lifts of ν_α starting with T with mutually distinct symbols, and

(2.2) for each $T = C_{\varphi,U}$ with $\varphi(a) = a$ such that φ is not isotopic to id_{S^n} and for each lift $\tau_i : [0,1] \to \mathcal{U}(S^n, \mathbb{R}^d)$ of ν_{α_i} starting with T, $i = 1, \dots n+1$, there exist i, j such that $\tau_i(1) \neq \tau_j(1)$.

Proposition 4 *Let M be a compact Riemannian manifold such that* $\mathrm{Isom}(M)$ *is a finite group (e.g. a hyperbolic surface). Take a point $a \in M$ and $\alpha : [0,1] \to \mathcal{K}_M$ be a path of disks with $\alpha(0) = \{a\}$. Also take $T = C_{\varphi,U} \in \mathcal{U}(\nu_\alpha(0))$ with $\varphi(a) = a$. Then there exists a continuous lift $\tau : [0,1] \to \mathcal{U}(M, \mathbb{R}^d)$ of ν_α starting with T if and only if $\varphi(\alpha(t)) = \alpha(t)$ for each $t \in (0,1)$. The symbol of $\tau(t)$ ($t \in [0,1]$) for each such lift τ is equal to φ.*

Proposition 5 *Let $T^n := \mathbb{R}^n/\mathbb{Z}^n$ be the flat torus and take a point $a \in T^n$. There exists a path of disks $\alpha : [0,1] \to \mathcal{K}_{T^n}$ such that for each $T = C_{\varphi,U} \in$*

$\mathcal{U}(\nu_\alpha(0))$ with $\varphi(a) = a$, there exists a lift $\tau : [0,1] \to \mathcal{U}(T^n, \mathbb{R}^d)$ of ν_α starting with T. The symbol of $\tau(t)$ ($t \in [0,1]$) for each such lift τ is equal to φ.

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