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Lau algebras defined by semisimple commutative Banach algebras of type I

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Abstract. This is an announcement of our research on semisimple commutative Banach algebras of type I and Lau algebras defined by them. We classify those algebras into four classes by means of BSE and BED algebras.

§1. Banach algebras of type I. Let A be a semisimple commutative Banach algebra with Gelfand space Φ_A. For any $x \in A$ we denote its Gelfand transform by $\widehat{x}$. We put $\widehat{A} = \{\widehat{x} : x \in A\}$. Let T be a bounded linear operator T from A into itself. We call T a *multiplier* of A if $T(xy) = xT(y)$ for all $x, y \in A$. The set of all multipliers of A becomes a unital commutative Banach algebra. We call it a *multiplier algebra* of A and denote it by $M(A)$. Obviously the Gelfand space of $M(A)$ contains Φ_A, and for any $T \in M(A)$, $\widehat{T}$ denotes the restriction of its Gelfand transform to Φ_A. We put $\widehat{M}(A) = \{\widehat{T} : T \in M(A)\}$. Let $C^b(\Phi_A)$ be the C^*-algebra of all bounded continuous complex-valued functions on Φ_A. Then we have

$$\widehat{A} \subset \widehat{M}(A) \subset C^b(\Phi_A)$$

(cf. [5]). If $\widehat{M}(A) = C^b(\Phi_A)$, then we say that A is a Banach algebra of *type I* (in short, "of type I").

Let A and B be semisimple commutative Banach algebras. Suppose that a mapping $T : b \mapsto T_b$ is a norm-decreasing homomorphism from B into $M(A)$. Then the product space $A \times B$ is a commutative Banach algebra with respect to multiplication

$$(a, b) \times_T (c, d) = \big(ac + T_d(a) + T_b(c),\, bd\big)$$

and norm $\|(a, b)\| = \|a\| + \|b\|$. This algebra is called a *Lau algebra* defined by $(A, B; T)$, and is written as $A \times_T B$ (see [6, 7, 11]). We have the following theorem.

Theorem 1. *Let A and B be semisimple commutative Banach algebras. Let $T : b \mapsto T_b$ be a norm-decreasing homomorphism from B into $M(A)$ such that $\{T_b : b \in B\} \subset A$. Then $A \times_T B$ is of type I if and only if both A and B are of type I.*

§2. BSE-algebras and BED-algebras. Let A be a semisimple commutative Banach algebra. By $\mathrm{span}(\Phi_A)$, we denote the linear span of Φ_A in the dual space A^* of A. Every functional p in $\mathrm{span}(\Phi_A)$ is uniquely represented as

$$p = \sum_{\varphi \in \Phi_A} \widehat{p}(\varphi)\,\varphi,$$

where $\widehat{p}$ is a complex-valued function on Φ_A with finite support. Let $\sigma \in C^b(\Phi_A)$. If there exists a positive constant β such that

$$\left| \sum_{\varphi \in \Phi_A} \widehat{p}(\varphi)\sigma(\varphi) \right| \leq \beta \, \|p\|_{A^*}$$

for all $p \in \mathrm{span}(\Phi_A)$, then we call σ a *BSE-function*, and define a *BSE-norm* of σ as the infimum of all such constants β's. With this norm, the set of all BSE-functions becomes a semisimple commutative Banach algebra. This algebra is written as $C_{BSE}(\Phi_A)$. If $\widehat{M(A)} = C_{BSE}(\Phi_A)$, then we say that A is a *BSE-algebra*. In [4], the fourth and first authors constructed a BSE-algebra of type I which is isomorphic to no C^*-algebras. This example gives a negative answer to the problem posed by the first author and Hatori ([10]). While it suggests further research on Banach algebras of type I. In [1], Dabhi took up a Lau algebra defined by BSE-algebras and proved the following theorem.

Theorem A. *Let A, B and T be as in Theorem 1. Then $A \times_T B$ is a BSE-algebra if and only if both A and B are BSE-algebras.*

Let $\mathcal{K}(\Phi_A)$ be the directed set of all compact subsets of Φ_A with the inclusion order. For each $\sigma \in C_{BSE}(\Phi_A)$ and $K \in \mathcal{K}(\Phi_A)$, we put

$$\|\sigma\|_{BSE,K} = \sup\left\{ \left| \sum_{\varphi \in \Phi_A} \widehat{p}(\varphi)\sigma(\varphi) \right| : p \in \mathrm{span}(\Phi_A),\ \|p\|_{A^*} \leq 1,\ \widehat{p}|_K = 0 \right\},$$

and

$$C^0_{BSE}(\Phi_A) = \left\{ \sigma \in C_{BSE}(\Phi_A) : \lim_{K \in \mathcal{K}(\Phi_A)} \|\sigma\|_{BSE,K} = 0 \right\}.$$

Then $C^0_{BSE}(\Phi_A)$ is a closed ideal of $C_{BSE}(\Phi_A)$. If $\widehat{A} = C^0_{BSE}(\Phi_A)$, then we say that A is a *BED-algebra* (cf. [2]). We have the following theorem.

Theorem 2. *Let A, B and T be as in Theorem 1. Then $A \times_T B$ is a BED-algebra if and only if both A and B are BED-algebras.*

§3. Classification of Lau algebras. We denote by $\mathcal{B}_{\mathrm{type\,I}}$ the collection of all semisimple commutative Banach algebras of type I. We classify $\mathcal{B}_{\mathrm{type\,I}}$ into four disjoint classes $\mathcal{B}^1_{\mathrm{type\,I}}$, $\mathcal{B}^2_{\mathrm{type\,I}}$, $\mathcal{B}^3_{\mathrm{type\,I}}$ and $\mathcal{B}^0_{\mathrm{type\,I}}$;

$$\mathcal{B}_{\mathrm{type\,I}} = \mathcal{B}^1_{\mathrm{type\,I}} \cup \mathcal{B}^2_{\mathrm{type\,I}} \cup \mathcal{B}^3_{\mathrm{type\,I}} \cup \mathcal{B}^0_{\mathrm{type\,I}}.$$

Here $\mathcal{B}^1_{\text{type I}}$ consists of elements in $\mathcal{B}_{\text{type I}}$ that are both BSE-algebras and BED-algebras; $\mathcal{B}^2_{\text{type I}}$ consists of ones that are BSE but not BED; $\mathcal{B}^3_{\text{type I}}$ consists of ones that are BED but not BSE; and $\mathcal{B}^0_{\text{type I}}$ consists of ones that are neither BSE nor BED.

Let us classify a Lau algebra $A \times_T B$ by means of the classes of A and B. Under the assumption in Theorems 1, 2 and A, we derive the following classification table of a Lau algebra $A \times_T B$.

$A \diagdown B$	$\mathcal{B}^1_{\text{type I}}$	$\mathcal{B}^2_{\text{type I}}$	$\mathcal{B}^3_{\text{type I}}$	$\mathcal{B}^0_{\text{type I}}$
$\mathcal{B}^1_{\text{type I}}$	$\mathcal{B}^1_{\text{type I}}$	$\mathcal{B}^2_{\text{type I}}$	$\mathcal{B}^3_{\text{type I}}$	$\mathcal{B}^0_{\text{type I}}$
$\mathcal{B}^2_{\text{type I}}$	$\mathcal{B}^2_{\text{type I}}$	$\mathcal{B}^2_{\text{type I}}$	$\mathcal{B}^0_{\text{type I}}$	$\mathcal{B}^0_{\text{type I}}$
$\mathcal{B}^3_{\text{type I}}$	$\mathcal{B}^3_{\text{type I}}$	$\mathcal{B}^0_{\text{type I}}$	$\mathcal{B}^3_{\text{type I}}$	$\mathcal{B}^0_{\text{type I}}$
$\mathcal{B}^0_{\text{type I}}$	$\mathcal{B}^0_{\text{type I}}$	$\mathcal{B}^0_{\text{type I}}$	$\mathcal{B}^0_{\text{type I}}$	$\mathcal{B}^0_{\text{type I}}$

This table can be seen to give a semilattice operation of order 4, where $\mathcal{B}^1_{\text{type I}}$ is an identity element and $\mathcal{B}^0_{\text{type I}}$ is an absorbing element.

§4. Four Classes. In this section, we investigate four classes $\mathcal{B}^1_{\text{type I}}$, $\mathcal{B}^2_{\text{type I}}$, $\mathcal{B}^3_{\text{type I}}$ and $\mathcal{B}^0_{\text{type I}}$. First we completely characterize $\mathcal{B}^1_{\text{type I}}$ as follows.

Theorem 3. *Let $A \in \mathcal{B}_{\text{type I}}$. Then A belongs to $\mathcal{B}^1_{\text{type I}}$ if and only if A is isomorphic to a certain commutative C^*-algebra.*

We have not obtained such characterizations of $\mathcal{B}^2_{\text{type I}}$, $\mathcal{B}^3_{\text{type I}}$ and $\mathcal{B}^0_{\text{type I}}$ yet. In the rest of this section, we exhibit some examples of algebras belonging to them. In the examples below, the concept of an extended Segal algebra will play an important role. It is a generalization of Reiter's Segal algebra. For Reiter's Segal algebras, see [8, 9]. For extended Segal algebras, see [3].

Example 4. Let X be a locally compact Hausdorff space which is not compact. Denote by $C_0(X)$ the commutative C^*-algebra of all continuous complex-valued functions on X vanishing at infinity. Let μ be a positive unbounded regular continuous Borel measure on X, and $L^p(X,\mu)$ the L^p-space on the measure space (X,μ), where $1 \le p < \infty$. Put

$$C_{0,p}(X,\mu) = C_0(X) \cap L^p(X,\mu).$$

Then $C_{0,p}(X,\mu)$ is a semisimple commutative Banach algebra with the ℓ^1-norm

$$\|f\|_{\infty,p} = \|f\|_\infty + \|f\|_p \qquad \bigl(f \in C_{0,p}(X,\mu)\bigr).$$

The algebra $C_{0,p}(X,\mu)$ belongs to $\mathcal{B}^2_{\text{type I}}$.

Example 5. Let X be a locally compact Hausdorff space which is not compact.

(i) Assume that X is σ-compact and take a sequence $\{K_1, K_2, \cdots\}$ in $\mathcal{K}(X)$ such that $K_1 \subsetneqq K_2 \subsetneqq \cdots$ and $\bigcup_{n=1}^{\infty} K_n = X$. For each $n \in \mathbb{N}$, choose $x_n \in K_n \setminus K_{n-1}$, where $K_0 = \emptyset$. Put

$$C_{0,p,\{x_i\}}(X) = \left\{ f \in C_0(X) : \sum_{i=1}^{\infty} |f(x_i)|^p < \infty \right\},$$

where $1 \leq p < \infty$. Then $C_{0,p,\{x_i\}}(X)$ is a semisimple commutative Banach algebra with the ℓ^1-norm

$$\|f\|_{\infty,p,\{x_i\}} = \|f\|_{\infty} + \left(\sum_{i=1}^{\infty} |f(x_i)|^p \right)^{1/p} \qquad \big(f \in C_{0,p,\{x_i\}}(X)\big).$$

The algebra $C_{0,p,\{x_i\}}(X)$ belongs to $\mathcal{B}^3_{\text{type I}}$.

(ii) Let τ be a real-valued function on X such that $\inf_{x\in X} \tau(x) \geq 1$. Put

$$C^b(X;\tau) = \left\{ f \in C^b(X) : \sup_{x\in X} |f(x)|\tau(x) < \infty \right\}.$$

Then $C^b(X;\tau)$ is a commutative Banach algebra with norm

$$\|f\|_{\infty,\tau} = \sup_{x\in X} |f(x)|\tau(x) \qquad \big(f \in C^b(X;\tau)\big).$$

Put

$$C_0(X;\tau) = \left\{ f \in C^b(X;\tau) : \lim_{K\in\mathcal{K}(X)} \sup_{x\notin K} |f(x)|\tau(x) = 0 \right\}.$$

If τ is upper semicontinuous and $\sup_{x\in X} \tau(x) = \infty$, then $C_0(X;\tau)$ belongs to $\mathcal{B}^3_{\text{type I}}$.

Example 6. Let X be a locally compact Hausdorff space. Let S_1 and S_2 be two Segal algebras in $C_0(X)$. Then $S_1 \cap S_2$ becomes a Segal algebra in $C_0(X)$ with norm $\|f\|_{S_1} + \|f\|_{S_2}$ $(f \in S_1 \cap S_2)$. We denote by $S_1 \wedge S_2$ such a Segal algebra in $C_0(X)$. Also, we denote by $S_1 \times S_2$ the usual product algebra of S_1 and S_2, that is, the Lau algebra in case that T is the zero homomorphism.

(i) *If $\tau(x) = |x|^{\alpha}+1$ $(x \in \mathbb{R}^n), 1 \leq p < n/\alpha$ and $0 < \alpha < n$, then $C_0(\mathbb{R}^n;\tau) \wedge C_{0,p}(\mathbb{R}^n)$ belongs to $\mathcal{B}^0_{\text{type I}}$.*

(ii) *The Banach algebras $C_{0,p}(X,\mu) \times C_{0,p,\{x_i\}}(X)$ and $C_{0,p}(X,\mu) \times C_0(X;\tau)$ belong to $\mathcal{B}^0_{\text{type I}}$, where τ is an upper semicontinuous function on X with $\sup_{x\in X} \tau(x) = \infty$.*

In order to complete our research on classification, we want to solve the isomorphism problem for $i = 1, 2, 3, 0$: Is every algebra in $\mathcal{B}^i_{\text{type I}}$ precisely isomorphic to any kind of Banach algebra? Indeed, if $i \neq j$, then each algebra in $\mathcal{B}^i_{\text{type I}}$ is not isomorphic to any algebra in $\mathcal{B}^{i}_{\text{type I}}$. For $i = 1$, we solved this problem in Theorem 3. This theorem provides the correspondence between Banach algebras in $\mathcal{B}^{1}_{\text{type I}}$ and locally compact Hausdorff spaces. For $i = 2, 3, 0$, it seems to be difficult to solve the isomorphism problem. That reminds us of a Japanese proverb "Hi kurete, michi tōshi (The day is short, and the work is much; My goal is still a long way off.)"

Note. In this announcement we wrote only the results without proofs. In the near future, we will prepare the manuscript containing the detail and submit it to some other journal.

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