Title: METRIZABILITY OF SPACES OF LIPSCHITZ FUNCTIONS (Researches on isometries from various viewpoints)

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METRIZABILITY OF SPACES OF LIPSCHITZ FUNCTIONS

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1. INTRODUCTION

Let Lip₀(E) be the linear space of all scalar-valued Lipschitz functions vanishing at 0 on a normed space E. Let τ be a locally convex topology on Lip₀(E) such that τ₀ ≤ τ ≤ τ₃, where τ₀ and τ₃ denote the compact-open topology and the Nachbin–Couré topology on Lip₀(E).

We prove in this note that (Lip₀(E), τ₀) is a metrizable space if and only if E has finite dimension. Motivated by a positive answer in the setting of holomorphic mappings, the following question is raised: Is it true that (Lip₀(E), τ) is metrizable only if E is finite-dimensional?

2. PRELIMINARIES

Let E be a normed space and let Lip₀(E) denote the linear space of all Lipschitz mappings f from E into K for which f(0) = 0. We refer the reader to Weaver’s book [6] for the basic theory of Lip₀(E).

Let X be a topological space and let C(X) be the linear space of all continuous mappings from X into K. We recall the following topologies on C(X).

The compact-open topology on C(X) is the locally convex topology generated by the seminorms

\[ |f|_K = \sup_{x \in K} |f(x)|, \quad f \in C(X), \]

where K varies over the family of all compact subsets of X.

A seminorm p on C(X) is ported by the compact subset K of X if for every open neighborhood V of K in X, there is a constant cᵥ > 0 such that \( p(f) \leq cᵥ \sup_{x \in V} |f(x)| \) for all \( f \in C(X) \). The Nachbin topology on C(X) is the locally convex topology generated by the seminorms on C(X) which are ported by the compact subsets of X.

The Nachbin–Couré topology on C(X) is the locally convex topology generated by the seminorms p on C(X) which satisfy the following property: for each increasing countable open cover \( \{V_n\}_{n \in \mathbb{N}} \) of X, there are \( m \in \mathbb{N} \) and \( c_m > 0 \) such that \( p(f) \leq c_m \sup_{x \in V_n} |f(x)| \) for all \( f \in C(X) \).

We will denote by τ₀, τ₁, and τ₃ the compact-open topology, the Nachbin-ported topology and the Nachbin–Couré topology on C(X), or on any linear subspace of C(X).

Now we prove the following result.

**Theorem 2.1.** If E is a Banach space, then (Lip₀(E), τ₀) is metrizable if and only if E has finite dimension.

**Proof.** Suppose that (Lip₀(E), τ₀) is metrizable. Then there exists a sequence \( \{K_n\}_{n \in \mathbb{N}} \) of compact subsets of E, containing the origin, such that the sequence of seminorms \( |\cdot|_{K_n} \) defines the topology τ₀ on Lip₀(E).

We claim that there exists a constant \( c > 0 \) such that E is included in \( \bigcup_{n \in \mathbb{N}} \text{cl}(K_n) \), where \( \text{cl}(K_n) \) denotes the closed, convex, balanced hull of \( K_n \) in E. Indeed, given \( x \in E \), it is clear that \( |(x)|_{K_n} \) defined on Lip₀(E) is a continuous seminorm on (Lip₀(E), τ₀), so there are \( m \in \mathbb{N} \) and \( c > 0 \) such that \( |f(x)| \leq c|f|_{K_m} \) for all \( f \in \text{Lip}_0(E) \). It follows that \( |f(x)| \leq c|f|_{\text{cl}(K_m)} \) for all \( f \in \text{Lip}_0(E) \). Notice that each \( \text{cl}(K_n) \) is compact by the Mazur theorem. Since the dual space \( E' \) is a subset of Lip₀(E), we have \( |f(x)| \leq c|f|_{\text{cl}(K_m)} \) for all \( f \in E' \). By the Hahn–Banach separation theorem, we infer that \( x \) is in \( \text{cl}(K_m) \) as we wanted. Since E is a Baire space, our claim implies that there exists \( p \in \mathbb{N} \) such that \( \text{cl}(K_p) \) has no empty interior in E. Hence there is a compact neighborhood of 0 in E and therefore E has finite dimension by the Riesz theorem.

Conversely, if E is finite dimensional, then (C(E), τ₀) is metrizable (see the proof of [3, Theorem 16.9]) and therefore so is (Lip₀(E), τ₀).

The results on the metrizabilities of spaces of holomorphic functions have an interesting history. In 1968, Alexander [1] proved the following theorem for Banach spaces with Schauder basis, which was generalized by Chae (see [3, Theorem 16.10]): If U is an open subset of an infinite dimensional Banach space E and
\( \tau \) is a topology on the space \( H(U) \) of all holomorphic functions on \( U \) finer than the topology of pointwise convergence, then \( (H(U), \tau) \) is not metrizable.

In 2007, this theorem probably motivated Ansemil and Ponte, whose paper [2] contains that if \( U \) is an open subset of an infinite-dimensional complex metrizable locally convex space \( E \), then \( (H(U), \tau_0) \) is not metrizable. This answered a question stated by Mujica in [5, Problem 11.9] thirty years ago. It is known that \( \tau_0 \leq \tau \leq \tau_5 \) on \( H(U) \).

In 2009, López-Salazar [4] improved this result showing that if \( U \) is an open subset of a complex metrizable locally convex space \( E \) and \( \tau \) is a locally convex topology on \( H(U) \) such that \( \tau_0 \leq \tau \leq \tau_5 \), then \( (H(U), \tau) \) is a metrizable space if and only if \( E \) has finite dimension.

Theorem 2.1 suggests to tackle the problem on the metrizability of \( \text{Lip}_0(E) \) equipped with other topologies, with an approach similar to that described above for spaces of holomorphic functions.

**REFERENCES**


