数理解析研究所講究録
第2035巻 2017年 168-172

HERMITIAN OPERATORS ON BANACH ALGEBRAS OF VECTOR-VALUED LIPSCHITZ MAPS

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ABSTRACT. Let H be a complex Hilbert space and $[\cdot,\cdot]$ an inner-product on H. A bounded linear operator T on H is a Hermitian operator if $[Tx,x]\in\mathbb{R}$ for each $x\in H$. In 1961, the Hermitian operator on a normed vector space was defined by means of the semi-inner product defined by Lumer [6]. Hermitian operators and their applications have been studied by many authors; a few of them are [1, 2, 5, 6, 7]. We exhibit forms of Hermitian operators on certain semisimple commutative Banach algebras.

1. INTRODUCTION

The notion of a Hermitian operator on a Banach space dates back to the seminal papers by Vidav [8] and Lumer [6]. Lumer considered a definition in terms of a semi-inner product.

Definition 1. Let V be a complex Banach space with the norm $\|\cdot\|_V$. A semi-innner product $[\cdot,\cdot]$ on V is a function from $V\times V$ into $\mathbb{C}$ with the following properties;

(1) $[u+v,w]=[u,w]+[v,w]$,
$[\lambda u,v]=\lambda[u,v]$ for $u,v,w\in V,\lambda\in\mathbb{C}$.
(2) $[v,v]\geq 0$ for all $v\in V$ and $[v,v]\neq 0$ if $v\neq 0$.
(3) $|[u,v]|^2\leq[u,u][v,v]$ for $u,v\in V$.

In addition, if $[v,v]=\|v\|_v^2$ for every v in V, then $[\cdot,\cdot]$ is said to be a semi-inner product compatible with the norm of V.

In this note we abbreviate a semi-inner product compatible with the norm as a semi-inner product.

Definition 2. Let $[\cdot,\cdot]$ be a semi-inner product on a complex Banach space V. Then a bounded linear operator T on V is said to be a Hermitian operator if $[Tv,v]\in\mathbb{R}$ for all $v\in V$.

It is well-known that any Banach space has a semi-inner product, which needs not to be unique. We note that the above definition of a Hermitian operator is independent of the semi-inner product chosen.

2. KNOWN RESULTS FOR HERMITIAN OPERATORS AND THE MAIN THEOREM

2.1. **Known results.** Let B be a unital Banach algebra. For each $a\in B$, M_a denotes the multiplication operator on B, which is defined by $M_a=a\cdot I$ with the identity operator I on B. We introduce a Hermitian element.

Definition 3. Let B be a unital Banach algebra. The numerical range of $a \in B$ is

$$V(a) := \{f(a); \|f\| = f(\mathbf{1}) = 1, f \in B^*\}.$$

Then $a \in B$ is said to be a Hermitian element if and only if $V(a) \subset \mathbb{R}$.

First proposition in this section summarizes some of the properties of Hermitian operators and Hermitian elements. In many situations this equivalent statements plays a pivotal role. The following is due to Theorem 5.2.6 in [3].

Proposition 2.1. *Let T be a bounded linear operator on a Banach space V. Then the following are equivalent.*

(1) *T is a Hermitian operator*
(2) $\|\exp(itT)\|_v = 1$ *for any* $t \in \mathbb{R}$
(3) $\exp(itT)$ *is an isometry for any* $t \in \mathbb{R}$
(4) *T is a Hermitian element in $\mathfrak{B}(V)$, which stands for the space of all bounded linear operators on V equipped with the operator norm.*

Proposition 2.2. *Let B be a unital Banach algebra. If $a \in B$ is a Hermitian element, then the multiplication operator M_a is a Hermitian operator on B.*

Proof. Let $a \in B$ be a Hermitian element. It is well-known that an element $a \in B$ is Hermitian if and only if $\|\exp(ita)\|_B = 1$ for any $t \in \mathbb{R}$. Thus, we deduce that

$$\|\exp(ita \cdot I)\| = 1$$

for all $t \in \mathbb{R}$. Applying Proposition 2.1, we conclude that M_a is a Hermitian operator on B. □

We are interested in a problem that under which circumstances the converse statement of Proposition 2.2 holds; when is a Hermitian operator on a unital Banach algebra a multiplication operator? Our purpose of this note is to give a partial answer to the problem. Now we recall two observations about Hermitian operators.

Theorem 4. [2, Theorem 4] *Let X be a compact Hausdorff space and E a complex Banach space. Suppose that $C(X, E)$ is the Banach space of all continuous functions on X with values in E with the supremum norm. A bounded linear operator T on $C(X, E)$ is a Hermitian operator if and only if for each $x \in X$ there is a Hermitian operator $A(x)$ on E such that for any $F \in C(X, E)$ we have*

$$TF(x) = A(x)F(x) \quad x \in X.$$

Theorem 5. [1, Theorem 3.1] *Let X be a compact metric space and* $\mathrm{Lip}(X)$ *a complex Banach algebra of complex-valued Lipschitz functions with the norm $L(\cdot) + \|\cdot\|_\infty$. A bounded linear operator T on* $\mathrm{Lip}(X)$ *is a Hermitian operator if and only if $T = \lambda \cdot I$ with $\lambda \in \mathbb{R}$.*

2.2. **The main theorem.** The following is the main theorem in this note.

Theorem 6. *Let B be a unital semisimple commutative Banach algebra. Suppose that every surjective unital isometry on B is multiplicative. If a bounded complex-linear operator T is a Hermitian operator, then*

$$T = M_{T(\mathbf{1})}$$

3. A proof of the main theorem

Proposition 3.1. *Let B be a unital Banach algebra. Suppose that T is a Hermitian operator on B. Then $T(\mathbf{1})$ is a Hermitian element in B.*

Proof. This proof is based on Lemma 3.2 in [1]. For any $f \in B^*$ with $\|f\| = f(1) = 1$, we define $\Phi_f : \mathfrak{B}(B) \to \mathbb{C}$ by

$$\Phi_f(S) = f(S(\mathbf{1})) \quad (S \in \mathfrak{B}(B)).$$

We infer that Φ_f is a bounded linear functional on $\mathfrak{B}(B)$ and satisfies $\|\Phi_f\| = \Phi_f(I) = 1$. Since T is a Hermitian element, this implies

$$f(T(\mathbf{1})) = \Phi_f(T) \in \mathbb{R}$$

for any $f \in B^*$ with $\|f\| = f(\mathbf{1}) = 1$. Thus, we obtain that $T(\mathbf{1})$ is a Hermitian element of B. □

Proposition 3.2. *Let T be a bounded complex linear operator on a unital semisimple commutative Banach algebra B. Then the following are equivalent.*

(1) $T = M_{T(\mathbf{1})}$

(2) $\exp(it(T - M_{T(\mathbf{1})}))$ *is multiplicative for every* $t \in \mathbb{R}$

Proof. Suppose that $T = M_{T(\mathbf{1})}$. Clearly we have

$$\exp(it(T - M_{T(\mathbf{1})})) = I$$

for every $t \in \mathbb{R}$.

In order to prove the converse, suppose that $\exp(it(T - M_{T(\mathbf{1})}))$ is multiplicative for every $t \in \mathbb{R}$. We define $H = T - M_{T(\mathbf{1})}$ and $U_t = \exp(itH)$ for every $t \in \mathbb{R}$. Differentiating U_t at $t = 0$, we get

$$U_t'|_{t=0}(ab) = iH(ab),$$

for any $a, b \in B$. As U_t is multiplicative, for any $a, b \in B$, we get

$$U_t'|_{t=0}(ab) = iaH(b) + iH(a)b.$$

It follows that H is a bounded derivation. By a theorem of Singer and Wermer, we observe that $H = 0$. □

We now proceed with the details for the proof of our main theorem.

A proof of the main theorem. Let T be a Hermitian operator on B. Applying Proposition 3.1, $T(\mathbf{1})$ is a Hermitian element of B. According to Proposition 2.2, we see that $M_{T(\mathbf{1})}$ and $T - M_{T(\mathbf{1})}$ are Hermitian operators on B. Therefore, $\exp it(T - M_{T(\mathbf{1})})$ is a unital surjective isometry for any $t \in \mathbb{R}$. By the assumption, every surjective unital isometry on B is multiplicative, thus $\exp it(T - M_{T(\mathbf{1})})$ is multiplicative. Hence Proposition 3.2 provides that $T = M_{T(\mathbf{1})}$.

□

4. Applications of the Main theorem

Let us begin with a definition of a vector-valued Lipschitz algebra. Let X be a compact metric space and A a uniform algebra on a compact Hausdorff space Y. A map F from X into A is said to be Lipschitz if it satisfies the following inequality

$$L(F) := \sup_{x \neq y \in X} \frac{\|f(x) - f(y)\|_\infty}{d(x, y)} < \infty.$$

The set of all Lipschitz maps is denoted by

$$\mathrm{Lip}(X,A) := \{F : X \to A;\ L(F) < \infty\}$$

and this is a unital semisimple commutative Banach algebra with the norm of $\|\cdot\|_L = \|\cdot\|_\infty + L(\cdot)$.

We exhibit a theorem of Jarosz in [4]. Let E be a linear subspace of $C(X)$ which separates the points in X and contains constants. We denote by $\mathrm{Ch}(E)$ the Choquet boundary for E. We call E a regular subspace of $C(X)$ if for any $\epsilon > 0$, $x_0 \in \mathrm{Ch}(E)$, and open neighborhood U of x_0, there exists an $f \in E$ with $\|f\|_\infty \leq 1+\epsilon$, $f(x_0) = 1$, $|f(x)| < \epsilon$ for $x \in X \setminus U$.

Theorem 7. [4] *Let X and Y be compact Hausdorff spaces. Let A, B be complex linear subspaces of $C(X)$ and $C(Y)$ respectively. We assume that A and B satisfy the following;*

(1) *A and B contain constant functions.*
(2) *A and B have $\|\cdot\|_A$ and $\|\cdot\|_B$:p-norm, q-norm.*
(3) *A and B are regular subspaces.*

If a bounded linear operator T from $(A, \|\cdot\|_A)$ onto $(B, \|\cdot\|_B)$ with $T(\mathbf{1}) = \mathbf{1}$ is a surjective linear isometry, then T is an isometry from $(A, \|\cdot\|_\infty)$ onto $(B, \|\cdot\|_\infty)$

Applying Theorem 7 by considering Lipschitz algebra $\mathrm{Lip}(X,A)$ to be a subspace of $C(X \times Y)$, we get the following corollary.

Corollary 8. *If U is a linear isometry from $\mathrm{Lip}(X,A)$ onto $\mathrm{Lip}(Y,B)$ with $U(\mathbf{1}) = \mathbf{1}$ then U is also an isometry with the supremum norm.*

Now, we give a characterization of Hermitian operators on vector-valued Lipschitz algebras.

Theorem 9. *Let X be a compact metric space and A be a uniform algebra. A bounded linear operator $T : \mathrm{Lip}(X,A) \to \mathrm{Lip}(X,A)$ is a Hermitian operator if and only if there exists a real-valued function $a \in A$ with $T(\mathbf{1}) = \mathbf{1} \otimes a$ such that*

$$T = M_{T(\mathbf{1})}.$$

Proof. Actually real-valued function $a \in A$ is a Hermitian element of A. Therefore, for a real-valued function $a \in A$, we see that $T(\mathbf{1}) = \mathbf{1} \otimes a$ is a Hermitian element of $\mathrm{Lip}(X,A)$. Applying Proposition 2.2, we get $T = M_{T(\mathbf{1})}$ is a Hermitian operator.

Now we consider the converse. Using Corollary 8, every surjective unital isometry on $\mathrm{Lip}(X,A)$ with the norm of $\|\cdot\|_L$ is an isometry with the supremum norm. Moreover, Nagasawa's theorem shows that it is also multiplicative. Thus, Theorem 6 follows every Hermitian operator on $\mathrm{Lip}(X,A)$ is a multiplication operator. □

Remark 10. As corollaries of Theorem 6, we also have Theorem 4 in [2] and Theorem 3.1 in [1].

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