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WHITTAKER FUNCTIONS ON $\mathrm{Sp}(2,\mathbf{R})$ AND ARCHIMEDEAN ZETA INTEGRALS

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1. Introduction

Let $\mathsf{G} = \mathrm{GSp}(2) = \{g \in \mathrm{GL}(4) \mid {}^t gJg = \nu(g)J \text{ for some } \nu(g) \in \mathrm{GL}(1)\}$, $J = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix}$ and $\Pi = \otimes_v' \Pi_v$ be a cuspidal automorphic representation of $\mathsf{G}(\mathbf{A})$ with $\mathbf{A} = \mathbf{A}_{\mathbf{Q}}$. We take a maximal unipotent subgroup N_0 of G by

$$\mathsf{N}_0 = \{n(x_0,x_1,x_2,x_3) = \left(\begin{array}{cc|cc} 1 & x_0 & & \\ & 1 & & \\ \hline & & 1 & \\ & & -x_0 & 1 \end{array}\right) \left(\begin{array}{cc|cc} 1 & & x_1 & x_2 \\ & 1 & x_2 & x_3 \\ \hline & & 1 & \\ & & & 1 \end{array}\right) \in \mathsf{G}\}.$$

We fix a nontrivial additive character $\psi = \Pi_v \psi_v$: $\mathbf{A}/\mathbf{Q} \to \mathbf{C}^{(1)}$, and define a nondegenerate unitary character ψ_{N_0} of $\mathsf{N}_0(\mathbf{A})$ by $\psi_{\mathsf{N}_0}(n(x_0,x_1,x_2,x_3)) = \psi(x_0 + x_3)$. For a cusp form $\varphi \in \Pi$, the global Whittaker function W_φ is defined by

$$W_\varphi(g) = \int_{\mathsf{N}_0(\mathbf{Q})\backslash \mathsf{N}_0(\mathbf{A})} \varphi(ng)\psi_{\mathsf{N}_0}(n^{-1})dn.$$

We assume that $W_\varphi \neq 0$ for some $\varphi \in \Pi$, that is, Π is (globally) generic. Then each local component Π_v is generic representation of $\mathsf{G}(\mathbf{Q}_v)$, that is,

$$\dim_{\mathbf{C}} \mathrm{Hom}_{\mathsf{G}(\mathbf{Q}_v)}(\Pi_v, \mathrm{Ind}_{\mathsf{N}_0(\mathbf{Q}_v)}^{\mathsf{G}(\mathbf{Q}_v)}(\psi_v)) = 1.$$

According to a result of Vogan [18], an irreducible generic representation Π_∞ of $\mathrm{GSp}(2,\mathbf{R})$ is isomorphic to one of the following:

- a (limit) of large discrete series representation;
- an irreducible principal series representation induced from proper parabolic subgroups $P_i = \mathsf{P}_i(\mathbf{R})$ $(i = 0,1,2)$ of $\mathsf{G}(\mathbf{R})$ where

$$\mathsf{P}_0 = \{\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & * & * \end{pmatrix} \in \mathsf{G}\},\quad \mathsf{P}_1 = \{\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & * & * & * \end{pmatrix} \in \mathsf{G}\},\quad \mathsf{P}_2 = \{\begin{pmatrix} * & * \\ 0_2 & * \end{pmatrix} \in \mathsf{G}\}.$$

For each generic representation above explicit formulas for Whittaker functions (at certain K-types) have been studied by several authors:

- Large discrete series / P_1-principal series: Oda [16] (LDS) and Miyazaki and Oda [10] (P_1) obtained system of partial differential equations for Whittaker functions, and gave explicit integral expressions for moderate growth Whittaker functions. Moriyama [12] gave another integral expression.
- P_0-principal series: Niwa [15] gave explicit formulas for class one principal series Whittaker functions. For general principal series, Miyazaki and Oda [11] obtained

a system of partial differential equations. The author [4] solved the system to get explicit integral expressions.

- P_2-principal series: Hasegawa [3] found a system of partial differential equations. Explicit integral expressions for Whittaker functions are given by the author [7].

Here is an application of explicit formulas to archimedean zeta integrals:

- Novodvorsky's zeta integrals: Moriyama [13] computed in the cases of large discrete series and P_1-principal series, to show the entireness of spinor L-functions and functional equations. Moriyama and the author [8] discussed P_0-case. The remaining P_2-case is treated in [7].
- Bump-Friedberg-Ginzburg zeta integrals [2]: This zeta integral contains two complex variables. In [2], it is shown that unramified zeta integrals become product of the standard and the spinor L-functions. At the archimedean places, the cases of class one principal series and large discrete series are treated in [5] and [6], respectively. The remaining cases are recently done by the author.

2. Representation theory of $\mathrm{GSp}(2,\mathbf{R})$

2.1. group structures. Let $G = \mathsf{G}(\mathbf{R}) = \mathrm{GSp}(2,\mathbf{R})$ and $G_0 = \mathrm{Sp}(2,\mathbf{R}) = \{g \in G \mid \nu(g) = 1\}$. We fix a maximal compact subgroup K (resp. K_0) of G (resp. G_0) by $K = G \cap \mathrm{O}(4)$ (resp. $K_0 = G_0 \cap \mathrm{O}(4)$) with $\mathrm{O}(4) = \{g \in \mathrm{GL}(4,\mathbf{R}) \mid {}^t gg = 1_4\}$. Then K_0 is isomorphic to the unitary group $\mathrm{U}(2) = \{g \in \mathrm{GL}(2,\mathbf{C}) \mid {}^t\bar{g}g = 1_2\}$ of degree two via the homomorphism

$$\kappa : \mathrm{U}(2) \ni A + \sqrt{-1}B \mapsto k_{A,B} := \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in K_0,$$

and we know $K = \{k_{A,B}, \gamma_0 k_{A,B} \mid A + \sqrt{-1}B \in \mathrm{U}(2)\}$ with $\gamma_0 := \mathrm{diag}(-1,-1,1,1)$.

2.2. Whittaker functions. A unitary character of the maximal unipotent subgroup $N_0 = \mathsf{N}_0(\mathbf{R})$ of G is of the form

$$\psi_{(c_0,c_3)}(n(x_0,x_1,x_2,x_3)) = \exp\{2\pi\sqrt{-1}(c_0x_0 + c_3x_3)\}$$

with real numbers c_0 and c_3. We assume that $\psi_{(c_0,c_3)}$ is nondegenerate, that is, $c_0c_3 \neq 0$. For a nondegenerate unitary character ψ of N_0, we denote by $C^\infty(N_0\backslash G, \psi)$ the space of smooth functions on G satisfying $f(ng) = \psi(n)f(g)$, forall $(n,g) \in N_0 \times G$. By the right translation the space $C^\infty(N_0\backslash G,\psi)$ becomes smooth $(\mathfrak{g}_\mathbf{C}, K)$-module ($\mathfrak{g}_\mathbf{C}$ is the complexification of the Lie algebra of G). We denote by $C^\infty_{\mathrm{mg}}(N_0\backslash G,\psi)$ the subspace of $C^\infty(N_0\backslash G,\psi)$ consisting of moderate growth functions on G. Let (π, H_π) be an irreducible admissible representation of G. Wallach's multiplicity one theorem [19] asserts that

$$\dim_\mathbf{C} \mathrm{Hom}_{(\mathfrak{g}_\mathbf{C},K)}(H_{\pi,K}, C^\infty_{\mathrm{mg}}(N_0\backslash G,\psi)) \leq 1.$$

Here $H_{\pi,K}$ means the space of K-finite vectors in H_π. For a nonzero intertwining operator $\Phi \in \mathrm{Hom}_{(\mathfrak{g}_\mathbf{C},K)}(H_{\pi,K}, C^\infty_{\mathrm{mg}}(N_0\backslash G,\psi))$ and a function $f \in H_{\pi,K}$, we call the image $\Phi(f)$ *(moderate growth) Whittaker function corresponding to* f, and denote by

$$\mathcal{W}(\pi,\psi) = \{\Phi(f) \mid \Phi \in \mathrm{Hom}_{(\mathfrak{g}_\mathbf{C},K)}(H_{\pi,K}, C^\infty_{\mathrm{mg}}(N_0\backslash G,\psi)), f \in H_{\pi,K}\}.$$

Let (τ, V_τ) be a K-type of (π, H_π). For $v \in V_\tau$, we denote by $W(v;*) \in \mathcal{W}(\pi,\psi)$ the image of v under K-embedding $V_\tau \to \mathcal{W}(\pi,\psi)$. Since we have

$$W(v;ngk) = \psi(n)W(\tau(k)v;g), \quad \forall (n,g,k) \in N_0 \times G \times K,$$

the Iwasawa decomposition $G = N_0AK$ implies that $W(v;*)$ is determined by its restriction $W(v;*)|_A$ to A, where $A = \{z\mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \mid z, a_1, a_2 > 0\}$. We call $W(v;*)|_A$ the *radial part* of $W(v;*)$.

2.3. **Representation theory of K.** Let $(\tau_\lambda^0, V_\lambda^0)$ be the irreducible finite dimensional representation of U(2) with highest weight $\lambda = (\lambda_1, \lambda_2)$, $(\lambda_1 \geq \lambda_2)$. Here $V_\lambda^0 = \{f \in \mathbf{C}[x_1, x_2] \mid \text{homogeneous}, \deg(f) = \lambda_1 - \lambda_2\}$ on which U(2) acts by $(\tau_\lambda^0(k)f)(x_1, x_2) = (\det k)^{\lambda_2} f((x_1, x_2)\cdot k)$ $(k \in \mathrm{U}(2),\ f \in V_\lambda^0)$. Via the isomorphism $\kappa : \mathrm{U}(2) \cong K_0$, we regard τ_λ^0 as a representation of K_0.

Let $\{v_i^{\lambda,0} \equiv v_i^0 = x_1^i x_2^{\lambda_1-\lambda_2-i} \mid 0 \leq i \leq \lambda_1 - \lambda_2\}$ be the standard basis of V_λ^0. We define U(2)-invariant inner product $\langle\ ,\ \rangle$ on V_λ^0 by $\langle v_i^0, v_j^0\rangle = \delta_{i,j}\binom{\lambda_1-\lambda_2}{i}^{-1}$. For $\lambda = (\lambda_1, \lambda_2)$, we put $\lambda^* = (-\lambda_2, -\lambda_1)$. Then the contragredient representation of τ_λ^0 is isomorphic to $\tau_{\lambda^*}^0$. We introduce a new basis $\{w_i^{\lambda,0} \equiv w_i^0 \mid 0 \leq i \leq \lambda_1 - \lambda_2\}$ by

$$w_{2j+\delta}^0 = \begin{cases} (x_1x_2)^\delta (x_1^2 + x_2^2)^{(\lambda_1-\lambda_2)/2-j-\delta}(x_2^2 - x_1^2)^j & \text{if } \lambda_1 - \lambda_2 \in 2\mathbf{Z}_{\geq 0}, \\ x_1^\delta x_2^{1-\delta}(x_1^2 + x_2^2)^{(\lambda_1-\lambda_2-1)/2-j}(x_2^2 - x_1^2)^j & \text{if } \lambda_1 - \lambda_2 \in 2\mathbf{Z}_{\geq 0} + 1 \end{cases}$$

with $\delta \in \{0, 1\}$.

Let $\tau_\lambda = \mathrm{Ind}_{K_0}^K \tau_\lambda^0$. Then τ_λ is irreducible if and only if $\lambda \neq \lambda^*$. In that case a basis of the representation space V_λ of τ_λ is $\{v_i, v_i^* \mid 0 \leq i \leq \lambda_1 - \lambda_2\}$ where the K-action is given by

$$\tau_\lambda(k_{A,B})v_i = \sum_{j=0}^{\lambda_1-\lambda_2} c_{ij}^\lambda(k_{A,B})v_j, \quad \tau_\lambda(k_{A,B})v_i^* = \sum_{j=0}^{\lambda_1-\lambda_2} c_{ij}^{\lambda^*}(k_{A,B})v_j^*,$$
$$\tau_\lambda(\gamma_0)v_i = (-1)^i v_{\lambda_1-\lambda_2-i}^*, \qquad \tau_\lambda(\gamma_0)v_i^* = (-1)^{\lambda_1-\lambda_2-i}v_{\lambda_1-\lambda_2-i},$$

where $c_{ij}^\lambda(k_{A,B}) = \langle\tau_\lambda^0(k_{A,B})v_i^{\lambda,0}, v_j^{\lambda,0}\rangle/\langle v_j^{\lambda,0}, v_j^{\lambda,0}\rangle$. Similarly we introduce another basis $\{w_i, w_i^* \mid 0 \leq i \leq \lambda_1 - \lambda_2\}$ of V_λ from the basis $\{w_i^0 \mid 0 \leq i \leq \lambda_1 - \lambda_2\}$ of V_λ^0.

When $\lambda = \lambda^*$, τ_λ has an irreducible decomposition $\tau_\lambda = \tau_\lambda^+ \oplus \tau_\lambda^-$. A basis of the representation space $V_\lambda^\pm$ of $\tau_\lambda^\pm$ is $\{v_i^\pm \mid 0 \leq i \leq \lambda_1 - \lambda_2 = 2\lambda_1\}$ where the K-action is given by

$$\tau_\lambda^\pm(k_{A,B})v_i^\pm = \sum_{j=0}^{\lambda_1-\lambda_2} c_{ij}^\lambda(k_{A,B})v_j^\pm, \quad \tau_\lambda^+(\gamma_0)v_i^+ = (-1)^i v_{2\lambda_1-i}^+, \quad \tau_\lambda^-(\gamma_0)v_i^- = (-1)^{i+1}v_{2\lambda_1-i}^-.$$

We denote by $\iota_\pm$ the isomorphism $V_{(\lambda_1,-\lambda_1)}^\pm \cong V_{(\lambda_1,-\lambda_1)}^0$ of $\mathbf{C}$-vector spaces given by $\iota_\pm(v_i^\pm) = v_i^0$.

2.4. **P_2-principal series representations.** Let $P_2 = \mathsf{P}_2(\mathbf{R}) = M_2A_2N_2$ be Siegel parabolic subgroup of G with $M_2 = \{\begin{pmatrix} \pm m & \\ & {}^t m^{-1}\end{pmatrix} \mid m \in \mathrm{SL}^\pm(2, \mathbf{R})\}$, $A_2 = \{z\,\mathrm{diag}(a_1, a_1, a_1^{-1}, a_1^{-1}) \mid z, a_1 > 0\}$, and $N_2 = \mathsf{N}_2(\mathbf{R})$. Let ε be a character of the group $\{1, \gamma_0\}$. We denote by $D_n = \mathrm{Ind}_{\mathrm{SL}(2,\mathbf{R})}^{\mathrm{SL}^\pm(2,\mathbf{R})}(D_n^+)$ where D_n^+ is the discrete series representation of $\mathrm{SL}(2, \mathbf{R})$ with Blattner parameter $n(\geq 2)$. For $c, \nu \in \mathbf{C}$, we define a quasi-character $\chi_{c,\nu}$ by $\chi_{c,\nu}(z\,\mathrm{diag}(a_1, a_1, a_1^{-1}, a_1^{-1})) = z^c a_1^{\nu+3}$. From the data above, we define P_2-principal series representation by $\pi_{\varepsilon,n,c,\nu} = \mathrm{Ind}_{P_2}^G((\varepsilon \otimes D_n) \otimes \chi_{c,\nu} \otimes 1_{N_2})$.

Via the Langlands parameters of P_2-principal series representation $\pi = \pi_{\varepsilon,n,c,\nu}$, we define L- and ε-factors for π by

$$L(s, \pi, \mathrm{spin}) = \Gamma_{\mathbf{R}}\Big(s + \frac{c+\nu}{2} + \delta_1\Big)\Gamma_{\mathbf{R}}\Big(s + \frac{c-\nu}{2} + \delta_2\Big)\Gamma_{\mathbf{C}}\Big(s + \frac{c+n-1}{2}\Big),$$

$$
\begin{aligned}
L(s,\pi,\mathrm{std}) &= \Gamma_{\mathbf{R}}(s)\Gamma_{\mathbf{R}}\Big(s+\frac{\nu+n-1}{2}\Big)\Gamma_{\mathbf{R}}\Big(s+\frac{-\nu+n-1}{2}\Big),\\
\varepsilon(s,\pi,\psi_\infty,\mathrm{spin}) &= (\sqrt{-1})^{\delta_1+\delta_2+n},\\
\varepsilon(s,\pi,\psi_\infty,\mathrm{std}) &= (-1)^n
\end{aligned}
$$

where $\delta_i \in \{0,1\}$ $(i=1,2)$ are determined by $(-1)^{\delta_1} = \varepsilon(\gamma_0)$ and $(-1)^{\delta_2} = (-1)^n\varepsilon(\gamma_0)$. Here we denote by $\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, $\Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$, and $\psi_\infty(x) = \exp(2\pi\sqrt{-1}x)$, $(x \in \mathbf{R}^\times)$.

3. Explicit formulas for Whittaker functions

We describe P_2-principal series Whittaker functions at certain multiplicity one K-types. More precisely we consider Whittaker functions at the following K-types.

- $n=2m$ and $\varepsilon(\gamma_0)(-1)^m = \pm 1$: $\tau^{\pm}_{(m,-m)}$;
- $n = 2m+1$: $\tau_{(m+1,-m)}$.

Hasegawa [3] obtained a system of partial differential equations for Whittaker functions belonging to the above K-types. For simplicity we assume $c_0 = c_3 = 1$ for $\psi_{(c_0,c_3)} \in \hat{N}_0$.

Proposition 3.1. ([3]) *Let*

$$W(v_i^{(m,-m),\pm}; z\,\mathrm{diag}(a_1,a_2,a_1^{-1},a_2^{-1})) = z^c a_1^2 a_2\,\varphi_i(a_1,a_2),\ \ (0 \le i \le n = 2m)$$

be the radial part of Whittaker function at K-type $\tau^{\pm}_{(m,-m)}$. If we set $y_1 = \pi a_1/a_2$, $y_2 = \pi a_2^2$, then $\{\varphi_i(y_1,y_2) \mid 0 \le i \le 2m\}$ satisfies the following.

- $(2\partial_2 - 2m + 1)(\varphi_i + \varphi_{i+2}) + 4y_2(\varphi_i - \varphi_{i+2}) = 0;$
- $(2\partial_1 - 2\partial_2 - i + 1)(\varphi_i - \varphi_{i+2}) + 2(-2y_2 + m - i - 1)(\varphi_i + \varphi_{i+2}) - 8\sqrt{-1}y_1\varphi_{i+1} = 0;$
- $\{\partial_1^2 + 2\partial_2^2 - 2\partial_1\partial_2 - 4y_1^2 - 8y_2^2 + 4(m-i)y_2 - \frac{1}{4}(\nu^2 + (2m-1)^2)\}\varphi_i$
  $-2\sqrt{-1}y_1\{(2m-i)\varphi_{i+1} - i\varphi_{i-1}\} = 0,$

where $\partial_i = y_i\frac{\partial}{\partial y_i}$.

Here is a Mellin-Barnes integral representation for P_2-principal series Whittaker function at the K-type $\tau^{\pm}_{(m,-m)}$. A convenience basis is $\{w_i^{(m,-m),\pm} \mid 0 \le i \le 2m\}$.

Theorem 3.2. ([7], *The case of $n=2m$) Up to a constant, we have*

$$
\begin{aligned}
&W(w_i^{(m,-m),\pm}; z\,\mathrm{diag}(a_1,a_2,a_1^{-1},a_2^{-1}))\\
&= \frac{z^c a_1^2 a_2}{(2\pi\sqrt{-1})^2}\int_{\sigma_2-\sqrt{-1}\infty}^{\sigma_2+\sqrt{-1}\infty}\int_{\sigma_1-\sqrt{-1}\infty}^{\sigma_1+\sqrt{-1}\infty} V_i(s_1,s_2)\Big(\pi\frac{a_1}{a_2}\Big)^{-s_1}(\pi a_2^2)^{-s_2}\,ds_1 ds_2,
\end{aligned}
$$

where

$$
\begin{aligned}
V_\delta(s_1,s_2) &= \frac{\pi^{s_1+s_2+2m}}{(2\pi\sqrt{-1})^2}\int_{\tau_2-\sqrt{-1}\infty}^{\tau_2+\sqrt{-1}\infty}\int_{\tau_1-\sqrt{-1}\infty}^{\tau_1+\sqrt{-1}\infty}\Gamma_{\mathbf{R}}(s_1+m+\delta)\Gamma_{\mathbf{R}}(s_1-t_1-t_2+m)\\
&\quad\times\Gamma_{\mathbf{R}}(s_2-t_1+m-\delta)\Gamma_{\mathbf{R}}(s_2-t_2+m)\\
&\quad\times\Gamma_{\mathbf{R}}(t_1+\nu/2)\Gamma_{\mathbf{R}}(t_1-\nu/2)\Gamma_{\mathbf{R}}(t_2+1/2)\Gamma_{\mathbf{R}}(t_2-1/2)\,dt_1dt_2,\\
V_{2j+\delta}(s_1,s_2) &= 2^{-j-\delta}(\sqrt{-1})^\delta(s_2-j+m-1/2)_j\cdot V_\delta(s_1,s_2-j),
\end{aligned}
$$

for $\delta \in \{0,1\}$. Here $(a)_n = \Gamma(a+n)/\Gamma(a)$, and $\sigma_i, \tau_i \in \mathbf{R}$ are taken so that $\sigma_1 > \tau_1+\tau_2-m$, $\sigma_2 > \max\{\tau_1,\tau_2\}$, $\tau_1 > |\mathrm{Re}(\nu)/2|$, $\tau_2 > 1/2$.

4. Novodvorsky's zeta integrals

Let $\Pi = \otimes'_v \Pi_v$ be a generic cuspidal automorphic representation of $\mathsf{G}(\mathbf{A})$. We denote by $\widetilde{\Pi} = \otimes'_v \widetilde{\Pi}_v$ its contragredient. We fix $\psi \in \hat{N}_0$ such that $\psi(n(x_0, x_1, x_2, x_3)) = \psi_\infty(x_0 + x_3)$ where $\psi_\infty(x) = \exp(2\pi\sqrt{-1}x)$. For $W \in \mathcal{W}(\Pi_\infty, \psi_\infty)$ and $s \in \mathbf{C}$, Novodvorsky's archimedean zeta integral $Z_\infty(s, W)$ is defined by

$$Z_\infty(s,W) = \int_{\mathbf{R}^\times}\int_{\mathbf{R}} W\left(\left(\begin{array}{cc|cc} y & & & \\ & y & & \\ \hline & & 1 & \\ & x & & 1 \end{array}\right)\right)|y|^{s-3/2}dx\frac{dy}{|y|},$$

which converges absolutely for $\mathrm{Re}(s) \gg 0$.

Theorem 4.1 (Moriyama [13] (Large d.s., P_1), Moriyama-I [8] (P_0), I [7] (P_2)). *For each irreducible generic representation Π_∞ of $G = \mathrm{GSp}(2, \mathbf{R})$, there exists $W \in \mathcal{W}(\Pi_\infty, \psi_\infty)$ such that*

$$\frac{Z_\infty(1-s,\widetilde{W})}{L(1-s,\widetilde{\Pi}_\infty,\mathrm{spin})} = \varepsilon(s,\Pi_\infty,\psi_\infty,\mathrm{spin})\frac{Z_\infty(s,W)}{L(s,\Pi_\infty,\mathrm{spin})},$$

and the ratio $Z_\infty(s,W)/L(s,\Pi_\infty,\mathrm{spin})(\neq 0)$ is an entire function of $s \in \mathbf{C}$. Here L-and ε-factors are defined by Langlands parameters of Π_∞, and $\widetilde{W}$ is contragredient Whittaker function defined by $\widetilde{W}(g) = \varpi_{\Pi_\infty}(\nu(g)^{-1})W(g\kappa\left(\begin{smallmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{smallmatrix}\right))$ where ϖ_{Π_∞} is the central character of Π_∞.

Example $\Pi_\infty \cong \pi_{\varepsilon,n.c.\nu}$ with $n = 2m$ and $\varepsilon(\gamma_0) = 1$: If we take $W(g) = W(w_{2m}^{(m,-m),\pm}; g)$, then we have

$$\frac{Z_\infty(s,W)}{L(s,\Pi_\infty,\mathrm{spin})} = \frac{C}{2\pi\sqrt{-1}}\int_{\tau-\sqrt{-1}\infty}^{\tau+\sqrt{-1}\infty}\frac{\Gamma_{\mathbf{R}}(t+\frac{\nu}{2})\Gamma_{\mathbf{R}}(t-\frac{\nu}{2})\Gamma_{\mathbf{C}}(t-\frac{1}{2})}{\Gamma_{\mathbf{R}}(t+s+\frac{\varepsilon}{2})\Gamma_{\mathbf{R}}(t+1-s-\frac{\varepsilon}{2})}dt,$$

with some constant C.

Remark 1. Miyazaki [9] obtained a similar result for the principal series of $\mathrm{GSp}(2, \mathbf{C})$.

Combined with non-archimedean results of Takloo-Bighash [17], we can find the following:

Corollary 4.2. *Let $\Pi = \otimes'_v \Pi_v$ be a generic cuspidal representation of $\mathrm{GSp}(2, \mathbf{A})$. Then the completed spinor L-function $L(s, \Pi, \mathrm{spin}) = \prod_{v\leq\infty} L(s, \Pi_v, \mathrm{spin})$ is continued to an entire function of $s \in \mathbf{C}$, and has the functional equation*

$$L(s,\Pi,\mathrm{spin}) = \varepsilon(s,\Pi,\mathrm{spin})L(1-s,\widetilde{\Pi},\mathrm{spin})$$

with $\varepsilon(s, \Pi, \mathrm{spin}) = \prod_{v\leq\infty} \varepsilon(s, \Pi_v, \psi_v, \mathrm{spin})$.

Remark 2. Asgari-Shahidi [1] proved the results above by Langlands-Shahidi method.

5. Bump-Friedberg-Ginzburg zeta integrals

We recall the zeta integral discovered by Bump, Friedberg and Ginzburg [2]. The unipotent radical N_i $(i = 1, 2)$ of P_i is given by $\mathsf{N}_1 = \{n(x_0, x_1, x_2, 0) \in \mathsf{G}\}$ and $\mathsf{N}_2 =$

$\{n(0,x_1,x_2,x_3) \in \mathsf{G}\}$. The Levi part of P_i is isomorphic to $\mathrm{GL}(2)\times\mathrm{GL}(1)$ embedded via the maps ι_i: $\iota_1(\alpha,g) = \begin{pmatrix} \alpha & & & \\ & a & & b \\ & & \alpha^{-1}\det g & \\ & c & & d \end{pmatrix}$, $\iota_2(\alpha,g) = \begin{pmatrix} \alpha g & \\ & {}^t g^{-1} \end{pmatrix}$, where $\alpha \in \mathrm{GL}(1)$ and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2)$. The modulus characters δ_i of P_i are given by and $\delta_1(\iota_1(\alpha,g)) = |\det g|^{-2}|\alpha|^4$ and $\delta_2(\iota_2(\alpha,g)) = |\det g|^3|\alpha|^3$. For a complex number s, we denote by $\mathrm{Ind}_{\mathsf{P}_i(\mathbf{A})}^{\mathsf{G}(\mathbf{A})}(\delta_i^s)$ the space of smooth functions $f_i(s,g)$ on $\mathsf{G}(\mathbf{A})$ satisfying $f_i(s,pg) = \delta_i^s(p)f_i(s,g)$ for all $p \in \mathsf{P}_i(\mathbf{A})$ and $g \in \mathsf{G}(\mathbf{A})$. For complex numbers s_1 and s_2, we take a global sections $f_1 \in \mathrm{Ind}_{\mathsf{P}_1(\mathbf{A})}^{\mathsf{G}(\mathbf{A})}(\delta_1^{s_1/2+1/4})$ and $f_2 \in \mathrm{Ind}_{\mathsf{P}_2(\mathbf{A})}^{\mathsf{G}(\mathbf{A})}(\delta_2^{(s_2+1)/3})$. We define Eisenstein series $E_i(s_i,f_i,g)$ as usual manner: $E_i(s_i,f_i,g) = \sum_{\gamma\in\mathsf{P}_i(\mathbf{Q})\backslash\mathsf{G}(\mathbf{Q})} f_i(s_i,\gamma g)$.

For a generic cusp form $\varphi \in \Pi$, the global zeta integral is defined by

$$Z(s_1,s_2,\varphi,f_1,f_2) = \int_{\mathsf{Z}(\mathbf{A})\mathsf{G}(\mathbf{Q})\backslash\mathsf{G}(\mathbf{A})} \varphi(g)E_1(s_1,f_1,g)E_2(s_2,f_2,g)\,dg.$$

Here we denote by Z the center of G. Unfolding two Eisenstein series, one can find the basic identity:

$$Z(s_1,s_2,\varphi,f_1,f_2) = \int_{\mathsf{Z}(\mathbf{A})\mathsf{N}_{12}(\mathbf{A})\backslash\mathsf{G}(\mathbf{A})} W_\varphi(g)f_1(s_1,w_2g)f_2(s_2,w_1g)\,dg$$

for $\mathrm{Re}(s_1)$ and $\mathrm{Re}(s_2)$ sufficiently large. Here $\mathsf{N}_{12} = \mathsf{N}_1 \cap \mathsf{N}_2 = \{n(0,x_1,x_2,0) \in \mathsf{G}\}$, $w_1 = \left(\begin{array}{cc|cc} 1 & & & \\ & & & -1 \\ \hline & & 1 & \\ & 1 & & \end{array}\right)$ and $w_2 = \left(\begin{array}{cc|cc} & 1 & & \\ 1 & & & \\ \hline & & & 1 \\ & & 1 & \end{array}\right)$. Suppose that Π, f_1 and f_2 are factorizable. Then the global zeta integral is the product of local zeta integrals

$$Z_v(s_1,s_2,W_v,f_{1,v},f_{2,v}) = \int_{\mathsf{Z}(\mathbf{Q}_v)\mathsf{N}_{12}(\mathbf{Q}_v)\backslash\mathsf{G}(\mathbf{Q}_v)} W_v(g)f_{1,v}(s_1,w_2g)f_{2,v}(s_2,w_1g)\,dg,$$

where the subscripts denote the local analogues. Bump, Friedberg and Ginzburg performed the unramified computation.

As for the archimedean zeta integrals we can show the following.

Theorem 5.1. *For each generic reprensetation Π_∞ of $G = \mathrm{GSp}(2,\mathbf{R})$, there exists a tuple $\{W_\infty, f_{1,\infty}, f_{2,\infty}\}$ such that*

$$Z_\infty(s_1,s_2,W_\infty,f_{1,\infty},f_{2,\infty}) = L(s_1,\Pi_\infty,\mathrm{spin})L(s_2,\Pi_\infty,\mathrm{std}),$$

and

$$\begin{aligned}&\frac{\widetilde{Z}_\infty(s_1,s_2,W_\infty,f_{1,\infty},f_{2,\infty})}{L(1-s_1,\widetilde{\Pi}_\infty,\mathrm{spin})L(1-s_2,\widetilde{\Pi}_\infty,\mathrm{std})}\\ &= \varepsilon(s_1,\Pi_\infty,\psi_\infty,\mathrm{spin})\varepsilon(s_2,\Pi_\infty,\psi_\infty,\mathrm{std})\frac{Z_\infty(s_1,s_2,W_\infty,f_{1,\infty},f_{2,\infty})}{L(s_1,\Pi_\infty,\mathrm{spin})L(s_2,\Pi_\infty,\mathrm{std})},\end{aligned}$$

where

$$\widetilde{Z}_\infty(s_1,s_2,W_\infty,f_{1,\infty},f_{2,\infty}) = \int_{\mathsf{Z}(\mathbf{R})\mathsf{N}_{12}(\mathbf{R})\backslash\mathsf{G}(\mathbf{R})} W_\infty(g)M^*_{1,\infty}f_{1,\infty}(s_1,w_2g)M^*_{2,\infty}f_{2,\infty}(s_2,w_1g)\,dg,$$

with normalized intertwining operators $M^*_{i,\infty}$.

Example $\Pi_\infty \cong \pi_{\varepsilon,n,c,\nu}$ with $n = 2m$ and $(-1)^m\varepsilon(\gamma_0) = 1$: If we take $\{W_\infty, f_{1,\infty}, f_{2,\infty}\}$ as

- $W_\infty(g) = W(v; g)$, $v \in V^+_{(m,-m)}$;
- $f_{1,\infty}(s_1, k_0) = 1$ for $k_0 \in K_0$;
- $f_{2,\infty}(s_2, k_0) = \overline{\langle \tau^0_{(m,-m)}(k_0)v', w_0^{(m,-m),0}\rangle}$ for $k_0 \in K_0$, $v' \in V^0_{(m,-m)}$,

then we have

$$Z_\infty(s_1, s_2, W_\infty, f_{1,\infty}, f_{2,\infty}) = C\langle \iota_+(v), v'\rangle \cdot \frac{L(s_1, \Pi_\infty, \mathrm{spin})L(s_2, \Pi_\infty, \mathrm{std})}{\Gamma_{\mathbf{R}}(2s_1+1)\Gamma_{\mathbf{R}}(s_2+m+1)\Gamma_{\mathbf{R}}(2s_2+2m)}.$$

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