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THE KOHNEN PLUS SPACE AND JACOBI FORMS

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The Kohnen plus space, simply called plus space, is a subspace of the space of modular forms with half-integral weight in which the modular forms satisfy some restriction on whose Fourier coefficients. The concept was initially brought up by Kohnen [4] in 1980. The original definition was only for the classical modular forms, that is, which is in one variable. It was shown by Eichler and Zagier [1] that the plus space of weight $k+1/2$ is isomorphic to the space of Jacobi forms of weight $k+1$ and index 1 if k is odd. Later, the plus space was generalized to the case of Siegel modular forms by Ibukiyama [2] in 1992. Ibukiyama also showed that in the case we can still construct an isomorphism between the plus space and the space of Siegel-Jacobi forms. On the other hand, in 2013, the concept of plus space and its relation with the space of Jacobi forms was brought into the case of Hilbert modular forms by Hiraga and Ikeda [3]. They used Weil representation to characterize the plus space and showed that it is actually the fixed subspace of some Hecke operator E^K on the whole space of Hilbert modular forms of weight $k+1/2$. And here, we want to state the similar results for the Hilbert-Siegel case. The definition of plus space in this case is based on the ones from Ibukiyama, Hiraga and Ikeda.

Let F be a totally real field of degree n over $\mathbb{Q}$ with ring of integers $\mathfrak{o}$ and the different $\mathfrak{d}$. Denote the n embeddings of F in $\mathbb{R}$ by ι_i. An element $\xi \in F$ will be considered as a real n-tuple.

Let us fix a positive integer m. The Siegel upper half-plane of genus m is defined by

$$\mathfrak{h}_m = \{X+\sqrt{-1}Y \in M_m(\mathbb{C}) \mid X, Y \in \mathrm{Sym}_m(\mathbb{R}), Y>0\}$$

where $Y>0$ means that Y is positive definite. The set $\mathfrak{h}_m^n$ consists of n-tuples whose components are in $\mathfrak{h}_m$. Also, the set $(\mathbb{C}^m)^n$ consists of n-tuples of complex column vectors with size m. Note that any vector with size m here is considered as a column vector.

The symplectic group of degree $2m$ is defined by

$$Sp_m(F) = \left\{ g \in GL_{2m}(F) \mid {}^t g \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} g = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} \right\}$$

where I_m is the identity matrix of size m. It acts on $\mathfrak{h}_m^n$ as

$$gz = \left((\iota_i(a)z_i + \iota_i(b))(\iota_i(c)z_i + \iota_i(d))^{-1}\right)_{i=1}^{n}$$

for

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp_m(F), \quad a, b, c, d \in M_m(F)$$

and $z = (z_i) \in (\mathfrak{h}_m)^n$.

To define the factor of automorphy with half-integral weight, we have to give the theta function.

Definition 0.1. *The theta function* Θ *is a function on* $\mathfrak{h}_m^n$ *defined by*

$$\Theta(z) = \sum_{p \in \mathfrak{o}^m} \exp\left(2\pi\sqrt{-1}\mathrm{Tr}\left({}^t\!pzp\right)\right)$$

where Tr *is the sum of the component of a complex* n*-tuple.*

Let us define the two congruence subgroups of $Sp_m(F)$:

(0.1)

$$\Gamma_0(1) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp_m(F) \,|\, a, c \in M_m(\mathfrak{o}), b \in M_m(\mathfrak{d}^{-1})c \in M_m(\mathfrak{d})\right\}$$

and

$$\Gamma_0(4) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp_m(F) \,|\, a, c \in M_m(\mathfrak{o}), b \in M_m(\mathfrak{d}^{-1})c \in M_m(4\mathfrak{d}).\right\}$$

The factor of automorphy of weight $1/2$ is a function $\tilde{j}$ on $\Gamma_0(4) \times \mathfrak{h}_m^n$ given by

$$\tilde{j}(\gamma, z) = \frac{\Theta(\gamma z)}{\Theta(z)}.$$

It satisfies

$$\tilde{j}(\gamma, z)^4 = N(cz + d)^2 \text{ if } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$$

where N is the product of the component of a complex n-tuple.

For simplicity, here we only consider the Hilbert-Siegel modular forms of parallel weight. We fix a positive integer k. Let $M_{k+1/2}(\Gamma_0(4))$ be the space of Hilbert-Siegel modular forms with respect to the factor of automorphy $\tilde{j}^{2k+1}$ and $S_{k+1/2}(\Gamma_0(4))$ be the subspace of $M_{k+1/2}(\Gamma_0(4))$ consisting of cusp forms. Then for any Hilbert-Siegel modular form $h \in M_{k+1/2}(\Gamma_0(4))$, it has Fourier expansion in the form

$$h(z) = \sum_{T \in L^*} c(T)\mathbf{e}(\mathrm{Tr}(\mathrm{tr}(Tz)))$$

where L^* is the set of all half-integral matrices in $M_m(F)$, the coefficient $c(T) = 0$ if T is not positive semi-definite and tr is the usual trace for matrices. Moreover, as usual, $\mathbf{e}(\tau) = \exp(2\pi\sqrt{-1}\tau)$ for $\tau \in \mathbb{C}$. For simplicity, we put $q^T = \mathbf{e}(\mathrm{Tr}(\mathrm{tr}(Tz)))$.

Now we are ready to define the Kohnen plus spaces.

Definition 0.2. *The Kohnen plus spaces with respect to the case above are defined by*

$$M^+_{k+1/2}(\Gamma_0(4)) = \{h \in M_{k+1/2}(\Gamma_0(4)) \mid c(T) = 0 \text{ unless there exists } \lambda \in \mathfrak{o}^m \text{ such that } (-1)^k T \equiv \lambda \cdot {}^t\lambda \mod 4L^*\}$$

and

$$S^+_{k+1/2}(\Gamma_0(4)) = M^+_{k+1/2}(\Gamma_0(4)) \cap S_{k+1/2}(\Gamma_0(4)).$$

Let $h = \sum_T c(T)q^T \in M^+_{k+1/2}(\Gamma_0(4))$. For any $\lambda \in (\mathfrak{o}/2\mathfrak{o})^m$, we set

$$h_\lambda(z) = \sum_{(-1)^k T \equiv \lambda \cdot {}^t\lambda \mod 4L^*} c(T) q^{T/4}.$$

It is easy to see the definition of h_λ does not depend on the choice of $\lambda \bmod 2\mathfrak{o}^m$. The functions h_λ are actually Hilbert-Siegel modular forms of weight $k + 1/2$ with respect to some congruence subgroups of $Sp_m(F)$ and some characters. From the definition of the plus space, we have

$$h(z) = \sum_{\lambda \in (\mathfrak{o}/2\mathfrak{o})^m} h_\lambda(4z).$$

Next, we want to give the definition of Jacobi forms. It is well-known that $Sp_m(F)$ acts on $\mathfrak{h}^n_m \times (\mathbb{C}^m)^n$ by

$$g(z, w) = \left((\iota_i(a)z_i + \iota_i(b))(\iota_i(c)z_i + \iota_i(d))^{-1}, {}^t(\iota_i(c)z_i + \iota_i(d))^{-1} w_i \right)_{i=1}^n$$

for

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp_m(F), \quad a, b, c, d \in M_m(F),$$

$z = (z_i) \in (\mathfrak{h}_m)^n$ and $w = (w_i) \in (\mathbb{C}^m)^n$.

Definition 0.3. *A holomorphic function G on $\mathfrak{h}^n_m \times (\mathbb{C}^m)^n$ is called a Jacobi form of weight k and index 1 if the following three statements hold.*

(1) $G(z, w + zx + y) = \mathbf{e}(-\mathrm{Tr}({}^t xzx + 2{}^t xw))G(z, w)$ *for any* $x \in \mathfrak{o}^m, y \in (\mathfrak{d}^{-1})^m$

(2) $G(\gamma(z, w)) = N(\det(cz + d))^k \mathbf{e}(\mathrm{Tr}({}^t w(cz + d)^{-1} cw))G(z, w)$ $\left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(1)\right)$,

(3) *G satisfies the cusp condition, for which we omit the detail here.*

The space of all such forms is denoted by $J_{k,1}$ and the subspace of cusp forms in $J_{k,1}$ is denoted by $J_{k,1}^{\mathrm{CUSP}}$.

For any $\lambda \in (\mathfrak{o}/2\mathfrak{o})^m$ we can refer a theta series on $\mathfrak{h}_m^n \times (\mathbb{C}^m)^n$ as

$$\theta_\lambda(z,w) = \sum_{p\in\mathfrak{o}^m} \mathbf{e}\left(\mathrm{Tr}\left({}^t\!(p+\frac{\lambda}{2})z(p+\frac{\lambda}{2}) + 2\cdot{}^t(p+\frac{\lambda}{2})w)\right)\right).$$

The right hand side above does not depend on the choice of λ mod $2\mathfrak{o}^m$. Now if $G \in J_{k,1}$ is a Jacobi form of weight k and index 1, then for any $\lambda \in (\mathfrak{o}/2\mathfrak{o})^m$, there exists a unique holomorphic function G_λ on $\mathfrak{h}_m^n$ such that

$$G(z,w) = \sum_{\lambda\in(\mathfrak{o}/2\mathfrak{o})^m} G_\lambda(z)\theta_\lambda(z,w).$$

This formula is called the theta expansion of G. In fact, G_λ are Hilbert-Siegel modular forms of weight $k+1/2$.

Now let k be odd. The main theorem tells us that the plus space and the space of Jacobi forms are actually isomorphic.

Theorem 0.1. *Assume $h \in M_{k+1/2}^+(\Gamma_0(4))$ and $G \in J_{k+1,1}$. With the notations given above, we have*

$$\sum_{\lambda\in(\mathfrak{p}/2\mathfrak{o})^m} h_\lambda(z)\theta_\lambda(z,w) \in J_{k+1,1}$$

and

$$\sum_{\lambda\in(\mathfrak{p}/2\mathfrak{o})^m} G_\lambda(4z) \in M_{k+1/2}^+(\Gamma_0(4)).$$

The two canonical mappings are the inverse of each other. Thus these give an isomorphism between $M_{k+1/2}^+(\Gamma_0(4))$ ($S_{k+1/2}^+(\Gamma_0(4))$) and $J_{k+1,1}$ (J_{k+1}^{CUSP}).

As mentioned in the beginning, the classical, Siegel and Hilbert case for this theorem were proved by Eichler & Zagier, Ibukiyama and Hiraga & Ikeda, respectively.

Finally, we want to state the key concept of this result. Let $\mathbb{A}$ be the adele ring of F and $\psi = \prod_v \psi_v : \mathbb{A}/F \to \mathbb{C}^\times$ be the unique additive character on $\mathbb{A}$ which is trivial on F and has $\psi_\infty(x) = \mathbf{e}(x)$ as whose local components for any infinite place ∞ of F. We denote the global Weil representation of $\widetilde{Sp_m(\mathbb{A}_f)}$, the finite part of the double metaplectic covering of $Sp_m(\mathbb{A})$, on the Schwartz space $\mathbb{S}(\mathbb{A}_f^m)$ of $\mathbb{A}_f^m$ by ω_ψ. For any finite place v, the group $K_v = \Gamma_0(1)_v$ is defined similarly as (0.1) and we put $K = \prod_{v<\infty} K_v$. It is known that if we restrict ω_ψ

on the inverse image $\widetilde{K}$ of K in $\widetilde{Sp_m(\mathbb{A}_f)}$, then $\mathbb{S}((2^{-1}\hat{\mathfrak{o}}/\hat{\mathfrak{o}})^m)$ forms an invariant irreducible subspace for the restricted representation. Here $\hat{\mathfrak{o}} = \prod_{v<\infty} \mathfrak{o}_v$ and $\mathbb{S}((2^{-1}\hat{\mathfrak{o}}/\hat{\mathfrak{o}})^m)$ consists of Schwartz functions Φ supported on $2^{-1}\hat{\mathfrak{o}}^m$ which satisfies $\Phi(X+Y) = \Phi(X)$ for $Y \in \hat{\mathfrak{o}}^m$. The deduced representation of $\widetilde{K}$ on $\mathbb{S}((2^{-1}\hat{\mathfrak{o}}/\hat{\mathfrak{o}})^m)$ is denoted by Ω_ψ. For $\lambda \in (\mathfrak{o}/2\mathfrak{o})^m$, we set $\Phi_\lambda \in \mathbb{S}((2^{-1}\hat{\mathfrak{o}}/\hat{\mathfrak{o}})^m)$ to be the characteristic function of $\lambda/2 + \hat{\mathfrak{o}}^m$. These 2^{nm} functions form a basis for $\mathbb{S}((2^{-1}\hat{\mathfrak{o}}/\hat{\mathfrak{o}})^m)$.

Note that any Hilbert-Siegel modular form of weight $k + 1/2$ can be uniquely lifted to an automorphic form on $\widetilde{Sp_m(\mathbb{A})}$, the metaplectic double covering of $Sp_m(\mathbb{A})$. If we denote the space of all the automorphic forms obtained by this way by $\mathcal{A}_{k+1/2}(Sp_m(F)\backslash\widetilde{Sp_m(\mathbb{A})})$, it forms a representation of $\widetilde{Sp_m(\mathbb{A}_f)}$ by the right translation ρ. The corresponding action of $\widetilde{Sp_m(\mathbb{A}_f)}$ on the union of all Hilbert-Siegel modular forms of weight $k + 1/2$ is also denoted by ρ.

Theorem 0.2. *Let k be odd. The three following statements are equivalent.*

(1) $h(z) = \sum_{\lambda\in(\mathfrak{o}/2\mathfrak{o})^m} h_\lambda(4z) \in M^+_{k+1/2}(\Gamma_0(4))$ *where h_λ is defined as above.*

(2) *Given a family $\{h_\lambda\}_{\lambda\in(\mathfrak{o}/2\mathfrak{o})^m}$ of 2^{nm} Hilbert-Siegel modular forms of weight $k+1/2$. The space $\sum_{\lambda\in(\mathfrak{o}/2\mathfrak{o})^m} \mathbb{C}\cdot h_\lambda$ forms a representation of $\widetilde{K}$ by ρ which is equivalent to $\overline{\Omega_\psi}$ via the intertwining map $h_\lambda \mapsto \Phi_\lambda$.*

(3) $\sum_\lambda h_\lambda(z)\theta_\lambda(z,w) \in J_{k+1,1}$.

The equivalence of (2) and (3) simply comes from the representative definition of Jacobi forms. So the efforts of the author on this research mainly focuses on the equivalence of (1) and (2), especially the (1)$\Rightarrow$(2) part.

References

[1] M. Eichler and D. Zagier, *The theory of Jacobi forms,* Springer (1985)

[2] T. Ibukiyama, *On Jacobi forms and Siegel modular forms of half integral weights,* Comment. Math. Univ. St. Paul. Vol. 41 No. 2, 109-124 (1992)

[3] K. Hiraga and T. Ikeda, *On the Kohnen plus space for Hilbert modular forms of half-Integral weight I,* Compositio Mathematica 149 (2013), 1963-2010

[4] W. Kohnen, *Modular forms of half-integral weight on* $\Gamma_0(4)$, Math. Ann. 248, 249-266 (1980)

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