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ON THE HAMILTONIAN FOR WATER WAVES

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ABSTRACT. Many equations that arise in a physical context can be posed in the form of a Hamiltonian system, meaning that there is a symplectic structure on an appropriate phase space, and a Hamiltonian functional with respect to which time evolution of their solutions can be expressed in terms of a Hamiltonian vector field. A normal forms transformation for a Hamiltonian dynamical system given by such a vector field is a change of variables in a neighborhood of a stationary point in phase space that eliminates inessential terms, retaining only essential nonlinearities while preserving the Hamiltonian structure of the system. It is known from the work of VE Zakharov that the equations for water waves can be posed as a Hamiltonian dynamical system, and that the equilibrium solution is an elliptic stationary point. This article discusses two aspects of the water wave equations in this context. Firstly, we generalize the Hamiltonian formulation of Zakharov to include overturning wave profiles, answering a question posed to the author by T. Nishida. Secondly, we will discuss the question of Birkhoff normal forms transformations for the water waves of equations, in the setting of spatially periodic solutions. Our results describe the function space mapping properties of the normal forms transformations, with and without inclusion of the effects of surface tension, and the dynamical implications of the normal forms. This latter is joint work with Catherine Sulem (University of Toronto).

1. INTRODUCTION

The equations for water waves describe the flow of an incompressible and irrotational fluid with a free surface, under the additional restoring forces of gravity and with the possibility to include the effects of surface tension. The fluid velocity $u(t,x,y)$, expressed in Eulerian coordinates, satisfies the conditions

$$\nabla \cdot u = 0 \ , \qquad \nabla \wedge u = 0 \ , \tag{1.1}$$

in a fluid domain $\Omega(t) \subseteq \mathbb{R}^{d-1}_x \times \mathbb{R}^1_y$ whose boundary consists of two components, a bottom described by a hypersurface $s \in \mathbb{R}^{d-1} \mapsto b(s) \in \mathbb{R}^d$ and a free surface given by a time dependent hypersurface $s \in \mathbb{R}^{d-1} \mapsto \gamma(t,s) \in \mathbb{R}^d$. It is possible that the bottom is unbounded below, and indeed it is common to consider the case that the bottom boundary lies at $\{y = -\infty\}$. The free surface of the fluid domain itself, defined by the hypersurface $\gamma(t,s)$, is one of the unknowns. Because of the constraints (1.1) the fluid motion is given

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by a *potential flow*

$$(1.2) \qquad u = \nabla\varphi \ , \qquad \Delta\varphi = 0 \ , \quad \text{in the fluid domain } \Omega(t) \ ;$$
$$\partial_N \varphi = 0 \ , \qquad \text{bottom boundary conditions on } (x,y) \in \{b(s) \ : s \in \mathbb{R}^{d-1}\} \ .$$

Denote the horizontal and vertical components of the hypersurface defining the free surface of $\Omega(t)$ by $\gamma(t,s) = (\gamma_1(t,s), \gamma_2(t,s))$ (that is, $x = \gamma_1 \in \mathbb{R}^{d-1}$ and $y = \gamma_2 \in \mathbb{R}^1$), and the space-time unit normal vector to the free surface to be $\mathbf{N}_{t,x,y}$. Furthermore define the space-time vector describing the fluid velocity field as $\mathbf{T}_{t,x,y} = (1, u(t,x,y))^T = (1, \nabla_{x,y}\varphi)^T$. Then the kinematic free boundary condition on the free surface $\{(x,y) \in \gamma\}$ is the geometrical condition that

$$(1.3) \qquad \mathbf{N}_{t,x,y} \cdot \mathbf{T}_{t,x,y} = 0 \ .$$

The physics of the flow of Euler's equations is described by the second nonlinear boundary condition;

$$(1.4) \qquad \partial_t \varphi \ = \ -g\gamma_2 - \tfrac{1}{2}|\nabla\varphi|^2 + \sigma H(\eta) \qquad \text{Bernoulli condition} \ .$$

The force of surface tension is given by the term σH, where $H(\eta)$ is the mean curvature of the free surface. We study both of the cases $\sigma > 0$ and $\sigma = 0$.

In the case that the the free surface is given as a graph, $y = \eta(t,x)$, $x \in \mathbb{R}^{d-1}$ the conditions (1.3)(1.4) can be rewritten as

$$(1.5) \qquad \begin{aligned} \partial_t \eta \ &= \ \partial_y \varphi - \partial_x \eta \cdot \partial_x \varphi \qquad && \text{kinematic boundary conditions} \\ \partial_t \varphi \ &= \ -g\eta - \tfrac{1}{2}|\nabla\varphi|^2 + \sigma H(\eta) && \text{Bernoulli condition} \ . \end{aligned}$$

This dynamic free boundary problem was recognized by VE Zakharov ([15] 1968) to be a Hamiltonian PDE, which is to say that equations (1.2)(1.5) can be given the form of a Hamiltonian system

$$\dot{z} = X^H(z) \ , \qquad \text{where} \quad X^H(z) = J \operatorname{grad}_z H(z) \ ,$$

with the Hamiltonian function H the total energy of the system (1.1)(1.2)(1.5), namely

$$(1.6) \qquad H = \frac{1}{2} \iint_{\Omega(t)} |\nabla\varphi|^2 \, dydx + \frac{g}{2} \int_{\mathbb{R}^{d-1}} \eta^2 \, dx + \sigma \int_{\mathbb{R}^{d-1}} \sqrt{1 + |\partial_x \eta|^2} - 1 \, dx \ .$$

A more subtle aspect is the choice of canonical variables for the phase space, which as in [15] is normally given by $z(x) := (\eta(x), \xi(x) := \varphi(x, \eta(x)))$, namely

$$\partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \delta_\eta H \\ \delta_\xi H \end{pmatrix} \ , \qquad \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} := J \ ,$$

which is in Darboux coordinates in that the symplectic form is defined by J as given above.

The goals of this article are (1) to explain this fact, (2) to extend the formulation of the equations of water waves as a Hamiltonian system to the general case of equations (1.1)(1.2) and boundary conditions (1.3)(1.4), which allows for overturning wave profiles, (3) to use the resulting Hamiltonian formulation in an analysis of Birkhoff normal forms transformations, which permits one to gain useful information about such free surface flows, and (4) to justify on a rigorous level some of the function space mapping properties of the normal forms transformations. This latter is intended for use in studying the dynamics of solutions of the water waves equations, including long time existence results, KAM theoretic constructions, and analysis of the principal model equations that are commonly used to describe ocean wave dynamics. The topic of Birkhoff normal forms for the equations for water waves was the topic of the author's lecture at the Kyoto RIMS workshop.

2. Hamiltonian for overturning waves

During the RIMS Symposium on Mathematical analysis in fluid and gas dynamics in Kyoto, T. Nishida asked the author whether Zakharov's formulation of the water waves problem (1.5) as a Hamiltonian PDE could be extended to take into consideration the case of geometries of free surfaces that are not graphs, and in particular waves that are overturning. A subtle issue in Zakharov's formulation is the specific choice of canonical conjugate variables, which appears to require remarkable insight, but in retrospect can be deduced from a *principle of least action* à la Lagrange and a subsequent Legendre transform [3]. It turns out that similar considerations are useful when seeking to describe the water wave problem in general coordinates.

We will address this question essentially on a formal level, and in the case $d = 2$, for which $\Omega(t) \subseteq \mathbb{R}^2$. Configuration space is taken to be the space of curves $\Gamma := \{s \mapsto \gamma(s) : s \in \mathbb{R}^1\}$. To actually perform analysis for data in this configuration space, including solving Laplace's equation on the fluid domain $\Omega(t)$, we should give some topology to this space, such as $\gamma \in C^1(\mathbb{R}^1)$, and we should consider free surfaces $\gamma(s)$ that have a limit $\lim_{s\to\pm\infty} \gamma(s) = 0$. Furthermore we should ask that there be a uniform lower bound on the distance between γ and the bottom boundary $\{b(s)\}$, and also that γ satisfy a global chord - arc condition. However in the present context we will ignore these details.

2.1. **Free surface boundary conditions.**

Given a one parameter family of curves $\gamma(t, s)$, the velocity, which is the time derivative $\partial_t\gamma(t, s) = \dot\gamma(t, s)$ defines a vector field in the tangent space over the curve γ. A natural orthonormal frame for the tangent space over $\gamma(s)$ is given by $(T(s), N(s))$, where

$$T(s) = \frac{\partial_s\gamma(s)}{|\partial_s\gamma(s)|} , \quad N(s) = -JT(s) , \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

In the frame $(T(s), N(s))$ the velocity $\dot\gamma$ vector field can be represented by its coordinates

$$n(t, s) = N \cdot \dot\gamma , \quad \tau(t, s) = T \cdot \dot\gamma .$$

The equations for water waves determine the evolution of the fluid domain $\Omega(t)$ and the velocity field $u(t, x, y)$ defined in $\Omega(t)$. Because of the condition of irrotationality (1.1) they can be reduced to two nonlinear boundary conditions posed on the free surface $\gamma(t, s)$. The first of these is the kinematic condition (1.3), which in this frame is written

$$0 = \mathbf{N}_{t,x,y} \cdot \mathbf{T}_{t,x,y} = c(t, s)[N \cdot \dot\gamma - N \cdot u] ,$$

where $c(t,s)$ is a normalization irrelevant to the present discussion. When interpreted in terms of the velocity potential $\nabla_{x,y}\varphi(t,x,y) = u(t,x,y)$ this is the statement that

$$N \cdot \nabla\varphi\big|_\gamma = N \cdot \dot\gamma = n(t,s) \ . \tag{2.1}$$

With the foresight of Zakharov's formulation, define $\xi(s) = \varphi(\gamma(s))$ to be the boundary values of the velocity potential on the free surface $\gamma(s)$. Then the function $n(s)$ can be expressed in terms of the Dirichlet – Neumann operator for the fluid domain

$$n(s) = G(\gamma)\xi(s) \tag{2.2}$$

(and these quantites will depend parametrically on time t). Namely, φ is the solution of Laplace's equation on the fluid domain Ω satisfying Neumann boundary conditions on the bottom $(x,y) = b(s)$ and with boundary data on the free surface $\gamma(s)$ given by $\varphi(\gamma(s)) = \xi(s)$, where the operator $G(\gamma)$ is then defined by

$$\xi(s) \mapsto \varphi(x,y) \mapsto \nabla\varphi \cdot N := G(\gamma)\xi(s) \ .$$

The normalization for the Dirichlet – Neumann operator is that $|N| = 1$ (which differs slightly from what is commonly used for the problem posed in graph coordinates). This is an elliptic boundary value problem which can be solved for $\varphi(x,y)$, hence the map $\xi \mapsto G(\gamma)\xi$ is well defined.

The Bernoulli condition (1.4) expresses the physics described by the Euler equations on the free surface. Written in terms of $\xi(t,s) = \varphi(\gamma(t,s))$, for which $\partial_t\xi(t,s) = \partial_t\varphi(t,\gamma(t,s)) = \varphi_t + \nabla\varphi\cdot\dot\gamma$, this is

$$\partial_t\xi - \nabla\varphi\cdot\gamma = -g\gamma_2 - \tfrac{1}{2}|\nabla\varphi|^2 \ . \tag{2.3}$$

Recalling the definition that $\dot\gamma = nN(t,s) + \tau T(t,s)$ (and using that $|T| = |N| = 1$ and $T\cdot N = 0$),

$$\nabla\varphi\cdot\dot\gamma = (\nabla\varphi\cdot N)n + (\nabla\varphi\cdot T)\tau = (G(\gamma)\xi)n + \frac{\partial_s\xi}{|\partial_s\gamma|}\tau \ .$$

Therefore (2.3) is rewritten as

$$\partial_t\xi = -g\gamma_2 + \tfrac{1}{2}\Big[(G(\gamma)\xi)^2 - \frac{1}{|\partial_s\gamma|^2}(\partial_s\xi)^2 + 2\frac{\partial_s\xi}{|\partial_s\gamma|}\tau\Big] \ , \tag{2.4}$$

where we have used the definition (2.2) for n in terms of ξ.

In general the geometry of the curve γ can be recovered from $T(s)$ (or equivalently from $N(s)$) by integration, but not its parametrization. However so far in this discussion we have not addressed the issues of ambiguity that have been introduced by allowing arbitrary (nonsingular) coordinatization of curves $\gamma(s)$. There exist numerous useful possibilities to specify this parametrization, one standard one being as a graph, but another is to parametrize by arc length. In this latter case

$$\partial_s\gamma = T \ , \quad |T(s)| = 1 \ ,$$
$$\partial_s T = \kappa(s)N \ , \quad \partial_s N = -\kappa(s)T \ ,$$

which describes the evolution in s of the Frenet frame, $\kappa(s)$ being the curvature. In these coordinates one recovers $\tau(s)$ from $n(s)$; indeed because

$$0 = \partial_t |T(t,s)|^2 = 2\partial_t T \cdot T = 2\partial_s \dot\gamma \cdot T$$

then one has

$$\partial_s \tau = \partial_s \dot\gamma \cdot T + \dot\gamma \cdot \partial_s T = \kappa \dot\gamma \cdot N = \kappa n \ . \tag{2.5}$$

In arc length coordinates, equation (2.4) is somewhat simpler, namely

$$\partial_t \xi = -g\gamma_2 + \tfrac{1}{2}\left[(G(\gamma)\xi)^2 - (\partial_s \xi)^2 + 2\partial_s \xi \tau \right] \ . \tag{2.6}$$

In this case, the tangential component of the velocity is recovered from (2.2)(2.5), namely $\partial_s \tau = \kappa n = \kappa G(\gamma)\xi$.

2.2. **Legendre transform.** The Lagrangian for free surface water waves corresponds to the total energy of the system, which consists of two terms, the kinetic energy K and the potential energy U;

$$L = K - U \ . \tag{2.7}$$

The Legendre transform is the classical approach to transfer a Lagrangian system into the canonical conjugate coordinates of a Hamiltonian system. When the Lagrangian functional L is expressed in terms of the variables γ and $\dot\gamma$, by analogy with classical mechanics one defines *conjugate momentum variables* via the Legendre transform as

$$\xi = \delta_{\dot\gamma} L \ .$$

The kinetic energy is given by the Dirichlet integral

$$K = \iint_\Omega \tfrac{1}{2} |\nabla \varphi(x,y)|^2 \, dydx$$

and the potential energy is respectively

$$U = \iint_\Omega gy \, dydx + C \ ,$$

which is, as usual, only defined up to an additive constant. If the effects of surface tension were to be included in the equations of motion, then the potential energy has an additional term, namely

$$U = \iint_\Omega gy \, dydx + \sigma \int_\gamma dS_\gamma + C' \ ,$$

where $dS_\gamma = |\partial_s \gamma(s)| \, ds$. Our derivation below is in the case that $\sigma = 0$, but by modifications of the argument the case $\sigma \neq 0$ is also able to be included.

Integrating by parts in K and using the boundary conditions, we can express the kinetic energy in terms of integrated quantities on the free surface

$$K = \int_\gamma \tfrac{1}{2} \xi G(\gamma) \xi \, dS_\gamma \ . \tag{2.8}$$

We note that the normalization for the Dirichlet – Neumann operator $G(\gamma)$ is different from that used in [15] and [4], so that it is Hermetian with respect to the line element dS_γ. Using (2.2) the kinetic energy can be written in terms of γ and $\dot\gamma$;

$$K(\gamma,\dot\gamma) := \int_\gamma \tfrac{1}{2} n G^{-1}(\gamma) n \, dS_\gamma \; . \tag{2.9}$$

The potential energy U can be expressed with respect to the divergence theorem, using a vector field $V(x,y) := (0, \frac{g}{2}y^2)^T$;

$$U(\gamma) = \iint_\Omega \nabla \cdot V(x,y) \, dvol = \int_\gamma V \cdot N \, dS_\gamma + C = \int_\gamma \frac{g}{2}\gamma_2^2 \frac{\partial_s \gamma_1}{|\partial_s \gamma|} \, dS_\gamma + C \; . \tag{2.10}$$

In arc length parametrization this would read

$$U(\gamma) = \int_\gamma \frac{g}{2}\gamma_2^2 \partial_s \gamma_1 \, ds + C \; .$$

In the case of general coordinates for the free surface $\gamma(s) = (\gamma_1(s), \gamma_2(s))$, gradients are expressed with respect to the metric given by $\int_\gamma \cdot \, dS_\gamma$. The tangent space T_γ at γ to the set of curves is given coordinates using the Frenet frame $(T(s), N(s))$. Variations of L with respect to vector fields $Y(s) \in T_\gamma$ along $\gamma(s)$ can be decomposed into their normal and tangential components, namely

$$\langle \delta_Y L, \delta Y \rangle_\gamma = \int_\gamma \mathrm{grad}_{N \cdot Y} L \, (N \cdot \delta Y) + \mathrm{grad}_{T \cdot Y} L \, (T \cdot \delta Y) \; dS_\gamma \; .$$

If the vector field $Y(s) = \dot\gamma(t,s)$ is the velocity of the curve $\gamma(t,s)$, this is written

$$\langle \delta_{\dot\gamma} L, \delta \dot\gamma \rangle_\gamma = \int_\gamma \mathrm{grad}_{N \cdot \dot\gamma} L \, (N \cdot \delta \dot\gamma) + \mathrm{grad}_{T \cdot \dot\gamma} L \, (T \cdot \delta \dot\gamma) \; dS_\gamma \; .$$

In the case of the kinetic energy K above, and because of the decomposition $\dot\gamma(s) = \tau(s)T(s) + n(s)N(s)$, this is

$$\delta_{\dot\gamma} K = \delta_n K + \delta_\tau K = G^{-1}(\gamma) n + 0 \; . \tag{2.11}$$

Thus $\xi = G^{-1}(\gamma)n$ is the canonical conjugate variable to normal perturbations of a given free surface γ, while τ remains undefined without further specification of the parametrization of the curve γ. This degeneracy will be resolved when a particular form of parametrization is imposed.

Following the prescription of the Legendre transform (2.11) the Hamiltonian is given by

$$H = K + U = \frac{1}{2}\int_\gamma \xi G(\gamma) \xi \, dS_\gamma + \frac{g}{2}\int_\gamma \gamma_2^2 \frac{\partial_s \gamma_1}{|\partial_s \gamma|} \, dS_\gamma \; . \tag{2.12}$$

The remaining questions are to how to best express the variables that are canonically conjugate to $\xi(s)$, and to show that the resulting equations of motion (2.2)(2.4) coincide with the Hamiltonian vector field, namely

$$\partial_t z = J \mathrm{grad} H(z) \; .$$

The gradient of the kinetic energy K with respect to ξ is

$$\mathrm{grad}_\xi K = G(\gamma)\xi \ ,$$

which corresponds to the conjugate of the normal variations of K with respect to $\dot{\gamma}$.

Using the expression (2.10), the gradient of the potential energy is given by

$$\begin{aligned}\langle \delta_\gamma U, \delta\gamma\rangle_\gamma &= \int_\gamma \frac{g}{2}\begin{pmatrix}-2\gamma_2\partial_s\gamma_2\\ 2\gamma_2\partial_s\gamma_1\end{pmatrix}\cdot\begin{pmatrix}\delta\gamma_1\\ \delta\gamma_2\end{pmatrix} ds \\ &= \int g\gamma_2(s)\, N\cdot\begin{pmatrix}\delta\gamma_1\\ \delta\gamma_2\end{pmatrix} dS_\gamma \ ,\end{aligned} \tag{2.13}$$

corresponding to the gradient of U with respect to normal variation of γ itself, namely $\mathrm{grad}_{N\cdot\delta\gamma}U$.

The gradient of the kinetic energy K with respect to γ is the more subtle quantity in this formulation. Consider a fluid domain Ω with free surface $\gamma(s)$ and a family of nearby domains Ω_1 with nearby free surfaces $\gamma_1(s) = \gamma(s) + \delta\gamma(s)$. Denote the outward unit normal by $N(s)$ and $N_1(s)$ respectively. We consider the Dirichlet integrals

$$K(\gamma,\xi) = \tfrac{1}{2}\int_\gamma \xi(s)G(\gamma)\xi(s)\, dS_\gamma \ , \qquad K_1 = K(\gamma_1,\xi) = \tfrac{1}{2}\int_{\gamma_1}\xi(s)G(\gamma_1)\xi(s)\, dS_{\gamma_1} \ ,$$

for which we impose that the boundary values of the velocity potentials $\Phi_1(x,y)$ on γ_1 and $\Phi(x,y)$ on γ coincide

$$\Phi(\gamma(s)) = \xi(s) = \Phi_1(\gamma_1(s)) \ ,$$

while we vary the boundary curve $\gamma(s)$ to $\gamma_1(s) = \gamma(s) + \delta\gamma(s)$. This is to say that one takes the partial derivative of the kinetic energy with respect to variations of the domain, while fixing the boundary conditions for the velocity potential on the free surface. To this effect, the boundary values of $\Phi(x,y)$ on the curve $\gamma_1(s)$ are given by

$$\begin{aligned}\Phi(\gamma_1(s)) &= \Phi(\gamma(s)) + \nabla\Phi(\gamma(s))\cdot\delta\gamma(s) + \mathcal{O}(\delta^2) \\ &= \Phi(\gamma(s)) + (\nabla\Phi\cdot N)\, N\cdot\delta\gamma(s) + (\nabla\Phi\cdot T)\, T\cdot\delta\gamma(s) + \mathcal{O}(\delta^2) \ .\end{aligned}$$

Therefore

$$\Phi_1(\gamma_1(s)) - \Phi(\gamma_1(s)) = -(\nabla\Phi\cdot N)\, N\cdot\delta\gamma(s) - (\nabla\Phi\cdot T)\, T\cdot\delta\gamma(s) + \mathcal{O}(\delta^2) \tag{2.14}$$

Furthermore, given a harmonic function $\Phi(x,y)$ defined on a neighborhood that includes $\Omega\cup\Omega_1$, by Green's theorem the difference of the boundary integral expressions for their Dirichlet integrals is given by

$$\begin{aligned}&\tfrac{1}{2}\int_{\gamma_1}\Phi(\gamma_1(s))N_1\cdot\nabla\Phi(\gamma_1(s))\, dS_{\gamma_1} - \tfrac{1}{2}\int_\gamma \Phi(\gamma(s))N\cdot\nabla\Phi(\gamma(s))\, dS_\gamma \\ &= \tfrac{1}{2}\iint_{\Omega_1\setminus\Omega}|\nabla\Phi|^2\, dvol \simeq \tfrac{1}{2}\int_\gamma |\nabla\Phi|^2 N\cdot\delta\gamma(s)\, dS_\gamma \ .\end{aligned} \tag{2.15}$$

Therefore the variation of the kinetic energy K with fixed boundary data $\xi(s)$ is calculated as the limit in small δ of

$$\begin{aligned} K_1 - K =& \tfrac{1}{2}\int_{\gamma_1} \xi(s)G(\gamma_1)\xi(s)\,dS_{\gamma_1} - \tfrac{1}{2}\int_{\gamma} \xi(s)G(\gamma)\xi(s)\,dS_{\gamma} \\ =& \tfrac{1}{2}\int_{\gamma_1} \Phi_1(\gamma_1)N_1\cdot\nabla\Phi_1(\gamma_1)\,dS_{\gamma_1} - \tfrac{1}{2}\int_{\gamma}\Phi(\gamma)N\cdot\nabla\Phi(\gamma)\,dS_{\gamma} \\ =& \int_{\gamma_1}(\Phi_1-\Phi)(\gamma_1)N_1\cdot\nabla\Phi_1(\gamma_1)\,dS_{\gamma_1} \\ &+ \tfrac{1}{2}\int_{\gamma_1}\Phi(\gamma_1)N_1\cdot\nabla\Phi(\gamma_1)\,dS_{\gamma_1} - \tfrac{1}{2}\int_{\gamma}\Phi(\gamma)N\cdot\nabla\Phi(\gamma)\,dS_{\gamma} + \mathcal{O}(\delta^2)\ . \end{aligned}$$

Using (2.14) in the first term and (2.15) in the second and third,

$$\begin{aligned} K_1 - K =& \int_{\gamma} -(\nabla\Phi\cdot N)^2\,N\cdot\delta\gamma(s) - (\nabla\Phi\cdot N)(\nabla\Phi\cdot T)\,T\cdot\delta\gamma(s)dS_{\gamma} \\ &+ \tfrac{1}{2}\int_{\gamma}|\nabla\Phi|^2 N\cdot\delta\gamma\,dS_{\gamma} + \mathcal{O}(\delta^2)\ . \end{aligned}$$

Furthermore, both of the velocity potentials Φ and Φ_1 satisfy Neumann boundary conditions on the bottom $(x,y) = b(s)$. Thus $N\cdot\nabla\Phi(\gamma(s)) = G(\gamma)\xi(s)$ and $T\cdot\nabla\Phi(\gamma(s)) = \frac{1}{|\partial_s\gamma|}\partial_s\xi(s)$, giving an expression in the limit as $\delta\to 0$ for $\mathrm{grad}_{\delta\gamma}K$, namely

$$\begin{aligned} (2.16)\quad \langle\delta K\cdot\delta\gamma\rangle_{\gamma} =& \int_{\gamma}\mathrm{grad}_{\delta\gamma}K\cdot\delta\gamma\,dS_{\gamma} \\ =& \tfrac{1}{2}\int_{\gamma} -\big(G(\gamma)\xi\big)^2 N\cdot\delta\gamma + \Big(\frac{1}{|\partial_s\gamma|}\partial_s\xi\Big)^2 N\cdot\delta\gamma - 2\Big(\frac{1}{|\partial_s\gamma|}\partial_s\xi G(\gamma)\xi\Big)T\cdot\delta\gamma\,dS_{\gamma}\ . \end{aligned}$$

With these expressions in hand, we conclude that the equations of motion for the problem of water waves takes the canonical form of a Hamiltonian system;

$$\begin{aligned} N\cdot\partial_t\gamma &= \mathrm{grad}_{\xi}H \\ \partial_t\xi &= -\mathrm{grad}_{N\cdot\delta\gamma}H\ . \end{aligned} \tag{2.17}$$

In general the choice of coordinatization of the free surface is made separately from the decomposition of the tangent space T_γ into its normal and tangential components. Variations $Y(s) = \delta\gamma(s)$ of γ are necessarily constrained by the coordinate choice to the class of *admissible variations.* The choice of coordinitization determines the tangential component of the velocity $\tau = T\cdot\partial_t\gamma$ as a function of the normal component, through the constraints imposed by the coordinatization of the free surface. This applies in particular to the time derivative of the curve, $\dot\gamma(s)\in T_\gamma$. That is, coordinitization dictates a relation between $T\cdot\delta\gamma$ and $N\cdot\delta\gamma$, say $T\cdot\delta\gamma = \mathcal{T}(\gamma)(N\cdot\delta\gamma)$ in somewhat abstract terms. Thus, in terms of such a coordinate choice,

$$\mathrm{grad}_{N\cdot\delta\gamma}K = \tfrac{1}{2}\Big[\Big(\frac{1}{|\partial_s\gamma|}\partial_s\xi\Big)^2 - \big(G(\gamma)\xi\big)^2 - 2\Big(\frac{1}{|\partial_s\gamma|}\partial_s\xi G(\gamma)\xi\mathcal{T}(\gamma)\Big)\Big]\ . \tag{2.18}$$

This gradient is worked out in detail for several standard choices of parametrization in the subsection below.

2.3. **Particular coordinates.** Common choices for the parametrization of the free surface are: (1) the classical case of free surfaces given as a graph in $x \in \mathbb{R}^1$, which does not allow for overturning free surfaces. (2) arc length parametrization of $\gamma(s)$ which are able to describe overturning wave profiles. In these coordinates we have seen that $\partial_s \tau = \kappa n$. (3) Lagrangian coordinates, for which fluid particle positions are advected by the flow, $\partial_t(X(t), Y(t)) = u(X(t), Y(t)) = \nabla\varphi(\gamma(t,\cdot))$, or (4) conformal mapping coordinates as used in [10]. Specifying the coordinatization of free surface curves γ in cases (1)(2) and (4) gives rise to systems of constraints which may be considered to be holonomic as they are imposed independently of the velocity $\dot\gamma$. The parametric specification by Lagrangian coordinates in contrast is a nonholonomic constraint.

The traditional choice of parametrization is (1) to write the surface as a graph; in such graph coordinates, where $\gamma = (x, \eta(x))$, and the pair of variables $(\eta(x), \xi(x))$ are canonically conjugate as given by Zakharov [15]. With the expression for the kinetic energy K in terms of the Dirichlet – Neumann operator $G(\eta)$ as in [4], then

$$H(\eta, \xi) = \int_{\mathbb{R}^1} \tfrac{1}{2}\xi G(\eta)\xi\,\sqrt{1+(\partial_x\eta)^2}\,dx + \frac{g}{2}\int_{\mathbb{R}^1} \eta^2\,dx\ .$$

In these graph coordinates, $\gamma = (x, \eta(x))$ so that admissible variations are $\delta\gamma = (0, \delta\eta)$, and the relationship between $N\cdot\delta\gamma$ and $T\cdot\delta\gamma$ is given by

$$T\cdot\delta\gamma = \partial_x\eta\, N\cdot\delta\gamma\ .$$

The gradient of the kinetic energy is thus

$$\tfrac{1}{2}\Big[\Big(\frac{(\partial_x\xi)^2}{1+(\partial_x\eta)^2}\Big) - \big(G(\eta)\xi\big)^2 - 2\Big(\frac{1}{\sqrt{1+(\partial_x\eta)^2}}\partial_x\xi G(\eta)\xi\,\partial_x\eta\Big)\Big]\ .$$

Because of this,

$$N\cdot\dot\gamma = \frac{1}{\sqrt{1+(\partial_x\eta)^2}}\begin{pmatrix}-\partial_x\eta\\ 1\end{pmatrix}\cdot\begin{pmatrix}0\\ \partial_t\eta\end{pmatrix}$$

and the resulting equations (1.5) for water waves are given by

$$\frac{1}{\sqrt{1+(\partial_x\eta)^2}}\partial_t\eta = \delta_\xi H = G(\eta)\xi$$

$$\partial_t\xi = -\delta_\gamma H = -g\eta + \tfrac{1}{2}\Big[(G(\eta)\xi)^2 - \frac{(\partial_x\xi)^2}{1+(\partial_x\eta)^2} + \frac{2\partial_x\xi G(\eta)\xi\,\partial_x\eta}{\sqrt{1+(\partial_x\eta)^2}}\Big]\ .$$

Calculating for an independent verification of (2.4), one finds that in graph coordinates

$$\tau = \frac{\partial_t\eta\partial_x\eta}{\sqrt{1+(\partial_x\eta)^2}} = G(\eta)\xi\,\partial_x\eta\ .$$

This system of equations, modulo the difference in normalization of the Dirichlet – Neumann operator $G(\eta)$, appears in [4], and is used in the existence theory for water waves and many of its distinguished scaling limits in [11][12].

Coordinates given in terms of arc length along the free surface $\gamma(s)$ allow the system (1.3)(1.4) to describe overturning wave profiles. This choice of coordinates implies in particular that $\partial_s\gamma(s) = T(s)$ and $\partial_s T \perp T$, since

$$|\partial_s\gamma(s)|^2 = 1 \ , \quad 0 = \partial_s|\partial_s\gamma(s)|^2 = 2\partial_s\gamma \cdot \partial_s^2\gamma \ .$$

Indeed any vector field $Y(s)$ along the curve $\gamma(s)$ that arises from an infinitessimal motion which preserves the arc length parametrization must satisfy

$$0 = \frac{d}{d\delta}\Big|_{\delta=0} |\partial_s\gamma + \delta Y|^2 = 2\partial_s\gamma \cdot Y = 2T \cdot Y \ .$$

Admissible variations $\delta\gamma(s)$ are arc length preserving in the present case, implying that $Y = \partial_s\delta\gamma$ is as above, and hence

$$0 = \partial_s(T \cdot \delta\gamma) = \partial_s T \cdot \delta\gamma + T \cdot \partial_s\delta\gamma = \kappa N \cdot \delta\gamma \ .$$

This is the relationship between tangential and normal variations that applies to the gradient of the kinetic energy, an interesting geometrical aspect of this choice of coordinates. The resulting Bernoulli equations of motion are

$$\partial_t\xi = -g\gamma_2 - \tfrac{1}{2}\Big[\Big(\frac{1}{|\partial_s\gamma|}\partial_s\xi\Big)^2 - \big(G(\gamma)\xi\big)^2 - 2\Big(\frac{1}{|\partial_s\gamma|}\partial_s\xi G(\gamma)\xi\mathcal{T}(\gamma)\Big)\Big]$$

where $\mathcal{T}(\gamma)$ satisfies

$$\partial_s\big(G(\gamma)\xi\mathcal{T}\big) = \kappa G(\gamma)\xi \ .$$

3. Mapping properties of Birkhoff normal forms

Let us continue to restrict ourselves to the case of $d = 2$. In the coordinates above, for $x \in \mathbb{R}^1$, and in which the free surface is described as a graph $\gamma(t,x) = (x, \eta(t,x))$, and following Zakharov [15], the problem of water waves (1.1)(1.2)(1.4) can be rewritten

$$\partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \partial_\eta H \\ \partial_\xi H \end{pmatrix}$$

which is a Hamiltonian system in Darboux coordinates. One expresses the Hamiltonian $H = H(\eta, \xi)$ using the Dirichlet – Neumann operator

$$\xi(x) \mapsto \varphi(x,y) \mapsto N \cdot \nabla\varphi := G(\eta)\xi(x) \ ,$$

so that

$$H(\eta,\xi) = \int \tfrac{1}{2}\xi G(\eta)\xi\sqrt{1+(\partial_x\eta)^2}\,dx + \int \tfrac{g}{2}\eta^2\,dx + \sigma\int \sqrt{1+|\partial_x\eta|^2} - 1\,dx \ . \tag{3.1}$$

The Dirichlet – Neumann operator $G(\eta)\xi$ is linear in ξ and nonlocal in both η and ξ. It is admittedly quite complicated in its dependence upon η. However it is known [1] that for $\eta \in B_R(0) \subseteq C^1(\mathbb{R}^{d-1})$, then $G(\eta)$ with respect to η as an analytic operator $H^1 \to L^2$. At this point, and by convention, we normalize $\bar{G}(\eta) := G(\eta)\sqrt{1+|\partial_x\eta|^2}$ so that $\bar{G}(\eta)$ is Hermetian with respect to the metric dx. Therefore in this setting it can be described in terms of its Taylor series expansion

$$\bar{G}(\eta)\xi := G^{(0)}\xi + \sum_{m\geq 1} G^{(m)}(\eta)\xi \ .$$

This gives in turn an expression for the water waves Hamiltonian, which in the case of $\sigma = 0$ is

$$H(\eta,\xi) := H^{(0)}(\eta,\xi) + \sum_{m\geq 1} H^{(m)}(\eta,\xi) \tag{3.2}$$
$$H^{(0)} = \int \tfrac{1}{2}\xi G^{(0)}\xi + \tfrac{g}{2}\eta^2\, dx\ , \quad H^{(m)} = \int \tfrac{1}{2}\xi G^{(m)}(\eta)\xi\, dx\ .$$

A normal form for a Hamiltonian is the result of a change of variables $w := \tau(z)$ which eliminates inessential nonlinearities from the equations of motion. The Birkhoff procedure is to perform normal forms transformations for each order of the Taylor series in turn, while at each step employing a canonical transformation $\tau^{(m)}$ in order to preserve the Hamiltonian character of the evolution equations. The goal is to achieve the form of the transformed Hamiltonian $H^+(w)$, where $w = \tau^{(m)} \dots \tau^{(3)} z$

$$H^+(w) = H^{(2)} + \left(Z^{(3)} + \cdots + Z^{(m)}\right) + R^{(M+1)}$$

where the terms $Z^{(j)}$ are free of nonresonant terms, consisting only of resonant monomials of degree j. In this context we study the problem of water waves under periodic boundary conditions in $x \in \mathbb{R}^1$. In this case in particular, $Z^{(3)} = 0$ when $\sigma = 0$. The Birkhoff normal form for water waves in this setting was studied in [2], where the normal forms transformation $\tau^{(3)}$ is shown to be well defined as a mapping of analytic scales of Banach spaces. The proof is based on the abstract version of the Cauchy – Kowalevsky theorem by Nirenberg [13] and Nishida [14]. However in the case that $h = +\infty$ one can do better, exhibiting normal forms transformations that map a given Sobolev to itself; the importance of this class of results is that they bring significant information to the analysis of solutions to the equations of motion. This is the content of the following several theorems.

Theorem 3.1 ([6]). *Let $d = 2$, $\sigma = 0$ (and $h = +\infty$) and fix $r > 3/2$. There exists $R_0 > 0$ such that for any $R < R_0$, on every neighborhood $B_R(0) \subseteq H^r_\eta \times H^r_\xi$ the Birkhoff normal forms transformation $\tau^{(3)}$ is defined.*

$$\tau^{(3)} \ : \ B_R(0) \to B_{2R}(0)$$
$$(\tau^{(3)})^{-1} \ : \ B_{R/2}(0) \to B_R(0)$$

The result is that $w = \tau^{(3)}(z)$ transforms $H(z)$ to

$$H^+(w) = H^{(2)}(w) + 0 + R^{(4)}(w)$$

The fourth order Birkhoff normal form is also addressed in [6], in which case $\tau^{(4)}$ eliminates most resonant terms, however in order to remain a bounded transformation some fourth order interactions remain, coupling certain pairs of high frequency modes with two lower frequency modes. On a formal level this was addressed in the two articles [8][7], with the surprising result that the formal fourth order normal form is completely integrable. To describe this result, define complex symplectic coordinates based on the

Fourier mode decompositions of (η, ξ);

$$(\eta_k, \xi_k) := \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-ikx}(\eta(x), \xi(x))\, dx$$
$$z_k := \frac{1}{\sqrt{2}}\big(Q_k\eta_k + iQ_k^{-1}\xi_k\big) \ , \qquad Q_k = \Big(\frac{g}{|k|}\Big)^{1/4}$$

and action variables

$$I_1(k) = \tfrac{1}{2}(z_k\bar{z}_k + z_{-k}\bar{z}_{-k}) \ , \qquad I_2(k) = \tfrac{1}{2}(z_k\bar{z}_k - z_{-k}\bar{z}_{-k}) \ .$$

Theorem 3.2 (Dyachenko & Zakharov [8], Craig & Worfolk [7]). *The formal second Birkhoff normal form is*

$$\begin{aligned} \overline{H}^{+} =& \sum_k \omega_k I_1(k) - \frac{1}{2\pi}\sum_k |k|^3\big(I_1(k)^2 - 3I_2(k)^2\big) \\ &+ \frac{4}{\pi}\sum_{|k_4|<|k_1|} I_2(k_1)I_2(k_4) + \overline{R}^{(5)}(w) \\ =& H^{(2)}(I) + \overline{H}^{(4)}(I) + \overline{R}^{(5)}(w) \end{aligned} \tag{3.3}$$

In particular, Benjamin - Feir resonant interactions do not enter in the Hamiltonian, as the four wave interaction coefficients vanish. The quartet interactions occurring in the Hamiltonian H are indexed by the Fourier indices

$$\{(k_1, k_2, k_3, k_4) \in \mathbb{Z}^4 \ : \Sigma_{j=1}^4 k_j = 0\} \ .$$

The implication of the above calculation is that the coefficients on the transformed Hamiltonian $H^{(4)+}$ satisfy $c_{k_1k_2k_3k_4} = 0$ whenever the non-integrable resonant combinations of wavenumbers and frequencies occur, namely

$$k_1 : k_2 : k_3 : k_4 = n^2 : (n+1)^2 : n^2(n+1)^2 : -(n^2+n+1)^2)$$
$$\omega_1 : \omega_2 : \omega_3 : \omega_4 = n : -(n+1) : -n(n+1) : (n^2+n+1)$$

The only remaining resonant terms are in cases $k_1 + k_2 = 0$ and $k_3 + k_4 = 0$, or else $k_1 + k_3 = 0$ and $k_2 + k_4 = 0$, whose role in the Hamiltonian $H^{(4),+}$ is described by (3.3). Incidentally, $H^{(5)}$ contains resonant terms that depend upon angles as well as action variables, and are therefore nonintegrable in action - angle variables.

It is a fair question to ask about the function space mapping properties of the normal forms transformation to fourth order. Among other things these would determine the degree to which one controls the error terms resulting form the above formal calculations. To this end define the energy space $E^r := H^r_\eta \oplus H^{r+1/2}_\xi$. Due to Theorem 3.2 the *resonant* set is

$$R = \{k_1k_4, k_2k_3 > 0 : k_1 + k_2 = 0 = k_3 + k_4 \text{ or } k_1 + k_3 = 0 = k_2 + k_4\}$$

Define a quasihomogeneous neighborhood of R to be a set of near-resonant modes

$$C_R^+ :=\{(k_1,k_2,k_3,k_4) \in \mathbb{Z}^4 : \Sigma_{j=1}^4 k_j = 0 \quad \text{satisfying}$$
$$|k_1+k_2| < (|k_1|+|k_2|)^{1/4} \ , \ |k_3+k_4| < (|k_3|+|k_4|)^{1/4}$$
$$\text{and either } |k_3| < 1/10|k_1| \text{ or } |k_2| < 1/10|k_4|\}$$

The neighborhood $C_R^- \subseteq \mathbb{Z}^4$ is similar, exchanging the roles of $k_2 \leftrightarrow k_3$.

Theorem 3.3 (fourth order partial normal forms [6]). *Let $Q \subseteq \mathbb{Z}^4$ be any set of quartet interactions, such that*

$$Q \backslash B_\rho(0) \cap C_R^\pm = \emptyset \qquad \rho < +\infty$$

is symmetric under $(k \leftrightarrow -k)$, $(k_2 \leftrightarrow k_3)$ and $(k_1 \leftrightarrow k_4)$.

Then for $r > 3/2$ there exists a canonical transformation $\tau_Q^{(4)}$ on $B_R(0) \subseteq E^r$ such that

$$\tau_Q^{(4)} : H^{(2)} + \overline{H}^{(4)} + R^{(5)} \to \widetilde{H} = H^{(2)} + \widetilde{Z}^{(4)} + \widetilde{R}^{(5)} \ .$$

such that supp $\tilde{Z}^{(4)} \subseteq C_R^\pm$. For $(k_1,k_2,k_3,k_4) \in R$ then

$$\widetilde{Z}^{(4)}_{k_1,k_2,k_3,k_4} = Z^{(4)}_{k_1,k_2,k_3,k_4}(I)$$

This is to say that most but not all of the nonresonant interactions can be eliminated from the Hamiltonian $H^{(4)}$ by a canonical transformation on a fixed Sobolev space. the difficulty with the remaining terms is due to the loss of 1/4 derivative in the Hamiltonian vector field whose time-one flow is the canonical transformation to normal form. The four mode nonresonant interactions that are not able to be eliminated by the normal form are those that are asymptotically too close to resonance, as quantified by the quasihomogeneous sets $C_R^\pm$. While this analysis does not quite allow one to remove absolutely all of the resonant interactions from the Hamiltonian, it does allow one a partial normal form, which is useful to prepare the system of equations for further analysis, such as for a KAM theorem or a Nekhoroshev stability analysis.

In the case when the effects of surface tension are taken into account, $Z^{(3)}$ consists of special three-wave resonant terms having to do with the phenomenon of Wilton ripples. Indeed when $h < +\infty$ there are only a finite number of such three wave resonances. In this case a similar statement to that of Theorem 3.1 holds, for $r > 5/2$ with however a possible nonzero $Z^{(3)}$, this is the topic of reference [5].

Theorem 3.4 ([5]). *In the case of positive surface tension, with $0 < h \leq +\infty$, for $r > 1$, the normal forms transformation $\tau^{(3)}$: $B_R(0) \subseteq H_\eta^{r+1} \oplus H_\xi^{r+1/2} \to H_\eta^{r+1} \oplus H_\xi^{r+1/2}$, is a continuous mapping which eliminates all nonresonant terms from the equations of motion. However for certain values of the parameters (g,h,σ) the normal form $Z^{(3)}$ is nonzero. These are three wave interactions whose physical manifestation is the phenomeon of Wilton ripples.*

The transformations $\tau^{(m)}$ above, $m = 3,4$, are continuous mappings in the Hilbert spaces specified above. One may ask whether they are smooth mappings, in the standard sense that the Jacobian is a bounded operator. This is not the case, the flows which define these transformation being determined by unbounded vector fields. This is a situation that is expected in the setting of partial differential equations. However the transformations $\tau^{(m)}$ are smooth on a scale of Hilbert spaces. That is, in the case with surface tension the Jacobian $\partial_z\tau^{(3)}$ is a bounded map on energy spaces of slightly lower regularity, namely

$$\partial_z\tau^{(3)} \ : H^{r+1/2}_\eta \oplus H^r_\xi \to H^{r+1/2}_\eta \oplus H^r_\xi \ .$$

In the case of zero surface tension the Jacobian maps the analogous result is that

$$\partial_z\tau^{(3)} \ : H^{r-1}_\eta \oplus H^{r-1}_\xi \to H^{r-1}_\eta \oplus H^{r-1}_\xi \ .$$

A second comment is that the case $h < +\infty$ and $\sigma = 0$ is not covered by the Birkhoff normal forms results above. This has been discussed by Zakharov [16] in a formal setting. However the several special cancellations that occur in the calculation of $K^{(3)}$ in particular do not hold in the finite depth setting, and certain estimates of the resulting Hamiltonian vector field $X^{K^{(3)}}$ no longer hold. It would be important to understand the analysis of this setting in a deeper way. A third comment is the observation that these transformations generally mix the domain η and the potential ξ. This is an interesting parallel to the fact that canonical transformations mix the configuration space variables q with the momentum variables p, in accordance with the original ideas of Hamilton.

4. Outline of proof

This final section gives an outline of the proof of the above Theorems 3.1, 3.3 and 3.4 on the mapping properties of the normal forms transformations. Our source of canonical transformations is to make flows from auxiliary Hamiltonian vector fields, which are designed so that their time-one flow is the desired mapping to the desired normal form. In this section we will outline the mapping $\tau^{(3)}$ of Theorem 3.1, which is rather a special case. We also describe the strategy of proof for Theorem 3.4 where the main technique is to derive energy estimates for the flow of the auxiliary Hamiltonian.

4.1. **Proof of Theorem 3.4.** The goal is to identify the resonant terms in the term $H^{(3)}$ of the water waves Hamiltonian, in the case that $h = +\infty$ and $\sigma = 0$. In this case

$$\begin{aligned} H^{(3)}(\eta,\xi) &= \tfrac{1}{2}\int_0^{2\pi} \xi(D\eta D - G^{(0)}\eta G^{(0)})\xi \, dx \\ &= \frac{1}{2\pi}\sum_{k_1+k2+k_3=0} (k_1k_3 + |k_1||k_3|)\xi_{k_1}\eta_{k_2}\xi_{k_3} \end{aligned}$$

Proposition 4.1. *(Conservation of mass) One can choose initial data $\eta_0(x) = \eta(x,0)$ such that $M = \int_0^{2\pi} \eta(x)\,dx = 2\pi\eta_0 = 0$.*
(Conservation of momentum) Unless $\langle k, p-q\rangle = 0$ the interaction coefficients of the Hamiltonian satisfy

$$c_{pq} = 0$$

(Nonresonance) There are no nonzero $m = 3$ resonances for the water wave equations. Indeed

$$\omega(k_1) \pm \omega(k_2) \pm \omega(k_3) = 0 \qquad \text{and} \qquad k_1 + k_2 + k_3 = 0 \tag{4.1}$$

implies $k_\ell = 0$ *for some* $\ell = 1, 2, 3$

Our strategy follows classical lines, determining the desired canonical transformation as the time-one flow of an auxiliary Hamiltonian system. The auxiliary Hamiltonian is determined by the *cohomological equation*

$$\{K^{(3)}, H^{(2)}\} + H^{(3)} = 0 \tag{4.2}$$

which is to be solved for $K^{(3)}$. This equation is a linear equation in the space of homogeneous Hamiltonian functions. The transformation $\tau^{(3)}$ is constructed as the time $s = 1$ flow of the Hamiltonian vector field of $K^{(3)}$, that is

$$\frac{d}{ds} z = J\mathrm{grad}_z K^{(3)} := X^{K^{(3)}}(z) .$$

In case $h = +\infty$ the auxiliary Hamiltonian $K^{(3)}$ turns out to be remarkably simple

$$K^{(3)}(\eta, \xi) = \tfrac{1}{2} \int \big(i\mathrm{sgn}(D)\eta\big)^2 |D|\xi \, dx = \tfrac{1}{2} \int \tilde{\eta}^2 \partial_x \tilde{\xi} \, dx \tag{4.3}$$

where $(\tilde{\eta}, \tilde{\xi}) := -i\mathrm{sgn}(D)(\eta, \xi)$ the Hilbert transform of our original physical variables. Incidentally, the transformation $(\eta, \xi) \mapsto (\tilde{\eta}, \tilde{\xi})$ is canonical, as is the Fourier transform.

The auxiliary flow giving $\tau^{(3)}$ is thus the solution map of

$$\begin{aligned} \partial_s \tilde{\eta} = -\tilde{\eta}\partial_x \tilde{\eta} &= \mathrm{grad}_{\tilde{\xi}} K^{(3)} \\ \partial_s \tilde{\xi} = -\tilde{\eta}\partial_x \tilde{\xi} &= -\mathrm{grad}_{\tilde{\eta}} K^{(3)} \end{aligned}$$

This is the flow for Burger's equation for $\tilde{\eta}$, and its linearization for $\tilde{\xi}$, which is an interesting fact [6]. It is related to a similar result of Hunter & Ifrim [9], who also found that the Burger's equation eliminates the quadratic terms of the Burger's – Hilbert model equations, and as well it performs the same function for the water wave equations on deep water.

To find a solution $K^{(3)}$ that satisfies (4.2) is easier in hindsight than in foresight. Indeed one only has to check the Poisson bracket $\{H^{(2)}, K^{(3)}\}$ gives the right result. To this end,

$$\begin{aligned} \{H^{(2)}, K^{(3)}\} &= \tfrac{1}{2} \int \eta |D| \big(i\mathrm{sgn}(D)\eta\big)^2 + |D|\xi \big(i\mathrm{sgn}(D)(i\mathrm{sgn}(D)\eta |D|\xi)\big) \, dx \\ &= \int \frac{1}{6} \partial_x \big(i\mathrm{sgn}(D)\eta\big)^3 \, dx - \int \big(i\mathrm{sgn}(D)\eta\big) \partial_x \xi |D| \xi \, dx \end{aligned}$$

The first term vanishes as it is a total derivative. To address the second we use an identity for functions in Hardy space, that is, when $g = -i\mathrm{sgn}(D) f$ is the Hilbert transform, then $f + ig$ is a Hardy function, for which there is a formula for the Hilbery transform of products;

$$i\mathrm{sgn}(D)\big(fg\big) = \tfrac{1}{2}\big(f^2 - g^2\big) .$$

This expression gives

$$\{H^{(2)}, K^{(3)}\} = \tfrac{1}{2} \int \eta \big((\partial_x \xi)^2 - (|D|\xi)^2\big) \, dx = H^{(3)} ,$$

verifying the equation (4.2).

4.2. **Proof of Theorem 3.4.** Including the effects of surface tension, $\sigma > 0$ and allowing for $0 < h \leq +\infty$, the dispersion relation is

$$\omega^2(k) = (g + \sigma k^2)k \tanh(hk) \ .$$

With this dispersion relation there are cases of resonant triples for particular values of the parameters (g, h, σ), namely for certain indices (k_1, k_2, k_3) which are nonzero, $k_1 k_2 k_3 \neq 0$, then $(\omega(k_1), \omega(k_2), \omega(k_3))$ satisfies (4.1). However if $h < +\infty$ these lie in a compact set in k-space.

Aside from these resonances, solve the cohomological equation for $K^{(3)}$

$$\{H^{(2)}, K^{(3)}\} = H^{(3)} - Z^{(3)}$$

where $Z^{(3)}$ consists only of resonant terms, that is it Poisson commutes with $H^{(2)}$. The strategy of the proof is to show that the Hamiltonian vector field $X^{K^{(3)}}(\eta, \xi)$ satisfies energy estimates on neighborhoods $B_R(0)$ in the function space $H^{r+1}_\eta \oplus H^{r+1/2}_\xi$ for $r > 1$. The method of proof of Theorem 3.3 is similar, with the additional difficulty that the energy estimates for $X^{K^{(4)}}(\eta, \xi)$ loses 1/4 derivative unless the domain Q is restricted to lie outside of the quasihomogeneous sets $C^{\pm}_R$.

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