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1D BOLTZMANN EQUATION IN A PERIODIC BOX

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ABSTRACT. We study the nonlinear stability of the Boltzmann equation in the 1D periodic box with size $1/\varepsilon$, where $0<\varepsilon\ll 1$ is the Knudsen number. The convergence rate is $(1+t)^{-1/2}\ln(1+t)$ for small time region and exponential for large time region. Moreover, the exponential rate depends on the size of the domain (Knudsen number). This problem is highly nonlinear and hence we need more careful analysis to control the nonlinear term.

1. INTRODUCTION

1.1. The 1D Boltzmann equation. The 1D Boltzmann equation for the hard sphere model reads

$$\begin{cases} \partial_t F+\xi_1\partial_x F=\dfrac{1}{\varepsilon}Q(F,F)\,, \\ F(0,x,\xi)=F_0(x,\xi)\,, \end{cases} \tag{1}$$

where $Q(\cdot,\cdot)$ is the so-called collision operator given by

$$Q(g,h)=\frac{1}{2}\int_U\big[-g(\xi)h(\xi_*)-g(\xi_*)h(\xi)+g(\xi')h(\xi_*')+g(\xi_*')h(\xi')\big]|(\xi-\xi_*)\cdot\Omega|d\xi_*d\Omega$$

with

$$U=\big\{(\xi_*,\Omega)\in\mathbb{R}^3\times\mathbb{S}^2:(\xi-\xi_*)\cdot\Omega\geq 0\big\}$$

and

$$\xi'=\xi-[(\xi-\xi_*)\cdot\Omega]\Omega\,,\quad \xi_*'=\xi_*+[(\xi-\xi_*)\cdot\Omega]\Omega\,.$$

Here ε is the Knudsen number, the microscopic velocity $\xi\in\mathbb{R}^3$ and the space variable $x\in\mathbb{T}_1^1$, the 1D periodic box with unit size. In order to remove the parameter ε from the equation, we introduce the new scaled variables:

$$\widetilde{x}=\frac{1}{\varepsilon}x\,,\quad \widetilde{t}=\frac{1}{\varepsilon}t\,,$$

then after dropping the tilde, the equation (1) becomes

$$\begin{cases} \partial_t F+\xi_1\partial_x F=Q(F,F)\,,\quad (t,x,\xi)\in\mathbb{R}^+\times\mathbb{T}^1_{1/\varepsilon}\times\mathbb{R}^3\,, \\ F(0,x,\xi)=F_0(x,\xi)\,, \end{cases} \tag{2}$$

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where $\mathbb{T}^1_{1/\varepsilon}$ denotes the 1-dimensional periodic box with size $1/\varepsilon$. The conservation laws of mass, momentum and energy can be formulated as

$$\frac{d}{dt}\int_{\mathbb{T}^1_{1/\varepsilon}}\int_{\mathbb{R}^3}\left\{1,\xi,|\xi|^2\right\}F(t,x,\xi)d\xi dx = 0\,. \tag{3}$$

It is well-known that the Maxwellians are steady state solutions to the Boltzmann equation. Thus, it is natural to linearize the Boltzmann equation (2) around a global Maxwellian

$$w(\xi) = \frac{1}{(2\pi)^{3/2}}\exp\left(\frac{-|\xi|^2}{2}\right),$$

with the standard perturbation $F(t,x,\xi)$ and $F_0(x,\xi)$ to w as

$$F = w + w^{1/2}f\,, \quad F_0 = w + \eta w^{1/2}f_0\,, \quad \eta \ll 1\,.$$

Then after substituting into (2), we have the 1D Boltzmann equation near Maxwellian

$$\begin{cases} \partial_t f + \xi_1\partial_x f = Lf + \Gamma(f,f)\,, & (t,x,\xi)\in\mathbb{R}^+\times\mathbb{T}^1_{1/\varepsilon}\times\mathbb{R}^3\,,\\ f(0,x,\xi) = \eta f_0(x,\xi)\,, & \\ Lf = w^{-1/2}\left[Q(w,w^{1/2}f) + Q(w^{1/2}f,w)\right], & \\ \Gamma(f,f) = w^{-1/2}Q(w^{1/2}f,w^{1/2}f)\,. & \end{cases} \tag{4}$$

The null space of L is a five-dimensional vector space with the orthonormal basis $\{\chi_i\}_{i=0}^4$, where

$$\{\chi_0,\chi_i,\chi_4\} = \left\{w^{1/2},\xi_i w^{1/2},\frac{1}{\sqrt{6}}(|\xi|^2-3)w^{1/2}\right\}, \quad i=1,2,3\,.$$

Assuming the initial density distribution function $F_0(x,\xi)$ has the same mass, momentum and total energy as the Maxwellian w, we can further rewrite the conservation laws (3) as

$$\int_{\mathbb{T}^1_{1/\varepsilon}}\int_{\mathbb{R}^3} w^{1/2}(\xi)\left\{1,\xi,|\xi|^2\right\}f_0(x,\xi)d\xi dx = 0\,. \tag{5}$$

This means that the initial condition $f_0(x,\xi)$ satisfies the zero moments condition.

1.2. **Review of Previous Works.** There have been extensive investigations into the rate of convergence for the nonlinear Boltzmann equation, let us mention some of them. In the context of perturbed solutions, the first result was given by Ukai [10], where spectral analysis was used to obtain the exponential rates for the Boltzmann equation with hard potentials on the torus.

The so called $L^2 - L^\infty$ framework has been developed by Guo [4]. The name is self-descriptive: the coercive property of the linearized collision operator is captured in L^2 space, whereas the weighted L^∞ estimate is derived by careful analysis of the iterated Duhamel formula to control the bilinear perturbation. This idea can also be applied to the Boltzmann equation near rotational Maxwelloian [5] or relativistic Boltzmann equation [9].

Besides those methods mentioned above for the study of rates of convergence, the entropy method, which has general applications in existence theory for nonlinear equations. By using this method, as well as an elaborate analysis of functional inequalities, time-derivative estimates and interpolation, Desvillettes and Villani [1] first obtained the almost exponential rate of convergence of solutions to the Boltzmann equation on the torus with soft potentials for large initial data, under the additional regularity conditions that all the moments of f are uniformly bounded in time and f is bounded in all Sobolev spaces uniformly in time. By finding some proper Lyapunov functionals defined over the Hilbert space, Mouhot and Neumann [6] obtained exponential rates of convergence for some kinetic models with general structures in the case of the torus.

For the 1D Boltzmann equation, we need to mention the method of Green's functions, it was found by Liu and Yu [7, 8] to expose pointwise large time behavior of solutions to the Boltzmann equation and get detailed information on how varies types of fluid-kinetic waves propagate. In Liu and Yu's paper, they got the nonlinear stability of the Boltzmann equation in the 1D whole space case.

Under the same setting of this paper, the 3D case can be found in [12]. However, in 1D case, the nonlinear effect is much stronger than the 3D case.

1.3. **Main result.** Before the presentation of the main theorem, let us define some notations in this paper. For the microscopic variable ξ, since we consider the one-dimensional problem, by a shift of the variables ξ_2 and ξ_3, we can restrict the functional space to

$$L^2_\xi \equiv \left\{ f : \int_{\mathbb{R}^3} f\,\{\chi_2, \chi_3\}\, d\xi = 0, \quad \int_{\mathbb{R}^3} |f|^2 d\xi < \infty \right\},$$

and we denote

$$\|f\|_{L^2_\xi} = \left(\int_{\mathbb{R}^3} |f|^2 d\xi \right)^{1/2}.$$

The Sobolev space of functions with all its s-th partial derivatives in L^2_ξ will be denoted by H^s_ξ. The L^2_ξ inner product in $\mathbb{R}^3$ will be denoted by $\langle \cdot, \cdot \rangle_\xi$ and the weighted sup norm is denoted by

$$\|f\|_{L^\infty_{\xi,\beta}} = \sup_{\xi \in \mathbb{R}^3} |f(\xi)|(1 + |\xi|)^\beta .$$

For the space variable x, we have the similar notations. In fact, L^p_x, $1 \le p < \infty$ is the classical Banach space with norm

$$\|f\|_{L^p_x} = \left(\int_{\mathbb{T}^1_{1/\varepsilon}} |f|^p dx \right)^{1/p},$$

and the Sobolev space of functions with all its s-th partial derivatives in L^2_x will be denoted by H^s_x. We define the sup norm by

$$\|f\|_{L^\infty_x} = \sup_{x \in \mathbb{T}^1_{1/\varepsilon}} |f(x)| .$$

In this paper, if $f \in L^\infty_x L^\infty_{\xi,\beta} \cap L^1_x L^\infty_{\xi,\beta}$, we define the triple norm $|||\cdot|||_\beta$ by

$$|||f|||_\beta = \|f\|_{L^\infty_x L^\infty_{\xi,\beta}} + \|f\|_{L^1_x L^\infty_{\xi,\beta}}.$$

For simplicity of notations, hereafter, we abbreviate " $\le C$ " to " $\lesssim$ ", where C is a positive constant depending only on fixed number.

In the following, we describe our main result.

Theorem 1. *Assuming that* $0 < \varepsilon \ll 1$, $\beta > 5/2$. *Then there exists* $\eta > 0$ *such that if* $F_0(x,\xi) = w + \eta w^{1/2} f_0(x,\xi)$ *with* $f_0 \in L^p_x L^\infty_{\xi,\beta}$, $1 \le p \le \infty$ *and satisfies the zero moments condition* (5), *there exists a unique solution* $F(t,x,\xi) = w + w^{1/2} f(t,x,\xi)$ *to the Boltzmann equation* (2) *such that*

$$\|f\|_{L^\infty_x L^\infty_{\xi,\beta}} \lesssim \eta e^{-\overline{a}\varepsilon^2 t}(1+t)^{-1/2}\ln(1+t)|||f_0|||_\beta$$

for some constant $\overline{a} > 0$.

1.4. **Method of proof and plan of the paper.** Motivated by [7], we want to estimate the "main part" of the solution carefully. More precisely, we decompose our solution as the fluid part and non-fluid part, then one can estimate the leading part of the fluid and non-fluid parts separately, which are the "main part" of the solution. Once the estimate of the leading parts completes, we subtract it and then estimate the tail part. This careful analysis will help us control the nonlinear term.

The paper is organized as follows: we list some properties of the linearized collision operator and some basic estimates in section 2, then proof the main theorem in section 3.

2. Preliminaries

Let us review some basic properties of the linearized collision operator L:

Lemma 2. ([3] Grad's decomposition) *The collision operator* L *consists of a multiplicative operator* $\nu(\xi)$ *and an integral operator* K: $Lf = -\nu(\xi)f + Kf$, *where*

$$Kf = \int_{\mathbb{R}^3} W(\xi,\xi_*) f(\xi_*) d\xi_*$$

is the linear integral operator with kernel

$$W(\xi,\xi_*) = \frac{2}{\sqrt{2\pi}|\xi-\xi_*|}\exp\left\{-\frac{(|\xi|^2-|\xi_*|^2)^2}{8|\xi-\xi_*|^2} - \frac{|\xi-\xi_*|^2}{8}\right\} - \frac{|\xi-\xi_*|}{2}\exp\left\{-\frac{|\xi|^2+|\xi_*|^2}{4}\right\},$$

and the multiplicative operator $\nu(\xi)$ *is given by*

$$\nu(\xi) = \frac{1}{\sqrt{2\pi}}\left[2e^{-\frac{|\xi|^2}{2}} + 2\left(|\xi| + |\xi|^{-1}\right)\int_0^{|\xi|} e^{-\frac{u^2}{2}} du\right].$$

Moreover, for multiplicative operator $\nu(\xi)$, *there exist* $\nu_0, \nu_1 > 0$ *such that*

$$\nu_0(1+|\xi|) \le \nu(\xi) \le \nu_1(1+|\xi|),$$

for some constants $\nu_0, \nu_1 > 0$. *The integral operator* K *has smoothing properties in* ξ, *i.e., there exist constants* C_K *and* $C_{K'}$ *such that*

$$\|Kf\|_{L^\infty_{\xi,0}} \le C_{K'}\|f\|_{L^2_\xi}\,, \quad \|Kf\|_{L^\infty_{\xi,\beta+1}} \le C_K\|f\|_{L^\infty_{\xi,\beta}}\,, \tag{6}$$

for any $\beta \ge 0$.

Lemma 3. (Spectrum of $-i\pi\varepsilon\xi_1 k + L$ [2]) *Given* $\delta > 0$,
(i) *there exists* $\tau_1 = \tau_1(\delta) > 0$ *such that if* $|\varepsilon k| > \delta$,

$$\mathrm{Spec}(\varepsilon k) \subset \{z \in \mathbb{C} : \mathrm{Re}(z) < -\tau_1\}\,. \tag{7}$$

(ii) *If* $|\varepsilon k| < \delta$, *the spectrum within the region* $\{z \in \mathbb{C} : Re(z) > -\tau_1\}$ *consisting of exactly three eigenvalues* $\{\sigma_j(\varepsilon k)\}_{j=1}^3$,

$$\mathrm{Spec}(\varepsilon k) \cap \{z \in \mathbb{C} : \mathrm{Re}(z) > -\tau_1\} = \{\sigma_j(\varepsilon k)\}_{j=1}^3\,, \tag{8}$$

and the corresponding eigenvectors $\{e_j(\varepsilon k)\}_{j=1}^3$. *They have the expansions*

$$\begin{aligned} \sigma_j(\varepsilon k) &= i a_{j,1}|\varepsilon k| - a_{j,2}|\varepsilon k|^2 + O(|\varepsilon k|^3)\,, \\ e_j(\varepsilon k) &= E_j + O(|\varepsilon k|)\,, \end{aligned} \tag{9}$$

here $a_{j,2} > 0$, $\langle e_j(-\varepsilon k), e_l(\varepsilon k)\rangle_\xi = \delta_{jl}$, $1 \le j, l \le 3$ *and*

$$\begin{cases} a_{11} = \sqrt{\tfrac{5}{3}}\,, \quad a_{21} = 0\,, \quad a_{31} = -\sqrt{\tfrac{5}{3}}\,, \\ E_1 = \sqrt{\tfrac{3}{10}}\chi_0 + \sqrt{\tfrac{1}{2}}\chi_1 + \sqrt{\tfrac{1}{5}}\chi_4\,, \\ E_2 = -\sqrt{\tfrac{2}{5}}\chi_0 + \sqrt{\tfrac{3}{5}}\chi_4\,, \\ E_3 = \sqrt{\tfrac{3}{10}}\chi_0 - \sqrt{\tfrac{1}{2}}\chi_1 + \sqrt{\tfrac{1}{5}}\chi_4\,. \end{cases} \tag{10}$$

More precisely, the semigroup $e^{(-i\pi\varepsilon\xi_1 k + L)t}$ *can be decomposed as*

$$\begin{aligned} e^{(-i\pi\varepsilon\xi_1 k + L)t} f &= e^{(-i\pi\varepsilon\xi_1 k + L)t}\Pi_k^\perp f \\ &\quad + \mathbf{1}_{\{|\varepsilon k| < \delta\}} \sum_{j=1}^3 e^{\sigma_j(\varepsilon k)t}\langle e_j(-\varepsilon k), f\rangle_\xi e_j(\varepsilon k)\,. \end{aligned} \tag{11}$$

where $\mathbf{1}_{\{\cdot\}}$ *is the indicator function. Moreover, there exist* $a(\tau_1) > 0$, $\overline{a}_1 > 0$ *such that* $\|e^{(-i\pi\varepsilon\xi_1 k + L)t}\Pi_k^\perp\|_{L^2_\xi} \lesssim e^{-a(\tau_1)t}$ *and* $e^{\sigma_j(\varepsilon k)t} \le e^{-\overline{a}_1|\varepsilon k|^2 t}$ *for all* $1 \le j \le 3$.

Lemma 4 and Lemma 5 are useful for the estimate of the fluid part.

Lemma 4. *If* $0 < \varepsilon \ll 1$, $k \in \mathbb{Z}$, $a > 0$, $s \ge 0$ *and* t *is in the short time region, i.e.,* $\varepsilon^2 t \ll 1$, *then*

$$\frac{1}{|\mathbb{T}^1_{1/\varepsilon}|} \sum_{|\varepsilon k| < \delta, k \ne 0} |\varepsilon k|^s e^{-a|\varepsilon k|^2 t} \lesssim e^{-a\varepsilon^2 t}(1+t)^{-(1+s)/2}\,.$$

Let $h(x,\xi)$ be any function with zero moments condition, one can define the fluid projection $\mathbb{P}_0$ and non-fluid projection $\mathbb{P}_1$ as follows:

$$\begin{cases} \mathbb{P}_0^j h(x,\xi) = \sum_{|\varepsilon k|<\delta, k\neq 0} e^{i\pi\varepsilon kx}\langle e_j(-\varepsilon k), (\widehat{h})_k\rangle_\xi e_j(\varepsilon k)\,, \\ \mathbb{P}_0 h(x,\xi) = \sum_{j=1}^{3}\mathbb{P}_0^j h(x,\xi)\,, \quad \mathbb{P}_1 h(x,\xi) = h(x,\xi) - \mathbb{P}_0 h(x,\xi)\,, \end{cases} \tag{12}$$

where

$$(\hat{h})_k(\xi) = \frac{1}{|\mathbb{T}^1_{1/\varepsilon}|}\int_{\mathbb{T}^1_{1/\varepsilon}} h(\cdot,\xi)e^{-i\pi\varepsilon kx}dx\,.$$

Let $\mathbb{G}^t_\varepsilon$ be the solution operator of the linearized Boltzmann equation, i.e., $g = \mathbb{G}^t_\varepsilon g_0$ and g satisfies the equation

$$\begin{cases} \partial_t g + \xi_1\partial_x g = Lg\,, \quad (t,x,\xi)\in\mathbb{R}^+\times\mathbb{T}^1_{1/\varepsilon}\times\mathbb{R}^3\,, \\ g(0,x,\xi) = g_0(x,\xi)\,. \end{cases} \tag{13}$$

We have the following linear and nonlinear estimates:

Lemma 5. *Assuming that* $0<\varepsilon\ll 1$, $\beta\geq 0$, *we have*

$$\|\mathbb{G}^t_\varepsilon\mathbb{P}_0 g_0\|_{L^\infty_x L^\infty_{\xi,\beta}} \lesssim (1+t)^{-1/2}e^{-\overline{a}\varepsilon^2 t}\|g_0\|_{L^1_x L^\infty_{\xi,\beta}} \tag{14}$$

and

$$\|\mathbb{G}^t_\varepsilon\mathbb{P}_0 g_0\|_{L^2_x L^\infty_{\xi,\beta}} \lesssim (1+t)^{-1/4}e^{-\overline{a}\varepsilon^2 t}\|g_0\|_{L^1_x L^\infty_{\xi,\beta}} \tag{15}$$

Moreover, for the nonlinear estimates, we have

$$\|\mathbb{G}^t_\varepsilon\mathbb{P}_0\Gamma(X_1,X_2)\|_{L^\infty_x L^\infty_{\xi,\beta}} \lesssim (1+t)^{-1}e^{-\overline{a}\varepsilon^2 t}\|X_1\|_{L^2_x L^\infty_{\xi,\beta}}\|X_2\|_{L^2_x L^\infty_{\xi,\beta}} \tag{16}$$

and

$$\|\mathbb{G}^t_\varepsilon\mathbb{P}_0\Gamma(X_1,X_2)\|_{L^2_x L^\infty_{\xi,\beta}} \lesssim (1+t)^{-3/4}e^{-\overline{a}\varepsilon^2 t}\|X_1\|_{L^2_x L^\infty_{\xi,\beta}}\|X_2\|_{L^2_x L^\infty_{\xi,\beta}}\,. \tag{17}$$

Proof. It is obvious that

$$\mathbb{G}^t_\varepsilon\mathbb{P}_0 g_0 = \sum_{j=1}^{3}\sum_{|\varepsilon k|<\delta, k\neq 0} e^{i\pi\varepsilon kx}e^{\sigma_j(\varepsilon k)t}\langle e_j(-\varepsilon k), (\widehat{g_0})_k\rangle_\xi e_j(\varepsilon k)\,.$$

For linear estimate (14), applying the zero moments condition (5) and lemma 4, we have

$$\begin{aligned}\|\mathbb{G}^t_\varepsilon\mathbb{P}_0 g_0\|_{L^\infty_{\xi,\beta}} &\leq \sum_{j=1}^{3}\sum_{|\varepsilon k|<\delta; |k|\neq 0}\left|e^{\sigma_j(\varepsilon k)t}\right|\|(\widehat{g_0})_k\|_{L^\infty_{\xi,\beta}} \\ &\lesssim \frac{1}{|\mathbb{T}^1_{1/\varepsilon}|}\sum_{|\varepsilon k|<\delta; |k|\neq 0} e^{-\overline{a}|k\varepsilon|^2 t}\|g_0\|_{L^1_x L^\infty_{\xi,\beta}} \\ &\lesssim (1+t)^{-1/2}e^{-\overline{a}\varepsilon^2 t}\|g_0\|_{L^1_x L^\infty_{\xi,\beta}}\,.\end{aligned}$$

For linear estimate (15), applying the zero moments condition (5), Cauchy-Schwarz inequality and lemma 4, we have

$$
\begin{aligned}
\|\mathbb{G}_\varepsilon^t \mathbb{P}_0 g_0\|^2_{L^2_x L^\infty_{\xi,\beta}} &\lesssim \Big\| \sum_{j=1}^{3} \sum_{|\varepsilon k|<\delta; |k|\neq 0} e^{\sigma_j(\varepsilon k)t} e^{i\pi\varepsilon k\cdot x} \big\langle e_j(-\varepsilon k), (\widehat{g_0})_k \big\rangle_\xi e_j(\varepsilon k) \Big\|^2_{L^2_x L^\infty_{\xi,\beta}} \\
&\lesssim |\mathbb{T}^1_{1/\varepsilon}| \sum_{j=1}^{3} \sum_{|\varepsilon k|<\delta; |k|\neq 0} |e^{\sigma_j(\varepsilon k)t}|^2 |\big\langle e_j(-\varepsilon k), (\widehat{g_0})_k \big\rangle_\xi|^2 \\
&\lesssim |\mathbb{T}^1_{1/\varepsilon}| \sum_{j=1}^{3} \sum_{|\varepsilon k|<\delta; |k|\neq 0} |e^{\sigma_j(\varepsilon k)t}|^2 \|(\widehat{g_0})_k\|^2_{L^\infty_{\xi,\beta}} \\
&\lesssim |\mathbb{T}^1_{1/\varepsilon}| \sum_{|\varepsilon k|<\delta; |k|\neq 0} e^{-2\overline{a}|k\varepsilon|^2 t} \|(\widehat{g_0})_k\|^2_{L^\infty_{\xi,\beta}} \\
&\lesssim \frac{1}{|\mathbb{T}^1_{1/\varepsilon}|} \sum_{|\varepsilon k|<\delta; |k|\neq 0} e^{-2\overline{a}|k\varepsilon|^2 t} \|g_0\|^2_{L^1_x L^\infty_{\xi,\beta}} .
\end{aligned}
$$

This means that

$$
\|\mathbb{G}_\varepsilon^t \mathbb{P}_0 g_0\|_{L^2_x L^\infty_{\xi,\beta}} \lesssim (1+t)^{-1/4} e^{-\overline{a}\varepsilon^2 t} \|g_0\|_{L^1_x L^\infty_{\xi,\beta}} .
$$

For the nonlinear estimate, note that

$$
\mathbb{G}_\varepsilon^t \mathbb{P}_0^j \Gamma(X_1, X_2) = \sum_{|\varepsilon k|<\delta, k\neq 0} e^{i\pi\varepsilon k x} e^{\sigma_j(\varepsilon k)t} \big\langle e_j(-\varepsilon k), \big(\Gamma(\widehat{X_1, X_2})\big)_k \big\rangle_\xi .
$$

We need to observe some cancelation properties from

$$
\big\langle e_j(-\varepsilon k), \big(\Gamma(\widehat{X_1, X_2})\big)_k \big\rangle_\xi .
$$

One can check that E_j are collision invariants of the operator Γ for all $1 \leq j \leq 3$. We then have

$$
\big\langle e_j(-\varepsilon k), \big(\Gamma(\widehat{X_1, X_2})\big)_k \big\rangle_\xi = \big\langle \widetilde{e}_j(-\varepsilon k), \big(\Gamma(\widehat{X_1, X_2})\big)_k \big\rangle_\xi ,
$$

where $\widetilde{e}_j(\varepsilon k) = e_j(\varepsilon k) - E_j$.

For nonlinear estimate (16), note that $e_j(\varepsilon k)$ decay faster than any polynomial, similar to the linear estimate, we have

$$
\begin{aligned}
\|\mathbb{G}_\varepsilon^t \mathbb{P}_0^j \Gamma(X_1, X_2)\|_{L^\infty_x L^\infty_{\xi,\beta}} &\leq \frac{1}{|\mathbb{T}^1_{1/\varepsilon}|} \sum_{|\varepsilon k|<\delta, k\neq 0} e^{-\overline{a}|\varepsilon k|^2 t} |\varepsilon k| \, \big\| \langle \xi \rangle^{-1} \Gamma(X_1, X_2) \big\|_{L^1_x L^\infty_{\xi,\beta}} \\
&\lesssim (1+t)^{-1} e^{-\overline{a}\varepsilon^2 t} \|X_1\|_{L^2_x L^\infty_{\xi,\beta}} \|X_2\|_{L^2_x L^\infty_{\xi,\beta}} .
\end{aligned}
$$

The estimate of (17) is similar and hence we omit the details. This completes the proof of the lemma. □

Lemma 6 is useful in the estimate of the nonfluid part.

Lemma 6. *Assuming that* $0 < \varepsilon \ll 1$, $\beta > 3/2$, *then* $g = \mathbb{G}^t_\varepsilon \mathbb{P}_1 g_0$ *has the following estimates*

$$\|\mathbb{G}^t_\varepsilon \mathbb{P}_1 g_0\|_{L^\infty_x L^\infty_{\xi,\beta}} \lesssim e^{-Ct}\|g_0\|_{L^\infty_x L^\infty_{\xi,\beta}} \tag{18}$$

and

$$\|\mathbb{G}^t_\varepsilon \mathbb{P}_1 g_0\|_{L^2_x L^\infty_{\xi,\beta}} \lesssim e^{-Ct}\|g_0\|_{L^2_x L^\infty_{\xi,\beta}}\,. \tag{19}$$

Moreover, if $\beta > 5/2$, *then*

$$\left\|\mathbb{G}^t_\varepsilon \mathbb{P}_1 \Gamma(X_1, X_2)\right\|_{L^\infty_x L^\infty_{\xi,\beta}} \lesssim e^{-Ct}\|X_1\|_{L^\infty_x L^\infty_{\xi,\beta}}\|X_2\|_{L^\infty_x L^\infty_{\xi,\beta}} \tag{20}$$

and

$$\left\|\mathbb{G}^t_\varepsilon \mathbb{P}_1 \Gamma(X_1, X_2)\right\|_{L^2_x L^\infty_{\xi,\beta}} \lesssim e^{-Ct}\|X_1\|_{L^2_x L^\infty_{\xi,\beta}}\|X_2\|_{L^\infty_x L^\infty_{\xi,\beta}}\,. \tag{21}$$

The proof of this lemma is based on the process in section II, part A and part B of [12] and hence we omit it.

3. Proof of the theorem

3.1. Fluid-nonfluid decomposition. Now, we decompose our solution as the fluid part (22) and nonfluid part (23):

$$\begin{cases} \partial_t u + \xi_1 \partial_x u = Lu + \mathbb{P}_0\Gamma(f, f)\,, & (t, x, \xi) \in \mathbb{R}^+ \times \mathbb{T}^1_{1/\varepsilon} \times \mathbb{R}^3\,, \\ u(0, x, \xi) = \eta \mathbb{P}_0 f_0(x, \xi)\,, \end{cases} \tag{22}$$

and

$$\begin{cases} \partial_t u^\perp + \xi_1 \partial_x u^\perp = Lu^\perp + \mathbb{P}_1\Gamma(f, f)\,, & (t, x, \xi) \in \mathbb{R}^+ \times \mathbb{T}^1_{1/\varepsilon} \times \mathbb{R}^3\,, \\ u^\perp(0, x, \xi) = \eta \mathbb{P}_1 f_0(x, \xi)\,. \end{cases} \tag{23}$$

This means that

$$f(t, x, \xi) = u(t, x, \xi) + u^\perp(t, x, \xi)\,.$$

3.2. Leading fluid part. We define the leading fluid part as follows:

$$\begin{cases} \partial_t U + \xi_1 \partial_x U = LU + \mathbb{P}_0\Gamma(U, U)\,, & (t, x, \xi) \in \mathbb{R}^+ \times \mathbb{T}^1_{1/\varepsilon} \times \mathbb{R}^3\,, \\ U(0, x, \xi) = \eta \mathbb{P}_0 f_0(x, \xi)\,. \end{cases} \tag{24}$$

In order to solve U, one can design the following iteration:

$$\begin{cases} \partial_t U_{n+1} + \xi_1 \partial_x U_{n+1} = LU_{n+1} + \mathbb{P}_0\Gamma(U_n, U_n)\,, & (t, x, \xi) \in \mathbb{R}^+ \times \mathbb{T}^1_{1/\varepsilon} \times \mathbb{R}^3\,, \\ U_{n+1}(0, x, \xi) = \eta \mathbb{P}_0 f_0(x, \xi)\,, \\ U_0 = 0\,. \end{cases}$$

We have

$$U_{n+1} = \eta \sum_{j=1}^{3} \sum_{|\varepsilon k|<\delta, k\neq 0} e^{i\pi\varepsilon kx} e^{\sigma_j(\varepsilon k)t} \langle e_j(-\varepsilon k), (\widehat{f_0})_k \rangle_\xi e_j(\varepsilon k)$$
$$+ \int_0^t \sum_{j=1}^{3} \sum_{|\varepsilon k|<\delta, k\neq 0} e^{i\pi\varepsilon kx} e^{\sigma_j(\varepsilon k)(t-s)} \langle e_j(-\varepsilon k), (\widehat{\Gamma(U_n, U_n)})_k \rangle_\xi (\cdot, s) e_j(\varepsilon k) ds\,.$$

Lemma 7. *Assuming that* $0 < \varepsilon \ll 1$, $\beta > 5/2$ *and* η *is small enough, then we have*

$$\|U_n\|_{L^2_x L^\infty_{\xi,\beta}} \lesssim \|U_1\|_{L^2_x L^\infty_{\xi,\beta}} + \eta^2 (1+t)^{-1/4} e^{-\overline{a}\varepsilon^2 t} \|f_0\|^2_{L^1_x L^\infty_{\xi,\beta}}$$
$$\lesssim \eta (1+t)^{-1/4} e^{-\overline{a}\varepsilon^2 t} |||f_0|||_\beta\,.$$

and

$$\|U_n\|_{L^\infty_x L^\infty_{\xi,\beta}} \lesssim \|U_1\|_{L^1_x L^\infty_{\xi,\beta}} + \eta^2 (1+t)^{-1/2} \ln(1+t) e^{-\overline{a}\varepsilon^2 t} \|f_0\|^2_{L^1_x L^\infty_{\xi,\beta}}$$
$$\lesssim \eta (1+t)^{-1/2} \ln(1+t) e^{-\overline{a}\varepsilon^2 t} |||f_0|||_\beta\,,$$

Proof. We prove $L^2_x L^\infty_{\xi,\beta}$ estimate first. The cases for $n = 1$ can be found in (15). For $n = 2$, by (17), we have

$$\|U_2\|_{L^2_x L^\infty_{\xi,\beta}} \lesssim \|U_1\|_{L^2_x L^\infty_{\xi,\beta}} + \int_0^t e^{-\overline{a}\varepsilon^2(t-s)} (1+t-s)^{-3/4} \|U_1\|^2_{L^2_x L^\infty_{\xi,\beta}} (\cdot, s) ds$$
$$\lesssim \|U_1\|_{L^2_x L^\infty_{\xi,\beta}} + \eta^2 \int_0^t e^{-\overline{a}\varepsilon^2(t-s)} (1+t-s)^{-3/4} e^{-2\overline{a}\varepsilon^2 s} (1+s)^{-1/2} ds |||f_0|||^2_\beta$$
$$\lesssim \eta (1+t)^{-1/4} e^{-\overline{a}\varepsilon^2 t} |||f_0|||_\beta + \eta^2 (1+t)^{-1/4} e^{-\overline{a}\varepsilon^2 t} |||f_0|||^2_\beta\,.$$

For $n \geq 2$, we claim our estimate by induction and omit the detail here.

On the other hand, in $L^\infty_x L^\infty_{\xi,\beta}$ estimate, the cases for $n = 1$ can be found in (14). For $n \geq 2$, by (16), we have

$$\|U_n\|_{L^\infty_x L^\infty_{\xi,\beta}} \lesssim \|U_1\|_{L^\infty_x L^\infty_{\xi,\beta}} + \int_0^t e^{-\overline{a}\varepsilon^2(t-s)} (1+t-s)^{-1} \|U_{n-1}\|^2_{L^2_x L^\infty_{\xi,\beta}} ds$$
$$\lesssim \|U_1\|_{L^\infty_x L^\infty_{\xi,\beta}} + \eta^2 \int_0^t e^{-\overline{a}\varepsilon^2(t-s)} (1+t-s)^{-1} e^{-2\overline{a}\varepsilon^2 s} (1+s)^{-1/2} ds |||f_0|||^2_\beta$$
$$\lesssim \eta (1+t)^{-1/2} e^{-\overline{a}\varepsilon^2 t} |||f_0|||_\beta + \eta^2 (1+t)^{-1/2} \ln(1+t) e^{-\overline{a}\varepsilon^2 t} |||f_0|||^2_\beta\,.$$

This completes the proof of the lemma. □

We apply this iteration scheme to get the following existence and uniqueness of the leading fluid part.

Proposition 8. *Assuming that* $0 < \varepsilon \ll 1$, $\beta \geq 0$ *and* η *small enough. Then there exists a unique solution* $U(t, x, \xi)$ *to the leading fluid part* (24) *such that*

$$\|U\|_{L^\infty_x L^\infty_{\xi,\beta}} \lesssim \eta e^{-\overline{a}\varepsilon^2 t} (1+t)^{-1/2} \ln(1+t) |||f_0|||_\beta$$

for some constant $\overline{a} > 0$.

3.3. **Leading nonfluid part.** We define the leading nonfluid part as:

$$(25)\quad \begin{cases} \partial_t U^\perp + \xi_1 \partial_x U^\perp = LU^\perp + \mathbb{P}_1 \Gamma(U,U)\,, & (t,x,\xi) \in \mathbb{R}^+ \times \mathbb{T}^1_{1/\varepsilon} \times \mathbb{R}^3\,, \\ U^\perp(0,x,\xi) = \eta \mathbb{P}_1 f_0(x,\xi)\,, & \end{cases}$$

where U is the leading fluid part. Then

$$U^\perp = \eta \mathbb{G}^t_\varepsilon \mathbb{P}_1 f_0 + \int_0^t \mathbb{G}^{t-s}_\varepsilon \mathbb{P}_1 \Gamma(U,U)(\cdot,s)ds\,.$$

One has the following proposition:

Proposition 9. *Assuming that* $0 < \varepsilon \ll 1$, $\beta > 5/2$ *and* η *small enough. Then there exists a unique solution* $U^\perp(t,x,\xi)$ *to the leading nonfluid part* (25) *such that*

$$\|U^\perp\|_{L^\infty_x L^\infty_{\xi,\beta}} \lesssim \eta e^{-2\overline{a}\varepsilon^2 t}(1+t)^{-1}\ln^2(1+t)|||f_0|||_\beta$$

and

$$\|U^\perp\|_{L^2_x L^\infty_{\xi,\beta}} \lesssim \eta e^{-2\overline{a}\varepsilon^2 t}(1+t)^{-3/4}\ln(1+t)|||f_0|||_\beta\,.$$

3.4. **Estimate of the tail part.** We define the tail fluid part and tail nonfluid part as follows:

$$v = u - U\,, \quad v^\perp = u^\perp - U^\perp\,.$$

Then the tail fluid part v solves the equation

$$(26)\quad \begin{cases} \partial_t v + \xi_1 \partial_x v = Lv + \mathbb{P}_0 \Gamma(v+v^\perp, v+v^\perp) + 2\mathbb{P}_0 \Gamma(v+v^\perp, U+U^\perp) \\ \qquad + \mathbb{P}_0 \Gamma(U^\perp, U^\perp) + 2\mathbb{P}_0 \Gamma(U, U^\perp)\,, \\ v(0,x,\xi) = 0\,. \end{cases}$$

On the other hand, the tail nonfluid part $v^\perp$ solves the equation

$$(27)\quad \begin{cases} \partial_t v^\perp + \xi_1 \partial_x v^\perp = Lv^\perp + \mathbb{P}_1 \Gamma(v+v^\perp, v+v^\perp) + 2\mathbb{P}_1 \Gamma(v+v^\perp, U+U^\perp) \\ \qquad + \mathbb{P}_1 \Gamma(U^\perp, U^\perp) + 2\mathbb{P}_1 \Gamma(U, U^\perp)\,, \\ v^\perp(0,x,\xi) = 0\,. \end{cases}$$

Similar to the leading fluid part, one can design a iteration to get the following existence and uniqueness of the tail part.

Proposition 10. *Assuming that* $0 < \varepsilon \ll 1$, $\beta > 5/2$ *and* η *small enough. Then there exists a unique solution of the system* $(v, v^{\perp})$ *to the tail part* (26)-(27) *such that*

$$\begin{cases} \|v\|_{L^2_x L^\infty_{\xi,\beta}} \lesssim \eta^2 e^{-\overline{a}\varepsilon^2 t}(1+t)^{-3/4}\ln(1+t)|||f_0|||^2_\beta, \\ \|v^{\perp}\|_{L^2_x L^\infty_{\xi,\beta}} \lesssim \eta^2 e^{-\overline{a}\varepsilon^2 t}(1+t)^{-5/4}\ln^2(1+t)|||f_0|||^2_\beta, \\ \|v\|_{L^\infty_x L^\infty_{\xi,\beta}} \lesssim \eta^2 e^{-\overline{a}\varepsilon^2 t}(1+t)^{-1}\ln^2(1+t)|||f_0|||^2_\beta, \\ \|v^{\perp}\|_{L^\infty_x L^\infty_{\xi,\beta}} \lesssim \eta^2 e^{-\overline{a}\varepsilon^2 t}(1+t)^{-3/2}\ln^3(1+t)|||f_0|||^2_\beta. \end{cases}$$

for some constant $\overline{a} > 0$.

With Proposition 8, Proposition 9 and Proposition 10, we have our main theorem.

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