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INTEGRABLE MODULES OVER AFFINE LIE SUPERALGEBRAS $\mathfrak{sl}(1|n)^{(1)}$

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ABSTRACT. We describe the category of integrable $\mathfrak{sl}(1|n)^{(1)}$-modules with the positive central charge and show that the irreducible modules provide the full set of irreducible representations for the corresponding simple vertex algebra.

1. INTRODUCTION

Let $\mathfrak{g}$ be the Kac-Moody superalgebra $\mathfrak{sl}(1|n)^{(1)}, n \geq 2$. Recall that $\mathfrak{g}_{\bar{0}} = \mathfrak{gl}_n^{(1)}$. We call a $\mathfrak{g}$-module *integrable* if it is integrable over the affine Lie algebra $\mathfrak{sl}_n(1)$, locally finite over the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{gl}_n^{(1)}$ and with finite-dimensional generalized $\mathfrak{h}$-weight spaces.

We normalize the invariant form on $\mathfrak{g}$ in the usual way ($(\alpha, \alpha) = 2$ for the non-isotropic roots α). Let $\mathcal{F}_k$ be the category of the finitely generated integrable $\mathfrak{g}$-modules with central charge k. This category is empty for $k \notin \mathbb{Z}_{\geq 0}$. In this paper we study the category $\mathcal{F}_k$ for $k \in \mathbb{Z}_{>0}$. By [FR] (Theorem C) the irreducible objects in $\mathcal{F}_k$ are highest weight modules (for $k > 0$); these modules were classified in [KW]. We describe the blocks in $\mathcal{F}_k$ in Corollary 3.2.1 and Theorem 3.6.5; in Corollary 5.4 we show that Duflo-Serganova functor provides an invariant for the atypical blocks.

Recall the situation in the usual affine Lie algebra case. Let $\mathfrak{t}$ be an affine Lie algebra, $V^k(\mathfrak{t})$ be the affine vertex algebra with central charge k and $V_k(\mathfrak{t})$ denote its simple quotient. Let $k \neq 0$ be such that $V_k(\mathfrak{t})$ is integrable (as a $\mathfrak{t}$-module). Then the vertex algebra $V_k(\mathfrak{t})$ is rational and regular:

(a) the irreducible integrable $\mathfrak{t}$-modules of level k provide the full set of irreducible representations for $V_k(\mathfrak{t})$;

(b) there are finitely many (up to isomorphism) irreducible $V_k(\mathfrak{t})$-modules;

(c) any representation is completely reducible.

For positive energy modules (a), (c) are proven in [FZ]; (b) follows from (a) and the fact that there are finitely many irreducible $\mathfrak{t}$-integrable modules of level k. In [DLM] it is shown that any module is a direct sum of positive energy modules.

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Let $V^k(\mathfrak{g})$ be the affine vertex superalgebra (for $\mathfrak{g} = \mathfrak{sl}(1|n)^{(1)}, n \geq 2$) and let $V_k(\mathfrak{g})$ be its simple quotient. As a $\mathfrak{g}$-module, $V_k(\mathfrak{g})$ is integrable if and only if k is a non-negative integer. In Theorem 6.1 we will show that for $k \neq 0$ (a) holds for positive energy modules: the irreducible modules in $\mathcal{F}_k$ provide the full set of irreducible positive energy modules for $V_k(\mathfrak{g})$. Since $\mathfrak{g}$ has infinitely many irreducible integrable modules of level k (for $k \in \mathbb{Z}_{>0}$), (b) does not hold; (c) also does not hold. In this paper we classify the blocks of $\mathcal{F}_k$ and describe these blocks in terms of quivers with relations.

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2. Preliminaries

Let $\mathfrak{g} = \mathfrak{sl}(1|n)^{(1)}$. Recall that by definition an integrable $\mathfrak{g}$-module is integrable over the affine Lie subalgebra $\mathfrak{sl}_n^{(1)} \subset \mathfrak{g}_{\bar{0}}$ and locally finite over the Cartan subalgebra $\mathfrak{h}$. Recall also that $\mathfrak{h} \cap [\mathfrak{sl}_n^{(1)}, \mathfrak{sl}_n^{(1)}]$ acts diagonally on any integrable $\mathfrak{sl}_n^{(1)}$-module.

Note that $\mathcal{F}_k$ is the full subcategory in the thick category $\mathcal{O}$. In particular, it is equipped with a covariant duality functor $\mathcal{D}$ inherited from the contragredient duality in category $\mathcal{O}$. For any simple object L we have $\mathcal{D}(L) \simeq L$. In particular, $\mathrm{Ext}^1(L, L') = \mathrm{Ext}^1(L', L)$ for any two simple objects L and L'.

2.1. **Sets of simple roots.** A Dynkin diagram for $\mathfrak{g}$ is a cycle with $n+1$ nodes: there are two nodes which correspond to the odd isotropic roots and these nodes are adjacent. The minimal imaginary positive root δ is the sum of all simple roots.

We fix a triangular decomposition of $\mathfrak{g}_{\bar{0}}$ and consider only triangular decompositions of $\mathfrak{g}$ which are compatible with it (i.e., $\Delta^+_{\bar{0}}$ is fixed). We denote such sets of simple roots by Σ, Σ', etc.

For a fixed set of simple roots Σ we consider the standard partial order on $\mathfrak{h}^*$ given by $\lambda > \mu$ if and only if $\lambda - \mu \in \mathbb{Z}_{\geq 0}\Sigma$.

2.1.1. Let Π_0 be a set of simple roots for $\Delta^+_{\bar{0}}$ (recall that Π_0 is fixed). For any odd root β there exists a unique $\alpha \in \Pi_0$ such that $(\alpha, \beta) = -1$ and a unique $\alpha' \in \Pi_0$ such that $(\beta, \alpha') = 1$; the set

$$\Sigma = \{\beta, \alpha' - \beta\} \cup (\Pi_0 \setminus \{\alpha'\}).$$

is a unique set of simple roots containing β.

2.1.2. *Odd reflections.* Recall that for an odd root β belonging to a set of simple roots Σ, the odd reflection r_β gives another sets of simple roots $r_\beta\Sigma$ which contains $-\beta$, the roots

$\alpha \in \Sigma \setminus \{\beta\}$, which are orthogonal to β, and the roots $\alpha+\beta$ for $\alpha \in \Sigma$ which are not orthogonal to β. One has

$$\Delta^+(r_\beta\Sigma) = (\Delta^+(\Sigma) \setminus \{\beta\}) \cup \{-\beta\}.$$

Any two sets of simple roots are connected by a chain of odd reflections. We call a chain "proper" if it does not have loops (i.e. subsequences of the form $r_\beta r_{-\beta}$). Two sets of simple roots are connected by a unique "proper" chain of odd reflections.

Let Σ be a set of simple roots. One readily sees that the chain $r_{\beta_s}r_{\beta_{s-1}}\dots r_{\beta_1}\Sigma$ is proper if and only if $\beta_1,\dots,\beta_s \in \Delta^+(\Sigma)$. Let $\beta \notin \Sigma$ be an odd root and Σ' be a set of simple roots containing β (by above, Σ' is unique). If $\beta \in \Delta^+(\Sigma)$, then the proper chain which connects Σ and Σ' does not contain the reflections $r_{\pm\beta}$; if $\beta \in -\Delta^+(\Sigma)$, then the proper chain is of the form $\Sigma' = r_{\beta_s}r_{\beta_{s-1}}\dots r_{\beta_1}\Sigma$, where $\beta_s = \beta$.

2.2. **Simple modules.** For a set of simple roots Σ we denote by $L_\Sigma(\lambda)$ the irreducible module of the highest weight λ with respect to the Borel subalgebra corresponding to Σ. For an irreducible highest weight module L and a set of simple roots Σ we set $\rho wt_\Sigma L := \lambda$ if $L = L_\Sigma(\lambda - \rho_\Sigma)$ (where ρ_Σ is the Weyl vector for Σ, i.e. $(\rho_\Sigma, \alpha) = 1$ (resp. 0) for even (resp. odd) $\alpha \in \Sigma$). If $\alpha \in \Sigma$ is an odd root we have

$$\rho wt_{r_\alpha\Sigma} L = \begin{cases} \rho wt_\Sigma L & \text{if} \quad (\lambda,\alpha) \neq 0, \\ \rho wt_\Sigma L + \alpha & \text{if} \quad (\lambda,\alpha) = 0. \end{cases} \tag{1}$$

From (1), it follows that $L_\Sigma(\lambda)$ is integrable if and only if $(\lambda,\alpha) \in \mathbb{Z}_{\geq 0}$ for every even $\alpha \in \Sigma$, and for two odd roots $\beta_1, \beta_2 \in \Sigma$ one has either $(\lambda, \beta_1+\beta_2) \in \mathbb{Z}_{>0}$ or $(\lambda,\beta_1) = (\lambda,\beta_2) = 0$. Since δ is the sum of simple roots, the central charge of a highest weight module $L_\Sigma(\lambda)$ is $(\lambda, \sum_{\alpha\in\Sigma}\alpha)$. In particular, if $L_\Sigma(\lambda)$ is not one-dimensional, its central charge is a positive integer.

2.2.1. We fix a set of simple roots $\Sigma = \{\alpha_i\}_{i=0}^n$, where α_1, α_2 are odd. Note that $(\alpha_1,\alpha_2) = 1$. By above, the irreducible objects of $\mathcal{F}_k$ are the highest weight modules L where $\rho wt_\Sigma L$ satisfies the following condition. If $a_i = (\rho wt_\Sigma L, \alpha_i)$, then

(i) $a_i \in \mathbb{Z}_{>0}$ for $i = 0$ or $i = 3,\dots,n$;
(ii) $a_1 + a_2 \in \mathbb{Z}_{>0}$ or $a_1 = a_2 = 0$;
(iii) $a_0 + a_1 + \dots + a_n = k + n - 1$.

Notice that the numbers $\{a_i\}_{i=0}^n$ determines (up to isomorphism) L as $[\mathfrak{g},\mathfrak{g}]$-module (and thus $V^k(\mathfrak{g})$-module). For the $\mathfrak{g}$-modules $L(\lambda), L(\lambda+s\delta)$ the numbers $\{a_i\}_{i=0}^n$ are the same, however the Casimir element acts on $L(\lambda)$ and on $L(\lambda+s\delta)$ by different scalars.

2.2.2. **Lemma.** *Let $L_\Sigma(\lambda)$ be integrable and all $(\lambda,\alpha) \in \mathbb{R}$ for all $\alpha \in \Sigma$. Then there exists a set of simple roots Σ' such that $(\lambda + \rho_\Sigma, \alpha) \geq 0$ for every $\alpha \in \Sigma'$.*

Proof. Recall that $(\lambda+\rho_\Sigma,\delta)=k+n-1$. Note that $(\Sigma\setminus\{\alpha_1,\alpha_2\})\cup\{\alpha_1+s\delta,\alpha_2-s\delta\}$ is a set of simple roots for any $s\in\mathbb{Z}$. Therefore without loss of generality we may assume that

$$0\leq(\lambda+\rho_\Sigma,\alpha_1)<k+n-1 \tag{2}$$

If $0\leq(\lambda+\rho_\Sigma,\alpha_2)$, we take $\Sigma'=\Sigma$. Assume that $(\lambda+\rho_\Sigma,\alpha_2)<0$. For $r=2,\dots,n+1$ set $\beta_r:=\sum_{i=2}^r\alpha_i$ (where $\alpha_{n+1}:=\alpha_0$). Then $\delta=\beta_{n+1}+\alpha_1$, so (2) gives

$$(\lambda+\rho_\Sigma,\beta_2)<0,\quad(\lambda+\rho_\Sigma,\beta_{n+1})>0.$$

Let s be maximal such that $(\lambda+\rho_\Sigma,\beta_s)<0$. For $\Sigma':=r_{\beta_s}\dots r_{\beta_2}\Sigma$ the isotropic roots are $-\beta_s$ and β_{s+1}. Since $(\lambda+\rho_\Sigma,-\beta_s),(\lambda+\rho_\Sigma,\beta_{s+1})\geq 0$, Σ' is as required. □

2.2.3. *Definitions.* Let L be an irreducible highest weight module.

Recall that L is called *typical* if $(\rho wt_\Sigma L,\alpha)\neq 0$ for any (isotropic) odd root α and *atypical* otherwise. From (1), it follows that this notion does not depend on the choice of Σ and, moreover, $\rho wt_\Sigma L$ does not depend on Σ for typical L.

We say that L is Σ-*tame* if $(\rho wt_\Sigma L,\beta)=0$ for some odd $\beta\in\Sigma$. Any atypical L (for $\mathfrak{sl}(1,n)^{(1)}$) is tame with respect to some Σ.

Let β be an odd root. We call an odd reflection r_β L-*typical* if for Σ containing β one has $(\rho wt_\Sigma L,\beta)\neq 0$ (by 2.1.1, Σ is unique). Note that if Σ and Σ' are connected by a chain of odd L-typical reflections, then $\rho wt_\Sigma(L)=\rho wt_{\Sigma'}(L)$.

We say that $\lambda\in\mathfrak{h}^*$ is *regular* if $(\lambda,\alpha)\neq 0$ for any even real root and that λ is *singular* otherwise.

We say that L is Σ-*regular* if $\rho wt_\Sigma L$ is regular and that L is *regular* if it is Σ-regular for each Σ. We say that L is Σ-singular if it is not Σ-regular and that L is singular if it is not regular. By 2.2.1, L is Σ-singular if and only if $(\rho wt L,\alpha)=0$ for both odd roots $\alpha\in\Sigma$ (in particular, in this case L is Σ-tame).

2.2.4. *Character formulae.* If $L(\lambda)$ is typical, then $\operatorname{ch}L(\lambda)$ is given by the Kac-Weyl character formula; if $L(\lambda)$ is atypical and Σ-tame, $\operatorname{ch}L(\lambda)$ is given by Kac-Wakimoto formula, see [S2],[KW].

2.3. Fix Σ as in 2.2.1.

Lemma. *Let $L=L_\Sigma(\lambda)$ be atypical. Set $\rho=\rho_\Sigma$.*

(i) There exists Σ' such that L is Σ'-tame and Σ' is obtained from Σ by a sequence of L-typical odd reflections (in particular, $\rho wt_{\Sigma'}L=\lambda+\rho$).

(ii) L is Σ-regular if and only if there exists a unique odd $\beta\in\Delta^+(\Sigma)$ such that $(\lambda+\rho,\beta)=0$;

(iii) L is regular if and only if there exists a unique odd $\beta \in \Delta^+(\Sigma)$ such that $(\lambda+\rho, \beta) = 0$ and that $(\lambda+\rho, \alpha) \neq 1$ for $\alpha \in \Pi_0$ such that $(\beta, \alpha) = -1$.

Proof. (i) Since L is atypical, $(\rho wt_\Sigma L, \beta) = 0$ for some odd β. We can choose $\beta \in \Delta^+(\Sigma)$. There exists Σ'' obtained from Σ by the sequence of odd reflections such that $\beta \in \Sigma''$. Therefore we proceed applying the odd reflections to Σ until we obtain a base Σ' such that $(\rho wt_\Sigma L, \alpha) = 0$ for some $\alpha \in \Sigma'$. All the odd reflections which we applied are L-typical, so $\rho wt_\Sigma L = \rho wt_{\Sigma'} L$. Thus L is Σ'-tame.

(ii) Let $(\lambda+\rho, \beta_i) = 0$ for distinct odd roots $\beta_1, \beta_2 \in \Delta^+(\Sigma)$. Either $\beta_1 + \beta_2$ or $\beta_1 - \beta_2$ is an even root, so $(\lambda+\rho, \alpha) = 0$ for some $\alpha \in \Delta_0^+$. By 2.2.1, $\alpha = \alpha_1 + \alpha_2$ and $(\lambda+\rho, \alpha_1) = (\lambda+\rho, \alpha_2)$. This gives (ii).

For (iii) assume that L is Σ-regular. By (ii) β is unique and thus Σ' containing β is Σ' as in (i). By above, $\alpha \in \Sigma'$ and $r_\beta\Sigma$ contains the odd roots $\alpha+\beta$ and $-\beta$. One has

$$\rho wt_{r_\beta\Sigma'} L = \rho wt_{\Sigma'} L - \beta = \lambda + \rho - \beta.$$

In particular, $(\rho wt_{r_\beta\Sigma'} L, -\beta) = 0$ and

$$(\rho wt_{r_\beta\Sigma'} L, \alpha+\beta) = (\lambda+\rho-\beta, \alpha+\beta) = (\lambda+\rho, \alpha) - 1.$$

We conclude that L is Σ'-regular if and only if $(\lambda+\rho, \alpha) \neq 1$. In particular, if L is regular, then $(\lambda+\rho, \alpha) \neq 1$.

Now assume that L is singular and Σ-regular. Then there exists $\Sigma'' \neq \Sigma$ such that L is Σ''-singular. We will assume that Σ'' is the closest to Σ, i.e. that L is regular with respect to any set of simple roots between Σ and Σ''. Let β_1, β_2 be odd roots in Σ'' such that $(\rho wt_{\Sigma''} L, \beta_i) = 0$ for $i = 1, 2$. Let $\Sigma = r_{\gamma_s} \dots r_{\gamma_1} \Sigma''$ be a proper chain. Then γ_1 is β_1 or β_2 and $\gamma_i \in \Delta^+(r_{\gamma_1}\Sigma) \setminus \{-\gamma_1\}$ for $i = 2, \dots, s$. Let $\gamma_1 = \beta_1$. By above, L is tame and $r_{\beta_1}\Sigma''$-regular. Then for $i = 2, \dots, s$, r_{γ_i} is L-typical, so

$$\rho wt_\Sigma L = \rho wt_{r_{\beta_1}\Sigma''} L = \rho wt_{\Sigma''} L - \beta_1.$$

One has $-\beta_1 \in \Delta^+(\Sigma)$, $\beta_1+\beta_2 \in \Pi_0$ and $(-\beta_1, \beta_1+\beta_2) = -1$. By above, $(\rho wt_\sigma L, -\beta_1) = 0, (\rho wt_\Sigma L, \beta_1+\beta_2) = 1$ as required. □

3. The category of integrable $sl(1|n)^{(1)}$-modules with positive central charge

In this section we will describe $\mathcal{F}_k$ for $k > 0$.

Fix a set of simple roots Σ; let α_1, α_2 be odd roots in Σ.

We denote by $M_{\Sigma'}(\lambda)$ s Verma module of the highest weight λ for the Borel subalgebra corresponding to Σ'. We write $L(\mu)$ (resp., $M(\mu)$, ρ) for $L_\Sigma(\mu)$ (resp., for $M_\Sigma(\mu)$, ρ_Σ). Denote by $V(\mu)$ the maximal integral quotient of the Verma module $M(\mu)$.

3.1. Maximal integrable quotient of a Verma module. If $\lambda+\rho$ is typical, then for any set of simple roots Σ' one has $M(\lambda)=M_{\Sigma'}(\lambda')$, where $\lambda+\rho=\lambda'+\rho'$.

If $\lambda+\rho$ is atypical, then, by Lemma 2.3, there exists Σ' such that L is Σ'-tame and Σ' is obtained from Σ by L-typical odd reflections. In this case, $M(\lambda)=M_{\Sigma'}(\lambda')$ for λ' as above and $M_{\Sigma'}(\lambda')$ is Σ'-tame, i.e. $(\lambda'+\rho',\alpha)=0$ for some isotropic $\alpha\in\Sigma'$. In other words, any atypical Verma module is isomorphic to a tame Verma module for a suitable set of simple roots.

In [S2] the following lemma is proved (Lemma 14.3).

3.1.1. Lemma. *Let $L=L(\lambda)$ be an integrable module.*

(i) If $(\lambda,\alpha_i)=0$ for $i=1,2$, then $V(\lambda)=L(\lambda)$.

(ii) Assume that $(\lambda,\alpha_i)\neq 0$ for $i=1$ or $i=2$. Then the character of $V(\lambda)$ is given by typical formula

$$\operatorname{ch} V(\lambda)=\sum_{w\in W}\operatorname{sgn}(w)\operatorname{ch} M(w(\lambda+\rho)-\rho),$$

where W is the Weyl group of $\mathfrak{g}_{\bar{0}}$, and $V(\lambda)$ has a non-trivial self-extension.

If $L(\lambda)$ is typical, then $V(\lambda)=L(\lambda)$.

If $L(\lambda)$ is atypical and $(\lambda,\alpha_1)=0$, then $V(\lambda)$ has length two and can be described by the following exact sequence

$$0\to L(\lambda-\alpha_1)\to V(\lambda)\to L(\lambda)\to 0.$$

3.1.2. Corollary. *Let $L:=L(\lambda)$, $L(\mu)$ be integrable highest weight modules, $\mu\not\geq\lambda$ and*

$$\operatorname{Ext}^1(L(\lambda),L(\mu))\neq 0. \tag{3}$$

Then L is atypical. In addition,

(i) if $(\lambda+\rho,\alpha_1)=0$, then (3) is equivalent to the conditions $(\lambda+\rho,\alpha_2)\neq 0$ and $\mu=\lambda-\alpha_1$. (In particular, if L is Σ-tame, then it is Σ-regular).

(ii) If L is not Σ-tame, then for Σ' from Lemma 2.3, L is Σ'-regular and

$$L=L_{\Sigma'}(\lambda'),\ L(\mu)=L_{\Sigma'}(\mu-\beta),$$

where $\beta\in\Sigma'\cap\Delta^+(\Sigma)$ is an odd root orthogonal to $\lambda+\rho=\lambda'+\rho'$.

(iii) (3) implies $Ext^1(L(\lambda),L(\mu))=\mathbb{C}$.

Proof. Let N be a non-split extension given by the exact sequence

$$0\to L(\mu)\to N\to L(\lambda)\to 0.$$

Then N is an integrable quotient of $M(\lambda)$. From Lemma 3.1.1, we conclude that L is atypical and that (i), (iii) hold. For (ii) notice that since Σ' is obtained from Σ by L-typical odd reflections, $L(\lambda) = L_{\Sigma'}(\lambda')$ and $M(\lambda) = M_{\Sigma'}(\lambda')$, where $\lambda' + \rho' = \lambda + \rho$. Moreover, Σ' contains β such that $(\lambda' + \rho', \beta) = 0$ and $\beta \in \Delta^+(\Sigma)$. By Lemma 3.1.1, $L(\mu) = L_{\Sigma'}(\lambda' - \beta)$ as required. □

3.1.3. Lemma. *One has* $\mathrm{Ext}^1(L(\lambda), L(\lambda)) = 0$ *if* $L = L(\lambda)$ *is atypical and* $\mathrm{Ext}^1(L(\lambda), L(\lambda)) = \mathbb{C}$ *if* $L = L(\lambda)$ *is typical.*

Proof. Let L be Σ-atypical, i.e. $(\lambda, \alpha_1) = 0$ or $(\lambda, \alpha_2) = 0$. A non-trivial self-extension of $L(\lambda)$ induces a non-trivial self extension of $\dot{L}(\lambda)$ in the top degree component. However, an atypical irreducible $\dot{\mathfrak{g}}$-module does not have self-extension, see [G]. Hence $\mathrm{Ext}^1(L, L) = 0$ for an atypical irreducible $L \in \mathcal{F}_k$.

Let $L = L(\lambda)$ be typical. Consider a non-split exact sequence

$$0 \to L(\lambda) \to M \to L(\lambda) \to 0.$$

Recall that $\dot{\mathfrak{g}}_{\bar{0}} = \mathfrak{gl}_n$ contains a central element z. The λ-weight space M_λ is a non-split extension of $\mathbb{C}[z]$-modules

$$0 \to \mathbb{C}_\lambda \to M_\lambda \to \mathbb{C}_\lambda \to 0,$$

where $\mathbb{C}_\lambda$ is the one-dimensional $\mathbb{C}[z]$-module (z acts by $\lambda(z)$). Hence we have an injective homomorphism

$$\mathrm{Ext}^1(L(\lambda), L(\lambda)) \to \mathbb{C}.$$

By Lemma 3.1.1 we have a self-extension of $L(\lambda) = V_\Sigma(\lambda)$. Hence the statement. □

3.2. Typical blocks in $\mathcal{F}_k$. Recall that $\mathfrak{sl}(1|n)_{\bar{0}}$ has a non-trivial central element z; the centre of $\mathfrak{sl}(1|n)^{(1)}_{\bar{0}}$ is two-dimensional: it is spanned by K and z.

Let $\dot{L}$ be a typical finite-dimensional $\mathfrak{sl}(1|n)$-module of highest weight $\dot{\lambda}$ and let $\mathcal{F}(\dot{L})$ be the block containing $\dot{L}$ in the category of finitely generated $\mathfrak{sl}(1|n)$-modules. It is easy to deduce from [G] that the functor $N \mapsto N_\lambda$ provides an equivalence between $\mathcal{F}(\dot{L})$ and the category of finitely generated $\mathbb{C}[z]$-modules with a locally nilpotent action of $z - \lambda(z)$.

Using Corollary 3.1.2 we obtain the following

3.2.1. Corollary. *For any typical simple module $L := L(\lambda)$ in $\mathcal{F}_k$ there exists a block $\mathcal{F}_k(L)$ of $\mathcal{F}_k$ which has one up to isomorphism simple module L. The functor $N \mapsto N^{top}$ provides an equivalence between $\mathcal{F}_k(L)$ and the typical block of the category of finitely generated $\mathfrak{sl}(1|n)$-modules. The functor $N \to N_\lambda$ provides an equivalence between $\mathcal{F}_k(L)$ and the category of finitely generated $\mathbb{C}[z]$-modules with a locally nilpotent action of $z - \lambda(z)$.*

The inverse functors are given by the maximal integrable quotients of the corresponding induced modules $U(\mathfrak{g}) \otimes_{U(\dot{\mathfrak{g}})} -$ and $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} -$.

3.3. Atypical modules. Let $L \in \mathcal{F}_k$ be an atypical irreducible module. Let us describe $L' \in \mathcal{F}_k$ such that $Ext^1(L, L') \neq 0$. By duality, $Ext^1(L', L) \neq 0$, so we can assume that $\rho wt_\Sigma(L) \not\leq \rho wt_\Sigma(L')$. Using Corollary 3.1.2, we can describe L' is terms of Σ-s such that L is Σ-tame and Σ-regular. Below we show that there are exactly two such Σ-s and describe the $\rho wt_{\Sigma'}(L)$ (with respect to different Σ's).

3.4. Regular case. Recall that $L \in \mathcal{F}_k$ is regular if L is Σ-regular for every Σ. By 2.2.1, an atypical irreducible integrable highest weight module L is regular if and only if for every Σ there exists a unique $\beta \in \Delta^+_{\bar{1}}(\Sigma)$ such that $(\rho wt_\Sigma L, \beta) \neq 0$.

3.4.1. Lemma. *Let L be regular and atypical.*

(i) The set

$$S := \{\gamma \in \Delta |\ (\gamma, \rho wt_\Sigma L) = 0\}$$

does not depend on Σ and consists of two odd roots: $S = \{\pm\beta\}$.

In particular, L is Σ-tame for exactly two sets Σ.

(ii) Let Σ, Σ' be two sets of simple roots. One has

$$\rho wt_{\Sigma'} L = \begin{cases} \rho wt_\Sigma L & \text{if } \ S \cap \Delta^+(\Sigma) = S \cap \Delta^+(\Sigma') \\ \rho wt_\Sigma L + \beta & \text{if } \ S \cap \Delta^+(\Sigma) = \{\beta\}, S \cap \Delta^+(\Sigma') = \{-\beta\}. \end{cases} \tag{4}$$

(iii) Let $L = L_\Sigma(\lambda)$ be Σ-tame. Then $L_\Sigma(\lambda \pm \beta)$ are integrable and $Ext^1(L_\Sigma(\lambda \pm \beta), L) = \mathbb{C}$.

Proof. Fix Σ. Since L is atypical, S is not empty. Since L is regular and $k > 0$, $S \subset \Delta_{\bar{1}}$. If $\beta, \beta' \in \Delta_{\bar{1}}$ and $\beta \neq \pm\beta'$, then $\beta - \beta'$ or $\beta + \beta'$ is an even root. Hence $S = \{\pm\beta\}$. Recall that any two sets of simple roots are connected by a chain of odd reflections. One readily sees that the odd reflections do not change S; this gives (i); (ii) is straightforward. For (iii) let $\beta \in \Sigma$ and let $\beta' \in \Sigma$ be another odd root and $\alpha \in \Sigma$ be such that $(\alpha, \beta) = -1$. From 2.2.1, $L_\Sigma(\lambda \pm \beta)$ is integrable if and only if $(\lambda, \alpha), (\lambda, \beta') \geq 1$, which follows from regularity of L, see Lemma 2.3 (iii). From Corollary 3.1.1, $Ext^1(L_\Sigma(\lambda \pm \beta), L(\lambda)) = \mathbb{C}$. Hence (iii). □

3.4.2. If L is regular, then Σ' in Lemma 2.3 is unique (L is tame for two set of simple roots, connected by an odd reflections which are not L-typical).

3.5. Singular case. Let L be Σ-singular. By 2.2.1, $(\rho wt_\Sigma L, \beta_1) = (\rho wt_\Sigma L, \beta_2) = 0$, where β_1, β_2 are isotropic roots in Σ. Let $\tilde{\Sigma}$ be the maximal connected component in $\{\alpha \in \Sigma |\ (\rho wt_\Sigma L - \rho_\Sigma, \alpha) = 0\}$ which contains β_1, β_2. Since L has a non-zero central charge, $\tilde{\Sigma}$ is the set of simple roots of $\mathfrak{sl}(1|m)$ for some $m \leq n$.

We write

$$\tilde{\Sigma} = \{\alpha_1, \ldots, \alpha_s, \beta_1, \beta_2, \alpha_{s+1}, \ldots, \alpha_{m-2}\},$$

where the adjacent roots are not orthogonal (and for each i, α_i are non-isotropic).

Recall that any sets of simple roots are connected by a unique "proper" chain of odd reflections (the chain which does not contain subsequences of the form $r_\beta r_{-\beta}$). Thus, we can consider Σs "lying between" Σ', Σ'' (i.e., the proper chain from Σ' to Σ is a subchain of the proper chain from Σ' to Σ'').

3.5.1. Let $L = L_\Sigma(\lambda)$ be as above.

Lemma. *(i) There are exactly two sets of simple roots Σ_1, Σ_2 for which L is tame and regular.*

One has $\Sigma_1 = r_{\beta_1+\alpha_s+\ldots+\alpha_1} \cdots r_{\beta_1+\alpha_s} r_{\beta_1}\Sigma$ and

$$\rho wt_{\Sigma_1} L = \rho wt L + \beta_1 + (\beta_1 + \alpha_s) + \ldots + (\beta_1 + \alpha_s + \ldots + \alpha_1)$$

is orthogonal to the odd root $\beta := -(\beta_1 + \alpha_s + \ldots + \alpha_1) \in \Sigma_1$

(if $\tilde{\Sigma} = \{\beta_1, \beta_2, \ldots\}$, then $\Sigma_1 := r_{\beta_1}\Sigma$ and $\beta := -\beta_1$).

One has $\Sigma_2 = r_{\beta_2+\alpha_{s+1}+\ldots+\alpha_{m-2}} \cdots r_{\beta_2+\alpha_{s+1}} r_{\beta_2}\Sigma$ with the similar formulae for $\rho wt_{\Sigma_2} L$ and the orthogonal root β' in Σ_2.

(ii) L is Σ'-tame if and only if Σ' is obtained from Σ by a chain of odd reflections with respect to the roots in $\Delta^+(\tilde{\Sigma})$. In other words, Σ' lies between Σ and Σ_1 or Σ and Σ_2.

(iii) If L is Σ'-tame, then $L_\Sigma(\lambda) = L_{\Sigma'}(\lambda)$.

If L is not Σ'-tame, then $\rho wt_{\Sigma'} L = \rho wt_{\Sigma_i} L$, where $i = 1$ or $i = 2$ is such that $-\beta_i \in \Delta^+(\Sigma')$.

Proof. One readily sees that if Σ' is obtained from Σ by a proper chain of odd reflection $\Sigma' = r_{\gamma_j} r_{\gamma_{j-1}} \ldots r_{\gamma_1}\Sigma$, then $\gamma_1 = \beta_1$ or $\gamma_1 = \beta_2$ and $\gamma_i \in \Delta^+(\Sigma)$ for each $i = 1, \ldots, j$. Now the assertions follow from the observation that the odd reflection r_γ preserves $\rho wt L$ if this reflection is L-typical and preserves the highest weight of L otherwise. □

3.5.2. Corollary. *Let $L = L_\Sigma(\lambda)$ be as above. Then $Ext^1(L', L) \neq 0$ if and only if $L' \cong L_\Sigma(\lambda_\pm)$, where*

$$\lambda_- := \lambda + \beta_1 + (\beta_1 + \alpha_s) + (\beta_1 + \alpha_s + \alpha_{s-1}) + \ldots + (\beta_1 + \alpha_s + \ldots + \alpha_1),$$
$$\lambda_+ := \lambda + \beta_2 + (\beta_2 + \alpha_{s+1}) + (\beta_2 + \alpha_{s+1} + \alpha_{s+2}) + \ldots + (\beta_2 + \alpha_{s+1} + \ldots + \alpha_{m-2})$$

in the above notation.

Proof. Combining 3.1.2 and 3.5.1, we conclude that L' is isomorphic to $L_{\Sigma_1}(\lambda - \beta)$ or to a similar one for Σ_2. Let $L' = L_{\Sigma_1}(\lambda - \beta)$. One has

$$\Sigma = r_{-\beta_1} r_{-(\beta_1+\alpha_s)} \cdots r_{-(\beta_1+\alpha_s+\ldots+\alpha_1)}\Sigma_1.$$

One readily sees that all the reflections except $r_{-(\beta_1+\alpha_s+...+\alpha_1)} = r_\beta$ are L'-tame, so $\rho wt_\Sigma L' = \rho wt_{\Sigma_1} L' + \beta$ and $L' = L_\Sigma(\lambda')$ for

$$\lambda' = \rho wt_\Sigma L' - \rho = \lambda - \beta + \beta + \rho_1 - \rho = \lambda_-$$

as required (where ρ_1 stands for the Weyl vector for Σ_1). □

3.5.3. *Remark.* Note that the weight $\lambda + j\beta_1 + \rho$ is not regular for $j < s$ and is regular for $j = s$. One has

$$\lambda_- = r_{\alpha_1} \dots r_{\alpha_s}.(\lambda + s\beta_1),$$

where $w.\nu := w(\nu + \rho) - \rho$ is the standard ρ-shifted action.

3.6. Atypical blocks in $\mathcal{F}_k$. Fix a set of simple roots Σ and an atypical block. As we will see below, it contains a unique irreducible module $L_\Sigma(\lambda)$ with $(\lambda, \alpha_1) = (\lambda, \alpha_2) = 0$. Moreover, every irreducible module in this atypical block is $L_\Sigma(w.(\lambda + j\alpha_i))$ for $j > 0$, $i = 1, 2$ and $w \in W$ such that $\lambda + j\alpha_i$ is regular. Let us enumerate these modules as follows: set $\lambda^0 := \lambda$ and for $j > 0$ set $\lambda^j := w.(\lambda^{j-1} + s\alpha_1)$, where $s > 0$ is minimal such that this weight is regular; similarly, for $j < 0$ set $\lambda^j := w.(\lambda^{j-1} + s\alpha_2)$, where $s > 0$ is minimal such that this weight is regular. Then every irreducible module in the block is $L_\Sigma(\lambda^j)$ for a unique $j \in \mathbb{Z}$ and the non-zero extensions exist only between the adjacent modules: $Ext^1(L_\Sigma(\lambda^j), L_\Sigma(\lambda^s)) \neq 0$ if and only if $s = j \pm 1$.

3.6.1. Lemma. *For any set of simple roots Σ' the atypical block contains Σ'-singular module. Moreover, this module is unique.*

Proof. Let L be a simple module in the block and Σ be such that L is Σ-tame. We claim that it is enough to verify that

(1) the block contains a module which is tame for $r_\beta\Sigma$, where $\beta \in \Sigma$ is isotropic;

(2) the block contains a unique module $L_\Sigma(\lambda)$ which is Σ-singular.

Indeed, since any two sets of simple roots are connected by a chain of odd reflections (1) implies that any block contains a module tame with respect to any sets of simple roots Σ' and (2) implies the assertion .

Note that (2) implies (1), since $L_\Sigma(\lambda)$ is tame with respect $r_\beta\Sigma$. Hence it is enough to verify (2).

Let $L = L_\Sigma(\nu)$ and $(\alpha_1, \nu) = 0$, where $\alpha_1 \in \Sigma$ is odd. Let $\alpha_0, \alpha_2 \in \Sigma$ be such that $(\alpha_1, \alpha_0) = -1$ and $(\alpha_1, \alpha_2) = 1$ (α_2 is odd). The integrability of $L = L_\Sigma(\nu)$ implies that $(\nu, \alpha_i) \geq 0$ for $i = 0, 2$. If $(\nu, \alpha_2) = 0$, L is Σ-singular. Otherwise, by Lemma 3.4.1, $L_\Sigma(\nu - \alpha_1)$ is integrable and it lies in the same blocks as L; moreover, $(\nu - \alpha_1, \alpha_2) = (\nu, \alpha_1) - 1$. Thus the block contains a module $L_\Sigma(\lambda)$ with $(\lambda, \alpha_1) = (\lambda, \alpha_2) = 0$ as required.

Now let $L(\lambda)$, $L(\mu)$ be two Σ-singular modules which are in the same block and $\lambda \neq \mu$. Then there exists a set of weights $\nu_1, \dots, \nu_s$ such that $\mathrm{Ext}^1(L(\lambda), L(\nu_1)) \neq 0$,

$\mathrm{Ext}^1(L(\nu_i), L(\nu_{i+1})) \neq 0$ for all $i = 1, \dots, s-1$, and $\mathrm{Ext}^1(L(\nu_s), L(\mu)) \neq 0$. Without loss of generality we may assume that $L(\nu_1), \dots, L(\nu_s)$ are Σ-regular. By Lemma 3.1.1, Σ-singularity of $L(\lambda)$ implies $\lambda < \nu_1 < \dots < \nu_s < \mu$, i.e. $\lambda < \mu$. Similarly, Σ-singularity of $L(\nu)$ gives $\nu < \lambda$, a contradiction. □

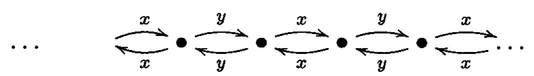
3.6.2. Proposition. *Let $\mathcal{B}$ be an atypical block in $\mathcal{F}_k$ and $L(\lambda^0) \in \mathcal{B}$ be a unique simple module which is Σ-singular.*

There exists a linear order λ^i, $i \in \mathbb{Z}$ of all simple modules $L^i = L(\lambda^i)$ such that

$$\mathrm{Ext}^1(L^i, L^j) = \begin{cases} \mathbb{C} & \text{if } j = i \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

For $i \geq 0$ one has $\lambda^i < \lambda^{i+1}$ and $\lambda^{-i} < \lambda^{-(i+1)}$.

The ext quiver of any atypical block is of the form



Proof. By 3.4.1 and 3.5.2, for any atypical module $L(\lambda)$ there exist two weights $\lambda_\pm$ such that $\mathrm{Ext}^1(L(\lambda), L(\lambda_\pm) \neq 0$. By Lemma 3.1.1, $\lambda_\pm > \lambda$ if $L(\lambda)$ is Σ-singular and $\lambda_- < \lambda < \lambda_+$ otherwise. Now the assertion follows from Lemma 3.6.1. We take $\lambda^{\pm 1}$ such that $Ext^1(L(\lambda^0), L(\lambda^{\pm 1})) \neq 0$. Suppose that $i > 0$ and λ^i is already constructed. Then λ^{i+1} is the unique weight such that $\lambda^{i+1} > \lambda^i$ and $\mathrm{Ext}^1(L^i, L^{i+1}) \neq 0$. If i is negative we define λ^{i-1} in the similar way. □

3.6.3. Let us show that the above quiver satisfies the relations $xy = yx = 0$.

Lemma. *There is no indecomposable module M in $\mathcal{F}_k$ such that $M/radM = L_1$, $radM/rad^2M = L_2$, $rad^2M = L_3$ for pairwise non-isomorphic irreducible modules L_1, L_2, L_3.*

Proof. Take Σ which contains the maximal possible number of odd roots orthogonal to $\rho wt_\Sigma L$: if L is regular (resp., singular) take Σ such that L is Σ-tame (resp., Σ-singular). Using Lemma 3.4.1 and Corollary 3.5.2, we conclude that for $i = 1, 3$ the differences $\rho wt_\Sigma(L_i) - \rho wt_\Sigma(L_2)$ are linear combinations of $\tilde{\Sigma}$, where $\tilde{\Sigma} \subsetneq \Sigma$ (for regular L, $\tilde{\Sigma}$ consists of one odd root). Consider the subalgebra $\dot{\mathfrak{g}} \subset \mathfrak{g}$ with the set of simple roots containing $\tilde{\Sigma}$; let $d \in \mathfrak{h}$ be the corresponding element (d acts on $\dot{\mathfrak{g}}t^r \subset \mathfrak{g}$ by rId). Let M^{top} be the generalized d-eigenspace with the maximal eigenvalue (maximal in a sense that $a + s$ is not an eigenvalue for $j \in \mathbb{Z}_{>0}$). Then M^{top} is an indecomposable $\dot{\mathfrak{g}}$-module which satisfies the same condition as M. This is impossible by [G]. □

3.6.4. Let us show that the above quiver does not have other relations except $xy = yx = 0$. This follows from [G]. Indeed, if there is another relation, it is of the form $P(x^2) = 0$ or $P(y^2) = 0$ for a non-zero polynomial P and x or y in $Ext^1(L^i, L^{i+1})$. Take Σ such that L^i, L^{i+1} are Σ-tame: $L^i = L(\lambda), L^{i+1} = L(\lambda - \beta)$ for $\beta \in \Sigma$. Consider the subalgebra $\dot{\mathfrak{g}} \subset \mathfrak{g}$ with the set of simple roots containing β. Define d and M^{top} as above. Then $(L^i)^{top}, (L^{i+1})^{top}$ are atypical $\dot{\mathfrak{g}}$-modules which satisfy the same relation; this contradicts to [G] (in the notation of [G], the quiver of the category C_r with r larger than degree P does not have relation given by P).

3.6.5. Theorem. *Any atypical block in $\mathcal{F}_k$ is equivalent to the category of finite-dimensional representations of the quiver of Proposition 3.6.2 with relations $xy = yx = 0$.*

4. The functor F_x

In this section we assume that $\mathfrak{g}$ is a Kac-Moody Lie superalgebra.

Take $x \in \mathfrak{g}_{\bar{1}}$ satisfying $[x, x] = 0$. The following construction is due to M. Duflo and V. Serganova, see [DS]. For a $\mathfrak{g}$-module N introduce

$$F_x(N) := Ker_N x / Im_N x.$$

Let $\mathfrak{g}^x$ be the centralizer of x in $\mathfrak{g}$. We view $F_x(N)$ as a module over $\mathfrak{g}^x$. Note that $[x, \mathfrak{g}] \subset \mathfrak{g}^x$ acts trivially on $F_x(N)$ and that $\mathfrak{g}_x := F_x(\mathfrak{g}) = \mathfrak{g}^x/[x, \mathfrak{g}]$ is a Lie superalgebra. Thus $F_x(N)$ is a $\mathfrak{g}_x$-module and F_x is a functor from the category of $\mathfrak{g}$-modules to the category of $\mathfrak{g}_x$-modules.

In [DS],[S1] the functor F_x was studied for finite-dimensional $\mathfrak{g}$. However, certain properties can be easily generalized to the affine case. In particular, F_x is a tensor functor, i.e. there is a canonical isomorphism $F_x(N_1 \otimes N_2) \simeq F_x(N_1) \otimes F_x(N_2)$.

4.1. Proposition. *Let $\mathfrak{g} = \dot{\mathfrak{g}}^{(1)}$ be the affinization of a Lie superalgebra $\dot{\mathfrak{g}}$ and assume that $x \in \dot{\mathfrak{g}}$. If $\dot{\mathfrak{g}}_x \neq 0$, then $\mathfrak{g}_x$ is the affinization of $\dot{\mathfrak{g}}_x$, If $\dot{\mathfrak{g}}_x = 0$ then $\mathfrak{g}_x$ is the abelian two-dimensional Lie algebra generated by K and d.*

Proof. Since

$$\mathfrak{g} = \mathbb{C}d \oplus \mathbb{C}K \oplus \bigoplus_{n \in \mathbb{Z}} \dot{\mathfrak{g}} \otimes t^n$$

and $\dot{\mathfrak{g}} \otimes t^n$ is isomorphic to the adjoint representation of $\dot{\mathfrak{g}}$ for every n, the statement follows. □

4.2. Let $\mathfrak{g} = \dot{\mathfrak{g}}^{(1)}$ be the affinization of a Lie superalgebra $\dot{\mathfrak{g}}$ and assume that $x \in \dot{\mathfrak{g}}$. Let $\dot{\Sigma}$ (resp., Σ) be the set of simple roots of $\dot{\mathfrak{g}}$ (resp., $\mathfrak{g}$).

Let $\beta_1, \dots \beta_r \in \dot{\Sigma}$ be a set of mutually orthogonal isotopic simple roots, fix non-zero root vectors $x_i \in \mathfrak{g}_{\beta_i}$ for all $i = 1, \dots, r$. Let $x := x_1 + \dots + x_r$. It is shown in [DS] that $\dot{\mathfrak{g}}_x$ is a finite-dimensional Kac-Moody superalgebra with roots

$$\dot{\Delta}^{\perp} := \{\alpha \in \dot{\Delta} |\ (\alpha, \beta_i) = 0, \alpha \neq \pm\beta_i\, i = 1, \dots, r\}$$

and the Cartan subalgebra

$$\mathfrak{h}_x := (\beta_1^{\perp} \cap \dots \cap \beta_r^{\perp}) / (\mathbb{C}h_{\beta_1} \oplus \dots \oplus \mathbb{C}h_{\beta_r}).$$

Assume that $\dot{\Delta}^{\perp}$ is not empty, then $\dot{\Delta}^{\perp}$ is the root system of the Lie superalgebra $\dot{\mathfrak{g}}_x$. One can choose a set of simple roots $\dot{\Sigma}_x$ such that $\Delta^+(\dot{\Sigma}_x) = \Delta^+ \cap \dot{\Delta}^{\perp}$. Let $\mathfrak{g}_x \subset \mathfrak{g}$ be the affinization of $\dot{\mathfrak{g}}_x$: the affine Lie superalgebra with a set of simple roots Σ_x containing $\dot{\Sigma}_x$ such that $\Delta^+(\Sigma_x) \subset \Delta^+$.

For example, if $\dot{\mathfrak{g}} = A(m|n), B(m|n)$ or $D(m|n)$, then $\dot{\mathfrak{g}} = A(m-r|n-r), B(m-r|n-r)$ or $D(m-r|n-r)$. If $\dot{\mathfrak{g}} = C(n)$, G_3 or F_4, then $r = 1$ and $\dot{\mathfrak{g}}_x$ is the Lie algebra of type C_{n-1}, A_1 and A_2 respectively. If $\dot{\mathfrak{g}} = D(2,1;\alpha)$, then $r = 1$ and $\mathfrak{g}_x = \mathbb{C}$.

4.3. Proposition. *Let $\mathfrak{g} = \dot{\mathfrak{g}}^{(1)}$ be the affinization of a Lie superalgebra $\dot{\mathfrak{g}}$ and assume that $x \in \dot{\mathfrak{g}}$. Let $x \in \dot{\mathfrak{g}}$ and N be a restricted $\mathfrak{g}$-module. If the Casimir element $\Omega_{\mathfrak{g}}$ acts on a N by a scalar C, then the Casimir element $\Omega_{\mathfrak{g}_x}$ acts on the $\mathfrak{g}_x$-module $F_x(N)$ by the same scalar C.*

Proof. Let us write the Casimir element $\Omega_{\mathfrak{g}}$ in the following form (see [K3], (12.8.3))

$$\Omega_{\mathfrak{g}} = 2(h^{\vee} + K)d + \Omega_0 + 2\sum_{i=1}^{\infty} \Omega(i),$$

where $\Omega(i) = \sum v_j v^j$ for some basis $\{v_j\}$ in $\dot{\mathfrak{g}} \otimes t^{-i}$ and the dual basis $\{v^j\}$ in $\dot{\mathfrak{g}} \otimes t^i$. Similarly we have

$$\Omega_{\mathfrak{g}_x} = 2(h^{\vee} + K)d + \Omega_0 + 2\sum_{i=1}^{\infty} \Omega_x(i).$$

We claim that $\Omega_x(i) \equiv \Omega(i)(\mathrm{mod}[x, U(\mathfrak{g})])$. Indeed, we use the decomposition $\dot{\mathfrak{g}} = \dot{\mathfrak{g}}_x \oplus \mathfrak{m}$, where $\mathfrak{m}$ is a free $\mathbb{C}[x]$-module. Using a suitable choice of bases we can write

$$\Omega(i) = \Omega_x(i) + \sum u_s u^s$$

for the pair of dual bases $\{u_s\}$ in $\mathfrak{m} \otimes t^{-i}$ and $\{u^s\}$ in $\mathfrak{m} \otimes t^i$. If $i > 0$, then $\sum u_s u^s$ is x-invariant element via the embedding $\mathfrak{m} \otimes \mathfrak{m} \hookrightarrow U(\mathfrak{g})$. If $i = 0$, then $\sum u_s u^s$ is x-invariant element via the embedding $S^2(\mathfrak{m}) \hookrightarrow U(\mathfrak{g})$. Since $\mathfrak{m} \otimes \mathfrak{m}$ and $S^2(\mathfrak{m})$ are free $\mathbb{C}[x]$-modules, we obtain in both cases that $\sum u_s u^s$ lies in the image of $\mathrm{ad}\, x$.

Now the statement follows immediately from the fact that $[x, U(\mathfrak{g})]$ annihilates $F_x(N)$. □

5. Invariants of simple objects in the same block

Now let $\mathfrak{g} = \mathfrak{sl}(1|n)^{(1)}$ with $n > 2$. Take a non-zero $x \in \mathfrak{g}_\beta$, where β is an odd isotropic root; then $[x, x] = 0$.

In this section we will show that for an irreducible modules $L, L' \in \mathcal{F}_k$ and non-zero $x \in \mathfrak{g}_\beta$ one has

(i) $F_x(L) = 0$ if and only if L is typical;

(ii) if L is atypical, then $F_x(L) \cong F_x(L')$ if and only if L and L' lie in the same block.

5.1. Fix a set of simple roots Σ; let $\alpha_1, \alpha_2 \in \Sigma$ be odd roots. Since for any odd root β the orbit $W\beta$ contains either α_2 or $-\alpha_2$, hence for integrable module M, $F_x(M) \simeq F_y(M)$ for some $y \in \mathfrak{g}_{\alpha_2}$ or $\mathfrak{g}_{-\alpha_2}$. Thus, we may assume that $x \in \mathfrak{g}_{\alpha_2}$ or $x \in \mathfrak{g}_{-\alpha_2}$. Then $\mathfrak{g}_x \cong \mathfrak{sl}_{n-1}^{(1)}$ with the set of simple roots

$$\Sigma_x := \{\alpha_0, \alpha_1 + \alpha_2 + \alpha_3, \alpha_4, \dots, \alpha_n\}.$$

Recall that, by Lemma 3.1.1, a Verma module $M(\lambda)$ has at most two integrable quotients: $L(\lambda)$ and N such that $N/L(\lambda - \beta) = L(\lambda)$.

5.2. Proposition. *Let L be an irreducible typical integrable highest weight module. Then $F_x(L) = 0$ for any non-zero $x \in \mathfrak{g}_\beta$, where β is an odd isotropic root.*

Proof. Set $\lambda := \rho wt_\Sigma L$; since L is typical, λ does not depend on Σ.

Let $F_x(L) \neq 0$ and let $v \in L$ be a preimage of a highest weight vector in $F_x(L)$; we can choose v to be a weight vector of weight ν. Then $(\nu, \alpha_2) = 0$. Note that if $(\lambda, \alpha_2) \notin \mathbb{Z}$, then such ν does not exist. Hence in this case $F_x(L) = 0$.

We assume now that $(\lambda, \alpha_2) \in \mathbb{Z}$ and $x \in \mathfrak{g}_{\pm\alpha_2}$. By Lemma 2.2.2, we can (and will) assume that $(\lambda, \alpha) > 0$ for each $\alpha \in \Sigma$. Let $\Sigma = \{\alpha_i\}_{i=0}^n$ and α_1, α_2 are odd. Set $\rho := \rho_\Sigma$. Set $a_i := (\nu + \rho, \alpha_i)$ for $i = 0, \dots, n$. Since $F_x(L)$ is $\mathfrak{g}_x$-integrable, and

$$\Pi_x = \{\alpha_0, \alpha_1 + \alpha_2 + \alpha_3, \alpha_4, \dots, \alpha_n\},$$

one has

$$a_2 = 0, \quad a_1 + a_3 \geq 0, \quad a_i > 0 \text{ for } i \neq 1, 2, 3. \tag{5}$$

Set $\lambda' := \nu + \rho - a_1\alpha_2, \quad \mu := \lambda - \lambda'$.

One has $(\lambda', \alpha_i) = 0$ for $i = 1, 2$ and $(\lambda', \alpha_i) \geq 0$ for $i = 0, \dots, n$.

Write $\lambda - \rho - \nu = \sum_{i=0}^n k_i\alpha_i$. Then $k_i \geq 0$ for each i (since $v \in L(\lambda - \rho)$). Since $a_1 = (\lambda, \alpha_1) + k_0 - k_2$, one has $k_2 + a_1 > 0$. Therefore

$$\mu \in \mathbb{Z}_{\geq 0}\Sigma.$$

By Proposition 4.3, $(\nu+2\rho_x,\rho_x)=||\lambda||^2-||\rho^2||$. One readily sees that $2(\rho-\rho_x)=(n-2)\alpha_2$, so $||\rho||^2=||\rho_x||^2$ and $||\nu+\rho_x||^2=||\nu+\rho||^2$.

This gives $||\lambda'||^2=||\lambda||^2$, that is

$$(\lambda,\mu)+(\lambda',\mu)=0.$$

Since $(\lambda,\alpha_i)>0$ and $(\lambda',\alpha_i)\geq 0$ for each $i=0,\dots,n$, we obtain $\lambda=\lambda'$. However, $(\lambda',\alpha_2)=0$, a contradiction. □

5.3. Proposition. *Let N be an integrable quotient of an atypical Verma module $M(\lambda)$.*

(i) $F_x(N)\cong L_{\mathfrak{g}_x}(\lambda|_{\mathfrak{h}_x})^{\oplus s}$, where $s=1$ if $N=L(\lambda)$ and $s=0$ or $s=2$ otherwise.

(ii) Let $(\lambda,\beta)=0$ for an isotropic simple root β. Then

$$F_x(N)\cong L_{\mathfrak{g}_x}(\lambda|_{\mathfrak{h}_x})^{\oplus s} \text{ where } \begin{cases} s=1 & \text{if } N=L(\lambda),\\ s=0 & \text{if } x\in\mathfrak{g}_{-\beta},\ N\neq L(\lambda),\\ s=2 & \text{if } x\in\mathfrak{g}_{\beta},\ N\neq L(\lambda).\end{cases}$$

Proof. By 3.1, $M(\lambda)=M_{\Sigma'}(\lambda')$, where $(\lambda',\alpha)=0$ for some isotropic $\alpha\in\Sigma'$. Thus for (i) we can assume that $(\lambda,\beta)=0$ for an isotropic simple root β. By above, we have $F_x(N)=F_y(N)$, where y in $\mathfrak{g}_\beta$ or in $\mathfrak{g}_{-\beta}$. Therefore (i) is reduced to (ii). Let us prove (ii). Clearly, $F_x(N)$ is $\mathfrak{g}_x$-integrable, so completely reducible. Assume that Ker_xN contains a vector v of weight $\lambda-\mu$ whose image in $F_x(N)$ is a $\mathfrak{g}_x$-singular vector. Since $v\in Ker_xN$ and $v\notin xN$, one has $(\lambda-\mu,\beta)=0$, that is $(\mu,\beta)=0$. Since $\mu\in\mathbb{Z}_{\geq 0}\Sigma$, we obtain $\mu\in\mathbb{Z}_{\geq 0}\Sigma_x+\mathbb{Z}\beta$.

Using Lemma 4.3 we get $||\lambda+\rho-\mu||^2=||\lambda+\rho||^2$, that is $(\lambda+\rho,\mu)+(\lambda+\rho-\mu,\mu)=0$.

Since N is integrable and $(\lambda,\beta)=0$, we get $(\lambda,\alpha)\geq 0$ for each $\alpha\in\Sigma$. Thus $(\lambda+\rho,\mu)\geq 0$ and so $(\lambda+\rho-\mu,\mu)\leq 0$.

Taking into account that $F_x(N)$ is $\mathfrak{g}_x$-integrable (where $\mathfrak{g}_x=\mathfrak{sl}_{n-1}^{(1)}$) and $\mu\in\mathbb{Z}_{\geq 0}\Sigma_x+\mathbb{Z}\beta$, $(\lambda+\rho-\mu,\mu)\geq 0$ and the equality holds if and only if $\mu\in\mathbb{Z}\beta$. Therefore $\mu\in\mathbb{Z}\beta$, that is $\mu\in\{0,\beta\}$. Hence

$$F_x(N)=L_{\mathfrak{g}_x}(\lambda|_{\mathfrak{h}_x})^{\oplus s},\quad \text{where } s:=\dim F_x(N_\lambda\oplus N_{\lambda-\beta}).$$

Note that $N':=N_\lambda\oplus N_{\lambda-\beta}$ is a module over a copy of $\mathfrak{sl}(1|1)$ generated by $\mathfrak{g}_{\pm\beta}$ (one has $x\in\mathfrak{sl}(1|1)$). If $N=L(\lambda)$, then N' is a trivial $\mathfrak{sl}(1|1)$-module; and if $N/L(\lambda-\beta)=L(\lambda)$ then N' is a Verma $\mathfrak{sl}(1|1)$-module of highest weight zero. The assertion follows. □

5.4. Corollary. *Let $L\in\mathcal{F}_k$ be an irreducible module. Then $F_x(L)=0$ if and only if L is typical. For atypical L, $F_x(L)$ is integrable $\mathfrak{sl}_{n-1}^{(1)}$-module and $F_x(L)\cong F_x(L')$ if and only L and L' lie in the same block.*

Proof. Retain notation of Proposition 3.6.2. If L^j, L^{j+1} are simple objects in an atypical block $\mathcal{B}$ and $j \geq 0$ (resp. $j < -1$), then there exists a Verma module $M(\lambda)$ such that its maximal integrable quotient $V(\lambda)$ such that $V(\lambda)/L^j \cong L^{j+1}$ (resp., $V(\lambda)/L^{j+1} \cong L^j$). From Proposition 5.3, we get $F_x(L^j) \cong F_x(L^{j+1})$, so $F_x(L)$ is a non-zero invariant of an atypical block.

Let us show that this invariant separates blocks. Fix a set of simple roots Σ and take $x \in \mathfrak{g}_{-\alpha_2}$. Let $\lambda^\# \in \mathfrak{h}_x$ be the highest weight of $F_x(L), F_x(L')$. Let us show that L, L' are in the same block. Indeed, each block contains a unique Σ-singular irreducible module. Thus we can (and will) assume that L, L' are Σ-singular. Let $L = L(\lambda), L' = L(\lambda')$. One has $\lambda^\# = \lambda|_{\mathfrak{h}_x} = \lambda'|_{\mathfrak{h}_x}$. Since λ, λ' are Σ-singular, $\lambda = \lambda'$, that is $L \cong L'$ as required. □

5.5. Let us calculate the highest weight of $F_x(L)$.

Let $L = L_\Sigma(\lambda)$ be an atypical integrable module of level k. Write $\Sigma = \{\alpha_0\} \cup \dot{\Sigma}$, where α_0 is even and $\dot{\Sigma}$ is a set of simple roots for $\mathfrak{sl}(1|n)$. Let $\{\varepsilon_i\}_{i=1}^n \cup \{\delta_1\}$ be the standard notation for $\mathfrak{sl}(1|n)$; then

$$\alpha_0 = \delta - \varepsilon_1 + \varepsilon_n,\ \alpha_1 = \varepsilon_1 - \delta_1, \alpha_2 = \delta_1 - \varepsilon_2,\ \ldots, \alpha_n = \varepsilon_{n-1} - \varepsilon_n.$$

Set $c_i := (\lambda + \rho, \varepsilon_i)$ for $i = 1, \ldots, n$ and $d := (\lambda, \delta_1)$. Note that these numbers determine L as a module over $[\mathfrak{g}, \mathfrak{g}]$.

We claim that either $c_1 = c_2 = b$ and $c_i - b$ is not divisible by $k + n - 1$; there exist a unique index i such that $c_i - b$ is divisible by $k + n - 1$. One has

$$F_x(L(\lambda)) = L_{\mathfrak{sl}^{(1)}_{n-1}}(\lambda^\#),$$

where $\lambda^\#$ has level k and the marks $(\lambda^\# + \rho, \varepsilon_i)$ are obtained from $(c_1, \ldots, c_n)$ by throwing away one element j with $c_j - b$ divisible by $k + n - 1$.

Indeed, set $L = L(\lambda)$. By Proposition 5.3, $F_x(L(\lambda)) = L_{\mathfrak{g}_x}(\lambda^\#)$ for some $\lambda^\# \in \mathfrak{h}_x$ (and $\mathfrak{g}_x \cong \mathfrak{sl}^{(1)}_{n-1}$). By Lemma 2.3, there exists Σ' such that L is Σ'-tame and Σ' is obtained from Σ by L-typical odd reflections, so $\rho wt_\Sigma L = \rho wt_{\Sigma'} L$. Let $\beta \in \Sigma'$ be such that $(\rho wt_{\Sigma'} L, \beta) = 0$. Take $y \in \mathfrak{g}_\beta$. By above, $F_x(L)$ is equivalent to $F_y(L)$, where $y \in \mathfrak{g}_\beta$ or $y \in \mathfrak{g}_{-\beta}$. Using Proposition 5.3 we get

$$F_y(L) = L_{\mathfrak{g}_y}(\lambda'|_{\mathfrak{h}_y}),$$

where $\lambda' = \lambda + \rho - \rho'$ and $\mathfrak{h}_y = \{h \in \mathfrak{h} \cap \mathfrak{sl}^{(1)}_n | \beta(h) = 0\}$.

Assume that $\lambda + \rho$ is regular. Then there exists a unique j such that $c_j - b$ is divisible by $k + n - 1$. By above, $\mathfrak{g}_y$ has a set of simple roots $\Sigma_y = \{\alpha \in \Sigma_0 |\ (\beta, \alpha) = 0\}$. From 5.1 it follows that for each $\alpha \in \Sigma_y$ one has

$$(\lambda^\# + 1, \alpha) = (\lambda' + \rho', \alpha) = (\lambda + \rho, \alpha).$$

Therefore $\lambda^\# + \rho^\#$ has the marks $\{c_i\}_{i=1}^n \setminus \{c_j\}$ as required.

Assume that $\lambda+\rho$ is singular. Then, by 2.2.1, $(\lambda+\rho,\alpha_1)=(\lambda+\rho,\alpha_2)=0$ (in particular, $\Sigma'=\Sigma$) and $\alpha_1+\alpha_2$ is the only even positive root orthogonal to $\lambda+\rho$. Then $c_1=c_2=b$ and c_i-b is divisible by $k+n-1$ if and only if $i=1,2$. Then $x\in\mathfrak{g}_{\alpha_j}$ for $j=1$ or $j=2$ and, as above, $\lambda^{\#}+\rho^{\#}$ corresponds to $\{c_i\}_{i=1}^n\setminus\{c_j\}$.

6. Modules over $V_k(\mathfrak{sl}(1|n))$ for $k\in\mathbb{Z}_{>0}$

View $\mathfrak{g}=\mathfrak{sl}(1|n)^{(1)}$ as the affinization of $\mathfrak{sl}(1|n)$. Let $\dot{\mathfrak{h}}$ be a Cartan subalgebra of $\mathfrak{sl}(1|n)$ and $\dot{\Pi}$ be the set of simple roots. Then $\mathfrak{h}=\dot{\mathfrak{h}}+\mathbb{C}d+\mathbb{C}K$ and $\Pi=\dot{\Pi}\cup\{\alpha_0\}$.

The modules over affine vertex superalgebra $V^k(\mathfrak{sl}(1|n))$ have the natural structure of $[\mathfrak{g},\mathfrak{g}]$-modules of level k. We say that $[\mathfrak{g},\mathfrak{g}]$-module M is graded if $M=\oplus_{m\in\mathbb{Z}}M_m$ with $(at^n)M_m\subset M_{m-n}$ (for $a\in\mathfrak{sl}(1|n)$).

The positive energy (in the sense of [DK]) $V^k(\mathfrak{sl}(1|n))$-modules are $\mathbb{Z}$-graded $[\mathfrak{g},\mathfrak{g}]$-module of level k with the grading bounded from the below. We also call such $[\mathfrak{g},\mathfrak{g}]$-modules also the modules of positive energy.

For such a module we extend the $[\mathfrak{g},\mathfrak{g}]$-action to the $\mathfrak{g}$-action by $dv:=-mv$ for $v\in M_m$.

Let V^k be the vacuum module of level k ($V^k:=\mathrm{Ind}^{\mathfrak{g}}_{\dot{\mathfrak{g}}+\mathfrak{n}^++\mathfrak{h}}\mathbb{C}_k$, where $\mathbb{C}_k$ is the trivial $\dot{\mathfrak{g}}+\mathfrak{n}^+$-module with K acting by kId and d acting by zero). Let V_k be the simple quotient of V^k and $|0)$ be the highest weight vector of V^k (and its image in V_k).

6.1. Theorem. *As a $\mathfrak{g}$-module, V_k is integrable if and only if $k\in\mathbb{Z}_{\geq 0}$. The irreducible positive energy $V_k(\mathfrak{sl}(1|n))$-modules are $L(\lambda)\in\mathcal{F}_k$, where $\lambda(d)\in\mathbb{Z}$.*

For $k\in\mathbb{Z}_{>0}$ the positive energy $V_k(\mathfrak{sl}(1|n))$-module are the positive energy $[\mathfrak{g},\mathfrak{g}]$-modules of level k, which are integrable over $\mathfrak{sl}_n^{(1)}$.

6.1.1. *Remark.* Let $k\in\mathbb{Z}_{\geq 0}$.

It is easy to see that $V_0(\mathfrak{sl}(1|n))$-modules are the direct sums of the trivial modules.

6.1.2. *Proof of Theorem 6.1.* Set $V^k:=V^k(\mathfrak{sl}(1|n))$, $V_k:=V_k(\mathfrak{sl}(1|n))$. We start from the following lemma (see, for example, [AM], Prop. 3.4).

Lemma. *If $I\subset V^k(\mathfrak{sl}(1|n))$ be a cyclic submodule generated by a vector a, then the $V^k(\mathfrak{sl}(1|n))/I$-modules are the $V^k(\mathfrak{sl}(1|n))$-modules annihilated by $Y(a,z)$.*

From 2.2.1 it follows that $V_k(\mathfrak{sl}(1|n))$ is integrable if and only if $k\in\mathbb{Z}_{\geq 0}$. Moreover, from 3.1 it follows that if V^k has an integrable quotient, then it is simple (i.e., is V_k). Let I be a submodule of V^k generated by $f_0^{k+1}|0)$, where f_0 is a non-zero element in $\mathfrak{g}_{-\alpha_0}$. One readily sees that V^k/I is integrable, so $V_k=V^k/I$. By Lemma 6.1.2, V_k-modules are V^k annihilated by $Y(f_0^{k+1}|0),z)$. Note that $Y(f_0^{k+1}|0),z)\in V^k(\mathfrak{sl}_n)$ and $V_k(\mathfrak{sl}_n):=V^k(\mathfrak{sl}_n)/I'$, where I' is the $\mathfrak{sl}_n^{(1)}$-submodule of $V^k(\mathfrak{sl}_n)$ generated by $f_0^{k+1}|0)$. Therefore the V_k-modules are exactly the V^k-modules which are the modules over $V_k(\mathfrak{sl}_n)$.

By [DLM], Thm. 3.7, the $V_k(\mathfrak{sl}_n)$-modules are direct sums of irreducible integrable highest weight $[\mathfrak{sl}_n^{(1)}, \mathfrak{sl}_n^{(1)}]$-modules of level k. Therefore the positive energy V_k-modules are the positive energy integrable $[\mathfrak{g}, \mathfrak{g}]$-modules of level k. If such module is irreducible, then, extending the action of $[\mathfrak{g}, \mathfrak{g}]$ to $\mathfrak{g}$ as above, we obtain an irreducible module in $\mathcal{F}_k$. Since d acts diagonally on each irreducible module in $\mathcal{F}_k$ the assertion follows. □

6.2. "Bad example". The following example shows that an indecomposable positive energy V_k-module may look rather wild.

Recall that $\dot{\mathfrak{g}} := \mathfrak{sl}(1|2)$ is a $\mathbb{Z}$-graded Lie algebra: $\dot{\mathfrak{g}} = \dot{\mathfrak{g}}_{-1} \oplus \dot{\mathfrak{g}}_0 \oplus \dot{\mathfrak{g}}_1$, where $\dot{\mathfrak{g}}_0 = \dot{\mathfrak{g}}_{\bar{0}}$ and $\dot{\mathfrak{g}}_{-1} \oplus \dot{\mathfrak{g}}_1 = \dot{\mathfrak{g}}_{\bar{1}}$. Let z be a central element in $\dot{\mathfrak{g}}_{\bar{0}} = \mathfrak{gl}_2$. View $\mathbb{C}[z]$ as a module over $\dot{\mathfrak{g}}_0 + \dot{\mathfrak{g}}_1$, where z acts by the multiplication and $\mathfrak{sl}_2 + \dot{\mathfrak{g}}_1$ act by zero; let E be the induced $\dot{\mathfrak{g}}$-module. Let e_θ be the highest root vector in $\mathfrak{sl}_2$. One readily sees that $e_\theta^2 E = 0$. From [DK], Thm. 2.30 (see also [Z], Thm. 2.2.1), it follows that there exists a $\mathbb{Z}_{\geq 0}$-graded $V_1(\mathfrak{sl}(1|2))$-module $N = \sum_{i=0}^{\infty} N_i$ with $N_0 = E$. This is a cyclic indecomposable module with infinite-dimensional graded components. This module is integrable over $\mathfrak{sl}_2^{(1)}$, but is not integrable over $\mathfrak{sl}(1|2)^{(1)}$ (since $z \in \mathfrak{h}$ acts freely on N_0).

The Sugawara construction equips N with an action of the Virasoro algebra $\{L_n\}_{n \in \mathbb{Z}}$, see [K3], 12.8 for details. The action of L_0 to N_0 is equal to the action of the Casimir operator $\dot{\Omega}$ of $\dot{\mathfrak{g}}$. View $\mathbb{C}[z]$ as a $(\dot{\mathfrak{g}}_0 + \dot{\mathfrak{g}}_1)$-submodule of $N_0 = E$. Since $\mathfrak{sl}_2 + \dot{\mathfrak{g}}_1$ act trivially, the action of $\dot{\Omega}$ is proportional to $z(z-1)$, so this is a free action. Since $[L_0, \dot{\mathfrak{g}}] = 0$, N_0 is a free L_0-module.

Defining the action of d on E by zero, we can view N as a $\mathfrak{g}$-module, which is an indecomposable integrable module with a free action of the Casimir element Ω; moreover, this module is bounded (the eigenvalues of d lie in $\mathbb{Z}_{\leq 0}$).

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