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Cubature formulas for great antipodal sets on complex Grassmann manifolds

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(joint work with Hirotake Kurihara†)

Abstract

Great antipodal subsets of compact symmetric spaces are defined by Chen–Nagano [Trans. Amer. Soc. Math. (1988)] as finite subsets satisfying certain geometric properties with maximum cardinalities. In this paper, we give a formulation of Delsarte theory for finite subsets of compact symmetric spaces, and as its application, we show that great antipodal subsets of complex Grassmannian manifolds give cubature formulas for certain functional spaces.

1 Main results

Throughout this paper, we use the following symbol for complex Grassnannian manifolds:

$$\mathrm{Gr}_k(\mathbb{C}^n) := \{k\text{-dimensional complex linear subspaces of } \mathbb{C}^n\} \quad (k \leq n/2).$$

Let us consider the standard Hermitian inner-product (,) on $\mathbb{C}^n$. Then the unitary group $U(n)$ of $\mathbb{C}^n$ acts on $\mathrm{Gr}_k(\mathbb{C}^n)$ transitively. Furthermore, for each point p of $\mathrm{Gr}_k(\mathbb{C}^n)$, the isotropy subgroup of $U(n)$ at p is isomorphic to the Lie group $U(k) \times U(n-k)$.

For each point $p \in \mathrm{Gr}_k(\mathbb{C}^n)$, the point symmetry on $\mathrm{Gr}_k(\mathbb{C}^n)$ at p will be denoted by s_p. That is, s_p is the isometry on $\mathrm{Gr}_k(\mathbb{C}^n)$ induced by the involution σ_p on $\mathbb{C}^n = p \oplus p^\perp$ with $\sigma_p|_p = \mathrm{id}_p$ and $\sigma_p|_{p^\perp} = -\mathrm{id}_{p^\perp}$.

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A finite subset X of $\mathrm{Gr}_k(\mathbb{C}^n)$ is said to be *antipodal* if X satisfies the following geometric property:

$$s_x(y) = y \quad \text{for any } x, y \in X.$$

It is known that the cardinality of any antipodal subset is bounded by $\binom{n}{k}$. An antipodal subset X of $\mathrm{Gr}_k(\mathbb{C}^n)$ is said to be *great* if $\#X = \binom{n}{k}$. By fixing an orthonormal basis $\mathcal{B} = \{e_1, \dots, e_n\}$ of $\mathbb{C}^n$, we obtain a great antipodal subset

$$X_{\mathcal{B}} := \{\mathrm{Span}_{\mathbb{C}}\{e_{i_1}, \dots, e_{i_k}\} \mid \{i_1, \dots, i_k\} \text{ is a } k\text{-subset of } \{1, \dots, n\}\}$$

of $\mathrm{Gr}_k(\mathbb{C}^n)$. It is also known that any great antipodal subset of $\mathrm{Gr}_k(\mathbb{C}^n)$ is of the form of $X_{\mathcal{B}}$ for some orthonormal bases $\mathcal{B}$ of $\mathbb{C}^n$, and thus great antipodal subsets of $\mathrm{Gr}_k(\mathbb{C}^n)$ are unique up to $U(n)$-conjugations. See [3] or Section 3.1 for the details of great antipodal subsets of complex Grassmannian manifolds.

In this paper, we study analytic properties of the great antipodal subsets of $\mathrm{Gr}_k(\mathbb{C}^n)$. In order to state our main results, we set up our notation of some functions as follows: For each pair $\mathcal{V} = (v_1, \dots, v_k), \mathcal{W} = (w_1, \dots, w_k) \in (\mathbb{C}^n)^k$ of the sets of ordered k-vectors in $\mathbb{C}^n$, we define the k-by-k matrix $A_{\mathcal{V},\mathcal{W}}$ by

$$(A_{\mathcal{V},\mathcal{W}})_{i,j} = (v_i, w_j).$$

Furthermore, for such $(\mathcal{V}, \mathcal{W})$, we also define the function $f_{\mathcal{V},\mathcal{W}}$ on $\mathrm{Gr}_k(\mathbb{C}^n)$ by

$$f_{\mathcal{V},\mathcal{W}}(p) := (\det A_{\mathcal{V},\mathcal{P}}) \cdot \overline{(\det A_{\mathcal{W},\mathcal{P}})} \quad \text{for each } p \in \mathrm{Gr}_k(\mathbb{C}^n)$$

where $\mathcal{P}$ is an ordered basis of the k-dimensional vector subspace p of $\mathbb{C}^n$ (then $f_{\mathcal{V},\mathcal{W}}(p)$ does not depend on the choice of $\mathcal{P}$). Remark that $f_{\mathcal{V},\mathcal{W}} \equiv 0$ if $\mathcal{V}$ or $\mathcal{W}$ are linearly dependent. We put

$$\mathcal{H} := \mathrm{Span}_{\mathbb{C}}\{f_{\mathcal{V},\mathcal{W}} : \mathrm{Gr}_k(\mathbb{C}^n) \to \mathbb{C} \mid \mathcal{V}, \mathcal{W} \text{ are sets of ordered} k\text{-vectors in } \mathbb{C}^n\}.$$

Then $\dim_{\mathbb{C}} \mathcal{H} = \binom{n}{k}^2$.

We remark that the functional space $\mathcal{H}$ on $\mathrm{Gr}_k(\mathbb{C}^n)$ defined above is a subrepresentation of the left regular representation of $U(n)$ on $C^\infty(\mathrm{Gr}_k(\mathbb{C}^n))$. As a finite-dimensional $U(n)$-representation, $\mathcal{H}$ can be decomposed as the direct sum of irreducible $U(n)$-representations $V_0, \dots, V_k$ such that the highest weight of V_l is of the form

$$(\underbrace{1, \dots, 1}_{l}, \underbrace{0, \dots, 0}_{n-2l}, \underbrace{-1, \dots, -1}_{l})$$

for each l.

As a main result of this paper, we show the following analytic property of the great antipodal subset X of $\mathrm{Gr}_k(\mathbb{C}^n)$:

Theorem 1.1. *Let X be a great antipodal subset of $\mathrm{Gr}_k(\mathbb{C}^n)$ and $\mathcal{H}$ the (finite-dimensional) functional space on $\mathrm{Gr}_k(\mathbb{C}^n)$ defined above. Then the following "cubature formula" holds:*

$$\frac{1}{\mathrm{vol}(\mathrm{Gr}_k(\mathbb{C}^n))} \int_{\mathrm{Gr}_k(\mathbb{C}^n)} f d\mu_{\mathrm{Gr}_k(\mathbb{C}^n)} = \frac{1}{\# X} \sum_{x \in X} f(x) \quad \text{for any } f \in \mathcal{H} \qquad (1)$$

where $\mu_{\mathrm{Gr}_k(\mathbb{C}^n)}$ is a $U(n)$-invariant Haar measure on $\mathrm{Gr}_k(\mathbb{C}^n)$ and $\mathrm{vol}(\mathrm{Gr}_k(\mathbb{C}^n))$ is the volume of $\mathrm{Gr}_k(\mathbb{C}^n)$ with respect to $\mu_{\mathrm{Gr}_k(\mathbb{C}^n)}$. Furthermore, any great antipodal subset X has the minimum cardinality as a finite subset of $\mathrm{Gr}_k(\mathbb{C}^n)$ such that the formula (1) holds.

Remark 1.2. *For some (k, n), there exists a finite subset X of $\mathrm{Gr}_k(\mathbb{C}^n)$ such that the formula (1) holds for any $f \in \mathcal{H}$, the cardinality of X is $\binom{n}{k}$ but not antipodal (see [6]).*

In Section 2, we give a fomulation of Delsarte theory for finite subsets of compact symmetric spaces. Theorem 1.1 is proved as an application of our Delsarte theory. In Section 3, we give a proof of the first harf part of Theorem 1.1. The details will be reported elsewhere.

2 Delsarte theory on compact symmeric spaces

Let G be a connected compact Lie group and σ an involutive automorphism on G. Fix a closed-open subgroup K of $G^\sigma := \{g \in G \mid \sigma(g) = g\}$. Then the compact homogeneous space $M := G/K$ has a structure of Riemannian symmetric space.

For the case where M is of rank one, Delsarte theory for finite subsets X of M shows that the spherical Fourier transform on M gives a correspondence between a certain geometric data of $X \subset M$ and a certain analytic data of $X \subset M$. The survey of Delsarte theory for finite subset of rank one compact symmetric space of rank one can be found in [2]. In this section, even for the case where M is of higher rank, we give a formulation of Delsarte theory for finite subsets of M.

2.1 Spherical Fourier transforms

In this subsection, we set up our notation for spherical Fourier transforms on M in a form that we shall need.

By definition, G acts on $M = G/K$ transitively. Let us consider the diagonal G-action on $M \times M$ and write $\mathcal{I}_M := (\operatorname{diag} G)\backslash(M \times M)$ for the quotient space of $M \times M$ by the diagonal G-action. The quotient map from $M \times M$ onto $\mathcal{I}_M$ will be denoted by

$$d_M : M \times M \twoheadrightarrow \mathcal{I}_M.$$

Remark 2.1. *The space $\mathcal{I}_M$ can be identified with the double coset space $K\backslash G/K$. Furthermore, let us fix a maximal totally geodesic flat submanifold T of M. Then we can define the Weyl group W acting on T and $\mathcal{I}_M$ can be identified with the quptient space $W\backslash T$. We omit the details here.*

Example 2.2. *Fix an integer n and k with $1 \leq k \leq n/2$. Let $G = U(n)$ and $K = U(k) \times U(n-k)$ and then $M := G/K$ can be identified with $\mathrm{Gr}_k(\mathbb{C}^n)$. For each $p \in \mathrm{Gr}_k(\mathbb{C}^n)$, we denote by $\mathrm{Proj}_p : \mathbb{C}^n \to \mathbb{C}^n$ the orthogonal projection from $\mathbb{C}^n$ onto p $(\subset \mathbb{C}^n)$. Furthermore, for each pair (p,q) of elements in $\mathrm{Gr}_k(\mathbb{C}^n)$, we define the endomorphism $P_{p,q}$ on $\mathbb{C}^n$ by*

$$P_{p,q} := \mathrm{Proj}_p \circ \mathrm{Proj}_q \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n).$$

Let us put $\alpha_s(P_{p,q})$ the s-th largest eigenvalue of the endomorphism $P_{p,q}$ for each $s = 1, \dots, k$. Then one can prove that $0 \leq \alpha_s(P_{p,q}) \leq 1$ for any $s = 1, \dots, k$ and we have

$$\mathcal{I}_{\mathrm{Gr}_k(\mathbb{C}^n)} \simeq \{\alpha = (\alpha_1, \dots, \alpha_k) \in [0,1]^k \mid \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k\}$$

with

$$\begin{aligned} d_{\mathrm{Gr}_k(\mathbb{C}^n)} : \mathrm{Gr}_k(\mathbb{C}^n) \times \mathrm{Gr}_k(\mathbb{C}^n) &\twoheadrightarrow \mathcal{I}_{\mathrm{Gr}_k(\mathbb{C}^n)}, \\ (p,q) &\mapsto (\alpha_1(P_{p,q}), \dots, \alpha_k(P_{p,q})) \end{aligned}$$

(See also the concept of "principal angles" explained in [1]).

In this paper, let us denote by $C^\infty(M)$ the set of all complex valued C^∞-functions on M. Then the left regular representation $L : G \to GL_{\mathbb{C}}(C^\infty(M))$ defined by

$$(L_g f)(p) := f(g^{-1}p) \quad \text{for each } f \in C^\infty(M), \text{ and } p \in M$$

gives a infinite-dimensional $\mathbb{C}$-linear representation of G. We define the set $\mathcal{J}_M$ by

$$\mathcal{J}_M := \{V \subset C^\infty(M) \mid V \text{ is an irreducible } G\text{-subrepresentation of } C^\infty(M)\}.$$

Remark 2.3. *We give some facts related to the set $\mathcal{J}_M$ below. Details can be found in [9].*

A finite-dimensional irreducible representation $\rho : G \to GL_{\mathbb{C}}(W)$ of G is said to be K-spherical if

$$W^K := \{w \in W \mid \rho(g)w = w \textit{ for any } g \in G\} \neq \{0\},$$

where W is the representation space of ρ. It is known that $\dim_{\mathbb{C}} W < \infty$ *and* $\dim_{\mathbb{C}} W^K = 1$ *for any spherical unitary irreducible representation $\rho : G \to GL_{\mathbb{C}}(W)$ since G is compact and (G, K) is a symmetric pair.*

We denote by $\widehat{G}$ the set of all equivalent classes of irreducible representations of G and put

$$\widehat{G}_K := \{[\rho] \in \widehat{G} \mid \rho \textit{ is } K\textit{-spherical}\}.$$

Then the set $\mathcal{J}_M$ can be identified with $\widehat{G}_K$. In particular, by the Peter–Weyl theorem, the left regular representation $L : G \to GL_{\mathbb{C}}(C^\infty(M))$ is multiplicity-free and the subrepresentation $\bigoplus_{V \in \mathcal{J}_M} V$ of L is dence in $C^\infty(M)$.

Example 2.4. *Let us consider the setting in Example 2.2. Then* $M \simeq \mathrm{Gr}_k(\mathbb{C}^n)$. *In this situation, one can identify $\mathcal{J}_{\mathrm{Gr}_k(\mathbb{C}^n)}$ as*

$$\mathcal{J}_{\mathrm{Gr}_k(\mathbb{C}^n)} \simeq \{\nu = (\nu_1, \dots, \nu_k) \in (\mathbb{Z}_{\geq 0})^k \mid \nu_1 \geq \nu_2 \geq \dots \geq \nu_k\}$$

such that the highest weight of the irreducible $U(n)$-representation V_ν (the subspace of $C^\infty(\mathrm{Gr}_k(\mathbb{C}^n))$ corresponding to ν) is

$$(\nu_1, \nu_2, \dots, \nu_k, 0, \dots, 0, -\nu_k, \dots, -\nu_1) \in \mathbb{Z}^n.$$

Let us fix $V \in \mathcal{J}_M$. To define the spherical Fourier transform on M, we fix our terminology of the reproducing kernel $\mathcal{K}_V$ and the spherical function $Q_V : \mathcal{I}_M \to \mathbb{C}$ of V as follows:

For each $p \in M$, we define the "delta function" δ_p^V in V by

$$\langle f, \delta_p^V \rangle = f(p) \quad \text{for each } f \in V,$$

where $\langle\ ,\ \rangle$ is the L^2-innerproduct on $C^\infty(M)$ with respect to the probability G-invariant Haar measure $\frac{1}{\mathrm{vol}(\mathrm{Gr}_k(\mathbb{C}^n))}\mu_{\mathrm{Gr}_k(\mathbb{C}^n)}$ on M. Then *the reproducing kernel* $\mathcal{K}_V : M \times M \to \mathbb{C}$ of V is defined by

$$\mathcal{K}_V : M \times M \to \mathbb{C}, \quad (p,q) \mapsto \langle \delta_p^V, \delta_q^V \rangle.$$

One can easily check that $\mathcal{K}_V$ is invariant by the diagonal G-action, that is, $\mathcal{K}_V(gp,gq) = \mathcal{K}_V(p,q)$ for any $p,q \in M$ and any $g \in G$. Therefore, $\mathcal{K}_V$ induces a function Q_V on $\mathcal{I}_M := (\mathrm{diag}\, G)\backslash(M \times M)$, that is, for each $(p,q) \in M \times M$, we put

$$Q_V(d_X(p,q)) := \mathcal{K}_V(p,q).$$

In this paper, such the function Q_V on $\mathcal{I}_M$ is called *the spherical function* for $V \in \mathcal{J}_M$.

It should be noted that $1_M := \{(p,p) \in M \times M \mid p \in M\}$ is an element of $\mathcal{I}_M := (\mathrm{diag}\, G)\backslash(M \times M)$ since G acts on M transitively, and one can easily show that $Q_V(1_M) = \dim_{\mathbb{C}} V$ for any $V \in \mathcal{J}_M$.

Throughout this paper, we use the following notation:

$$\begin{aligned}\mathbb{C}_{\mathcal{I}_M} &:= \text{ the complex vector space with its basis } \mathcal{I}_M, \\ \mathbb{C}^{\mathcal{J}_M} &:= \text{ the set of all complex functions on the set } \mathcal{J}_M.\end{aligned}$$

Let us give the definition of the spherical Fourier transform on M in a form that we shall need as follows:

Definition 2.5. *For each $\phi = (\phi_\alpha)_{\alpha\in\mathcal{I}_M} \in \mathbb{C}_{\mathcal{I}_M}$, we put*

$$\widehat{\phi}(j) := \sum_{\alpha\in\mathcal{I}_M} \phi_\alpha \overline{Q_V(\alpha)} \quad \textit{for each } V \in \mathcal{J}_M.$$

Throughout this paper,

$$\mathbb{C}_{\mathcal{I}_M} \to \mathbb{C}^{\mathcal{J}_M}, \quad \phi \mapsto \widehat{\phi}$$

is called the spherical Fourier transform on M.

Example 2.6. *As in Example* 2.2 *and* 2.4, *let us consider $M = U(n)/(U(k)\times U(n-k)) \simeq \mathrm{Gr}_k(\mathbb{C}^n)$ and then*

$$\begin{aligned}\mathcal{I}_{\mathrm{Gr}_k(\mathbb{C}^n)} &\simeq \{\alpha = (\alpha_1,\ldots,\alpha_k) \in [0,1]^k \mid \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k\}, \\ \mathcal{J}_{\mathrm{Gr}_k(\mathbb{C}^n)} &\simeq \{\nu = (\nu_1,\ldots,\nu_k) \in (\mathbb{Z}_{\geq 0})^k \mid \nu_1 \geq \nu_2 \geq \cdots \geq \nu_k\}.\end{aligned}$$

We put $\nu_l := (\underbrace{1,\dots,1}_{l},\underbrace{0,\dots,0}_{k-l}) \in \mathcal{J}_{\mathrm{Gr}_k(\mathbb{C}^n)}$ *for each* $l = 0,\dots,k$. *Then, by* [5], *the spherical function* Q_{ν_l} *on* $\mathcal{I}_{\mathrm{Gr}_k(\mathbb{C}^n)}$ *corresponding to* $\nu_l \in \mathcal{J}_{\mathrm{Gr}_k(\mathbb{C}^n)}$ *can be written as*

$$Q_{\nu_l}(\alpha) = \frac{(n-2l+1)\binom{n+1}{l}^2}{(n+1)\binom{n-k}{l}} \sum_{r=0}^{l} (-1)^{l-r} \frac{\binom{n-l+1}{r}\binom{k-r}{l-r}}{\binom{k}{r}} \sum_{1 \le i_1 < i_2 < \cdots < i_r \le k} \alpha_{i_1}\alpha_{i_2}\dots\alpha_{i_r}$$

for each $\alpha = (\alpha_1,\dots,\alpha_k) \in \mathcal{I}_{\mathrm{Gr}_k(\mathbb{C}^n)}$.

Remark 2.7. *A "spherical Fourier transform" on* M *gives an isomorphism between* $L_2(\mathcal{I}_M)$ *and* $L_2(\mathcal{J}_M)$ *for certain measures on* $\mathcal{I}_M$ *and* $\mathcal{J}_M$ *(see [9]). It should be noted that the domain* $\mathbb{C}_{\mathcal{I}_M}$ *of our transform is not a subset of* $L_2(\mathcal{I}_M)$. *We omit the details here.*

2.2 Delsarte theory

Let us fix a finite subset X of M. We shall define the vectors $\mathcal{A}_X \in \mathbb{C}_{\mathcal{I}_M}$ and $\mathcal{E}_X \in \mathbb{C}^{\mathcal{J}_M}$ reflected to a geometric property and a analytic property of $X \subset M$ as follows:

$$(\mathcal{A}_X)_\alpha := \frac{1}{(\#X)^2}\#\{(x,y) \in X \times X \mid d_M(x,y) = \alpha\} \quad \text{for each } \alpha$$

$$\mathcal{E}_X(V) := \max_{f \in V\setminus\{0\}} \frac{|\frac{1}{\#X}\sum_{x\in X} f(x)|}{\|f\|} \quad \text{for each } V \in \mathcal{J}_M$$

Theorem 2.8 (Delsarte theory for finite subsets of compact symmetric spaces). *For any finite subset* X *of* M, *the following equation holds:*

$$\widehat{\mathcal{A}_X} = |\mathcal{E}_X|^2,$$

where $\widehat{\mathcal{A}_X}$ *is the spherical Fourier transform of* $\mathcal{A}_X \in \mathbb{C}_{\mathcal{I}_M}$ *(see Definition 2.5) and* $|\mathcal{E}_X|^2 \in \mathbb{C}^{\mathcal{J}_M}$ *is defined by* $|\mathcal{E}_X|^2(V) = |\mathcal{E}_X(V)|^2$ *for each* $V \in \mathcal{J}_M$.

A proof of Theorem 2.8 will be reported elsewhere.

Remark 2.9. *By definition,* $|\mathcal{E}_X|^2$ *is non-negative function on* $\mathcal{J}_M$. *Thus by Theorem 2.8, the spherical Fourie transform* $\mathcal{A}_X$ *should be non-negative. This is the key idea of "Delsarte's linear programming method" (see [4]). We omit the details here.*

3 Main theorem as an example of Delsarte theory

In this section, we apply Theorem 2.8 for the great antipodal subset of $\mathrm{Gr}_k(\mathbb{C}^n)$.

3.1 Great antipodal subsets of compact symmetric spaces

First, we recall the definition of great antipodal subsets of compact symmetric spaces as follows.

Let M be a compact symmetric space as in Section 2. For each $p \in M$, we denote by $s_p : M \to M$ the point symmetry at p on M. A subset X of M is said to be *antipodal* if $s_x(y) = y$ for any $x, y \in X$. Any antipodal subset of M is finite and

$$\#_2 M := \sup\{\#X \mid X \text{ is an antipodal subset of } M\}$$

is also finite. An antipodal subset X of M is said to be *great* if $\#X = \#_2 M$.

Fact 3.1 (cf. Takeuchi [8], Sánchez [7] and Tanaka–Tasaki [10]). *Let us assume that $M = G/K$ is a symmetric R-space. Then the following holds:*

- *Great antipodal subsets of M are unique up to G-conjugations.*
- *$\#_2 M = \dim_{\mathbb{Z}/2\mathbb{Z}} \bigoplus_i H_i(M; \mathbb{Z}/2\mathbb{Z})$, where $H_i(M; \mathbb{Z}/2\mathbb{Z})$ is the i-th homology group of M with coefficients $\mathbb{Z}/2\mathbb{Z}$.*

Example 3.2. *Let $M = \mathrm{Gr}_k(\mathbb{C}^n)$ as in Section 1. Then $M = \mathrm{Gr}_k(\mathbb{C}^n)$ is a compact Hermitian symmetric space and hence a symmetric R-space. Therefore, by Fact 3.1, great antipodal subsets of $M = \mathrm{Gr}_k(\mathbb{C}^n)$ are unique up $U(n)$-conjugations. Concretely, as we mentioned in Section 1, $\#_2 \mathrm{Gr}_k(\mathbb{C}^n) = \binom{n}{k}$ and a great antipodal subset of $M = \mathrm{Gr}_k(\mathbb{C}^n)$ is of the form $X_{\mathcal{B}}$ for some orthonormal bases $\mathcal{B}$ of $\mathbb{C}^n$.*

3.2 Delsarte theory for the great antipodal subsets of complex Grassmannian manifolds

Fix a great antipodal subset X of $\mathrm{Gr}_k(\mathbb{C}^n)$. As we mentioned in Example 2.2, we can consider $\mathcal{I}_{\mathrm{Gr}_k(\mathbb{C}^n)}$ as

$$\{\alpha = (\alpha_1, \dots, \alpha_k) \in [0,1]^k \mid \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k\}$$

and the map $d_{\mathrm{Gr}_k(\mathbb{C}^n)}$ can be written by

$$d_{\mathrm{Gr}_k(\mathbb{C}^n)} : \mathrm{Gr}_k(\mathbb{C}^n) \times \mathrm{Gr}_k(\mathbb{C}^n) \twoheadrightarrow \mathcal{I}_{\mathrm{Gr}_k(\mathbb{C}^n)},$$
$$(p,q) \mapsto (\alpha_1(P_{p,q}), \dots, \alpha_k(P_{p,q})),$$

where $P_{p,q}$ is the composition of othrogonal projections Proj_p and Proj_q onto p and q, respectively, and $\alpha_s(P_{p,q})$ is the s-th largest eigenvalue of the endomorphism $P_{p,q} \in \mathrm{End}_{\mathbb{C}}(p)$ for each $s = 1, \dots, k$.

Then one can easily compute that for each $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathcal{I}_{\mathrm{Gr}_k(\mathbb{C}^n)}$,

$$(\mathcal{A}_X)_\alpha = \begin{cases} \dfrac{\binom{k}{s}\binom{n-k}{k-s}}{\binom{n}{k}} & \text{if } \alpha = (\underbrace{1,1,\dots,1}_{s}, \underbrace{0,\dots,0}_{k-s}) \text{ for } s = 0, \dots, k \\ 0 & \text{otherwise.} \end{cases}$$

Let us give a proof of the first half part of Theorem 1.1 as follows: As in Example 2.4, the set $\mathcal{J}_{\mathrm{Gr}_k(\mathbb{C}^n)}$ can be identified with

$$\{\nu = (\nu_1, \dots, \nu_k) \in (\mathbb{Z}_{\geq 0})^k \mid \nu_1 \geq \nu_2 \geq \cdots \geq \nu_k\}$$

and denote by V_ν the functional space on $\mathrm{Gr}_k(\mathbb{C}^n)$ corresponding to $\nu = (\nu_1, \dots, \nu_k) \in \mathcal{J}_{\mathrm{Gr}_k(\mathbb{C}^n)}$. Let us put

$$\nu_l := (\underbrace{1,\dots,1}_{l}, \underbrace{0,\dots,0}_{k-l}) \in \mathcal{J}_{\mathrm{Gr}_k(\mathbb{C}^n)}$$

for each $l = 0, \dots, 1$. Since by Theorem 2.8 and Example 2.6, one can compute that

$$\begin{aligned}
&|\mathcal{E}_X(V_{\nu_l})|^2 = \widehat{A_X}(V_{\nu_l}) \\
&= \sum_{\alpha \in \mathcal{I}_{\mathrm{Gr}_k(\mathbb{C}^n)}} Q_{\nu_l}(\alpha)(A_X)_\alpha \\
&= \frac{(n-2l+1)\binom{n+1}{l}^2}{(n+1)\binom{n-k}{l}} \sum_{r=0}^{l} (-1)^{l-r} \frac{\binom{n-l+1}{r}\binom{k-r}{l-r}}{\binom{k}{r}} \sum_{\alpha \in \mathcal{I}_{\mathrm{Gr}_k(\mathbb{C}^n)}} \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq k} \alpha_{i_1}\alpha_{i_2}\dots\alpha_{i_r} \cdot (A_X)_\alpha \\
&= \frac{(n-2l+1)\binom{n+1}{l}^2}{(n+1)\binom{n-k}{l}} \sum_{r=0}^{l} (-1)^{l-r} \frac{\binom{n-l+1}{r}\binom{k-r}{l-r}}{\binom{k}{r}} \sum_{s=0}^{k} \binom{s}{r} \frac{\binom{k}{s}\binom{n-k}{k-s}}{\binom{n}{k}} \\
&= 0
\end{aligned}$$

for each $l = 1, \ldots, 1$. (To prove the equation above, we need some formulas for binomial coefficients. We omit the details here.) This implies that $\sum_{x \in X} f(x) = 0$ for any $l = 1, \ldots, k$ and $f \in V_{\nu_l}$.

Recall that, as a representaion of $U(n)$, the functional space $\mathcal{H}$ defined in Section 1 can be decomposed as $\mathcal{H} = \bigoplus_{l=0}^{k} V_{\nu_l}$. Here, V_{ν_0} is the set of all constants on $\mathrm{Gr}_k(\mathbb{C}^n)$ and $V_{\nu_l} \perp V_{\nu_0}$ for $l = 1, \ldots, k$, and thus

$$\int_{\mathrm{Gr}_k(\mathbb{C}^n)} f d\mu_{\mathrm{Gr}_k(\mathbb{C}^n)} = 0$$

for any $f \in V_{\nu_l}$ for $l = 1, \ldots, k$. Therefore, the equation (1) in Theorem 1.1 holds. We omit the details of the last part of Theorem 1.1 (see also [6]). The details will be reported elsewhere.

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