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On Coxeter elements whose powers afford the longest word

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概要

The purpose of this note is to present a condition for the power of a Coxeter element of $\mathfrak{S}_{n+1}$ to become the longest word. To be precise, given a product C of n distinct adjacent transpositions of $\mathfrak{S}_{n+1}$ in any order, we describe a condition for C such that the $(n+1)/2$-th power $C^{(n+1)/2}$ of C becomes the longest element, in terms of the Amida diagrams.

1 Introduction

1.1 Coxeter elements

Consider a Coxeter system (W, S) of type A which is defined by the n generators:

$$W = \langle S\rangle, \quad S = \{s_1, s_2, \dots, s_n\}$$

and the relations:

$$\begin{aligned} s_i^2 &= 1 && (1 \le i \le n),\\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && (1 \le i \le n-1),\\ s_i s_j &= s_j s_i && (1 \le i, j \le n,\ |i-j| \ge 2). \end{aligned}$$

As is well known, this group is the same as the symmetric group $\mathfrak{S}_{n+1}$.

A Coxeter element C of (W, S) is a product of the n generators in any order.

Definition 1.

$$C = s_{i_1} s_{i_2} \cdots s_{i_n} \quad \{i_1, i_2, \dots, i_n\} = \{1, 2, \dots, n\}.$$

Of course C depends on the sequence of s_is. However we have the following:

REMARK 1. Coxeter elements are conjugate each other.

Since the symmetric group $\mathfrak{S}_{n+1}$ is finite, the order of C is also finite. Let H be the order of C.

Definition 2.

$$H = |\langle C \rangle| \quad (= \min e > 0 \text{ s.t. } C^e = 1).$$

Since Coxeter elements are conjugate, H does not depend on the choice of C. This H is called the *Coxeter number.* Usually to denote the Coxeter number, we use lower-case letter h. However the lower-case letter h will be used in another situation, so in this note we adopt the capital letter H.

For Coxeter system of type A, we know the following:

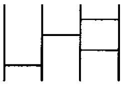
REMARK 2.

$$H = n + 1 \quad (\text{in case } W = \mathfrak{S}_{n+1}).$$

Although, Remark 1 and 2 are well known facts (see for example, [1, 3]), we gave an elementary proof for these facts in the paper [4] where we utilized Amida diagrams, which is defined in the following subsection.

1.2 Amida diagrams

The elements of the symmetric group are conveniently illustrated by *Amida diagrams.* Throughout this note, we will describe elements of $\mathfrak{S}_{n+1}$ drawing the following pictures called the Amida (Ghost legs) diagrams.



(1)

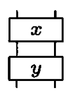
An *Amida diagram* consists of $n + 1$ vertical lines and horizontal segments placed between adjacent vertical lines like ladders so that the end points of each horizontal segment meet the vertical lines and so that they do not meet any other end points of the horizontal segments.

The $n+1$ runners who start at the bottoms of the vertical lines go up along the lines. If they find horizontal segments on their right [resp. left], they turn right [resp. left] and go along the segments. They necessarily meet the adjacent vertical lines. Then again they go up the vertical lines and iterate this trip until they arrive at the tops of the vertical lines. If the i-th runner arrives at the σ_i-th position $(i = 1, 2, \ldots, n+1)$, we consider the Amida diagram as one of the expressions of

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & i & \cdots & n+1 \\ \sigma_1 & \sigma_2 & \cdots & \sigma_i & \cdots & \sigma_{n+1} \end{pmatrix} \in \mathfrak{S}_{n+1}.$$

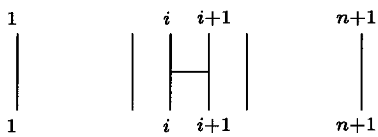
For example the Amida diagram as in (1) corresponds to $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$.

We can also consider the product of Amida diagrams. If x and y are Amida diagrams of $\mathfrak{S}_{n+1}$ then the product xy is defined to be an Amida diagram obtained from x and y by putting the former on the latter.



$= \quad xy$

A generator $s_i \in \mathfrak{S}_{n+1}$ corresponds to an Amida diagram which consists of $n+1$ vertical lines with only one horizontal segment between the i-th and the $(i+1)$-st vertical lines.



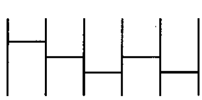
Since $s_1, s_2, \ldots, s_n$ generate $\mathfrak{S}_{n+1}$, any word in $\mathfrak{S}_{n+1}$ can be expressed as an Amida diagram. Conversely, we can read a word of the symmetric group from an Amida diagram. For example (1) denotes $s_3 s_2 s_3 s_1$. Of course we can read it as $s_3 s_2 s_1 s_3$.

$$(1) = s_3 s_2 s_3 s_1 = s_3 s_2 s_1 s_3$$

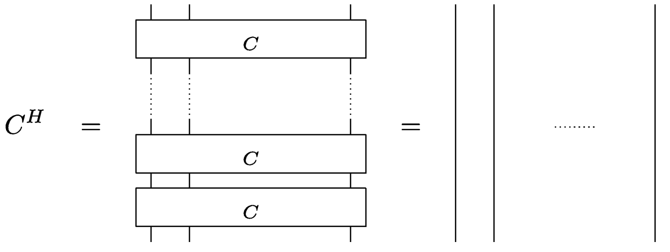
For the time being, this ambiguity is not necessary to worry about. But it will matter later.

Let us rephrase a Coxeter element and Coxeter number in terms of Amida diagrams. A Coxeter element C is an Amida diagram with n-segments each of which is placed between every adjacent pair of vertical lines one by one.

$C \quad =$



The Coxeter number H of C is the smallest positive integer such that the accumulation of H copies of C makes the identity.



In case of the symmetric group $\mathfrak{S}_{n+1}$, the Coxeter number is $n+1$ (Remark 2). This implies that the number of the horisontal segments in the Amida diagram of C^H is $n(n+1)$ and it works as the identity map. Let us pay attention to this figure:

$$n(n+1).$$

If we divide this by 2, then this number coincides with the length of the longest word w_0 in $\mathfrak{S}_{n+1}$.

$$\ell(w_0) = n(n+1)/2 \quad w_0 \in \mathfrak{S}_{n+1}.$$

So we may expect that the lower half of the Amida diagram of C^H affords the longest word, for any Coxeter element C.

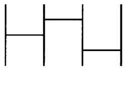
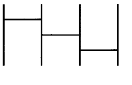
Conjecture 1. *The lower half of C^H affords the longest word.*

Unfortunately, this is wrong.

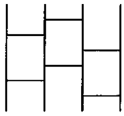
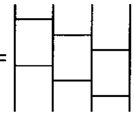
Let us check this by some examples. Note that the longest word w_0 maps the sequence $1, 2, \ldots, n, n+1$ to the inverse sequence $n+1, n, \ldots, 2, 1$:

$$w_0 = \begin{pmatrix} 1 & 2 & \cdots & i & \cdots & n & n+1 \\ n+1 & n & \cdots & n+2-i & \cdots & 2 & 1 \end{pmatrix}.$$

Consider the case $n = 3$ and accordingly $H = 4$. In this case, there are essentially two Coxeter elements.

$s_2 s_1 s_3 =$ , $s_1 s_2 s_3 =$ 

The lower half of C^4 is C^2. The double of the left diagram affords the longest word, while the right one does not afford the longest word.

$(s_2s_1s_3)^2 =$  $= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = w_0$, $\quad (s_1s_2s_3)^2 =$  $\neq w_0$

So we divide all Coxeter elements into two classes:

Definition 3. One is the class of

Good Coxeter elements

which afford the longest word by taking the lower half of C^H.

The other is the class of

Bad Coxeter elements

which do not afford the longest word.

Our aim of this note is to characterize the good Coxeter elements. First consider the case n is odd.

2 Good Coxeter element (n odd, $H = n+1$ even)

In this case $H/2$ is an integer. So the lower half of C^H is literally $C^{H/2}$. Suppose that C is a good Coxeter element. Then $C^{H/2}$ affords the longest word. Then we have this:

$$C^{H/2} = w_0 = \begin{pmatrix} 1 & 2 & \cdots & i & \cdots & n & n+1 \\ n+1 & n & \cdots & n+2-i & \cdots & 2 & 1 \end{pmatrix}.$$

The number i will be mapped to $n+2-i$:

$$C^{H/2}(i) = n+2-i, \quad i = 1, 2, \ldots, n+1.$$

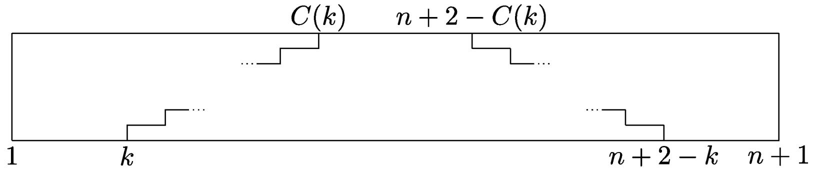
Setting i to $C(k)$ we have the following equation.

$$C^{H/2}(C(k)) = C(C^{H/2}(k)) = C(n+2-k) = n+2-C(k).$$

Namely, $n+2-k$ is mapped to $n+2-C(k)$.

$$\begin{cases} k \mapsto C(k) \\ n+2-k \mapsto n+2-C(k) \quad k = 1, 2, \ldots, n+1 \end{cases}$$

This implies that in the Amida diagram of C, a runner who starts at the k-th position from the left arrives at the $C(k)$-th position from the left, while a runner who starts at the k-th position from the right arrives at the $C(k)$-th position from the right.



Since C is a Coxeter element, every runner goes his way rightward or leftward only. This means the Amida diagram of a good Coxeter element is symmetric with respect to the vertical axis.

Theorm 1. *If n:odd, $H = n+1$:even, $C \in \mathfrak{S}_{n+1}$. Then*

C is a good Coxeter element
$\Leftrightarrow$ the Amida diagram of C is symmetric w.r.t. the vertical axis.

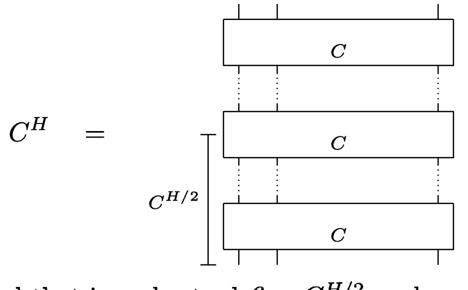
Next consider the case n is even and accordingly H is odd.

3 Good Coxeter element (n even, $H = n+1$ odd)

In this case $H/2$ is a half integer, so first we have to define $C^{H/2}$.

3.1 Definition of $C^{H/2}$ for odd H

Consider the Amida diagram of C^H again. Since $H/2$ is a half integer, we have the following diagram.



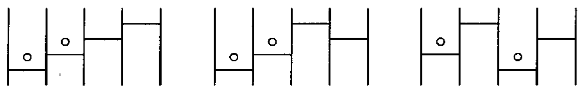
By this diagram, we find that in order to define $C^{H/2}$ we have to divide C itself into the lower half and the upper half. So C should be written as w_2 times w_1.

$$C = w_2 w_1 \quad (\ell(w_2) = \ell(w_1) = n/2)$$

Then $C^{H/2}$ may be defined as follows:

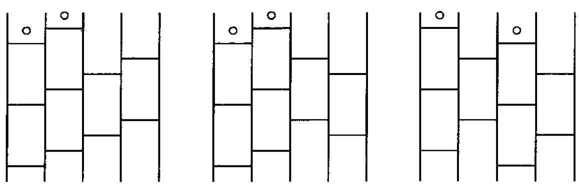
$$C^{H/2} = w_1 C^{[H/2]} = w_1(w_2w_1)(w_2w_1)\cdots(w_2w_1). \qquad (2)$$

Consider the case $n = 4$ accordingly $H = 5$ and $H/2 = 2.5$. In this case there are essentially three Coxeter elements.



(3)

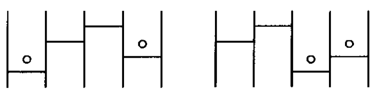
We take the marked horizontal segments as the lower halves. Obeying the expression (2), we consider the 2.5-th powers of these elements.



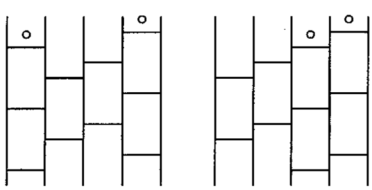
(4)

Then the middle one and the right one make the longest word. So we can say the middle one and the right one are good Coxeter elements. But the left one does not make the longest word.

Suppose that we take the lower halves of these good Good Coxeter elements in (3) the other way as follows.



Then the 2.5-th powers of them do not become the longest word.



This implies that the definition of $C^{H/2}$ depends on the choice of w_1. Accordingly, whether $C^{H/2}$ affords the longest word or not also depends on the choice of w_1. So rigorously, $C^{H/2}$ should have a subscript w_1.

$$C^{H/2}_{w_1} = w_1 C^{[H/2]} = w_1(w_2w_1)\cdots(w_2w_1) \qquad (5)$$

If we take these situations into consideration, it would be appropriate to define good Coxeter elements as follows.

Definition 4.

Good Coxeter elements

If we can properly choose w_1, $C_{w_1}^{H/2}$ in (5) afford the longest word.

Bad Coxeter elements

$C_{w_1}^{H/2}$ in (5) never afford the longest word for any possible choice of w_1.

Under this definition we characterize the good Coxeter elements. Before that, we introduce the notion of heights.

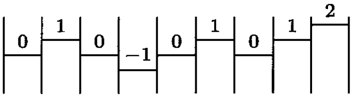
3.2 Height of C

Let us get back to the case $n = 4$. Consider why the left Coxeter element in (3) does not afford the longest word while the middle one and the right one do.

On the left diagram in (4), the left most runner arrives at the right most position "too early". He can arrive at the right most position tracking only one Coxeter element. After that he is forced to turn back leftward. On the other hand, on the middle diagram in (4), the left most runner needs two copies of the Coxeter element to reach at the right most position.

In other words, if there exists a "steep staircase" in the Amida diagram of C, it cannot become a good Coxeter element. That's the outline of the idea.

To measure the "total gradient" of Coxeter elements, we define the "heights" of Coxeter elements.



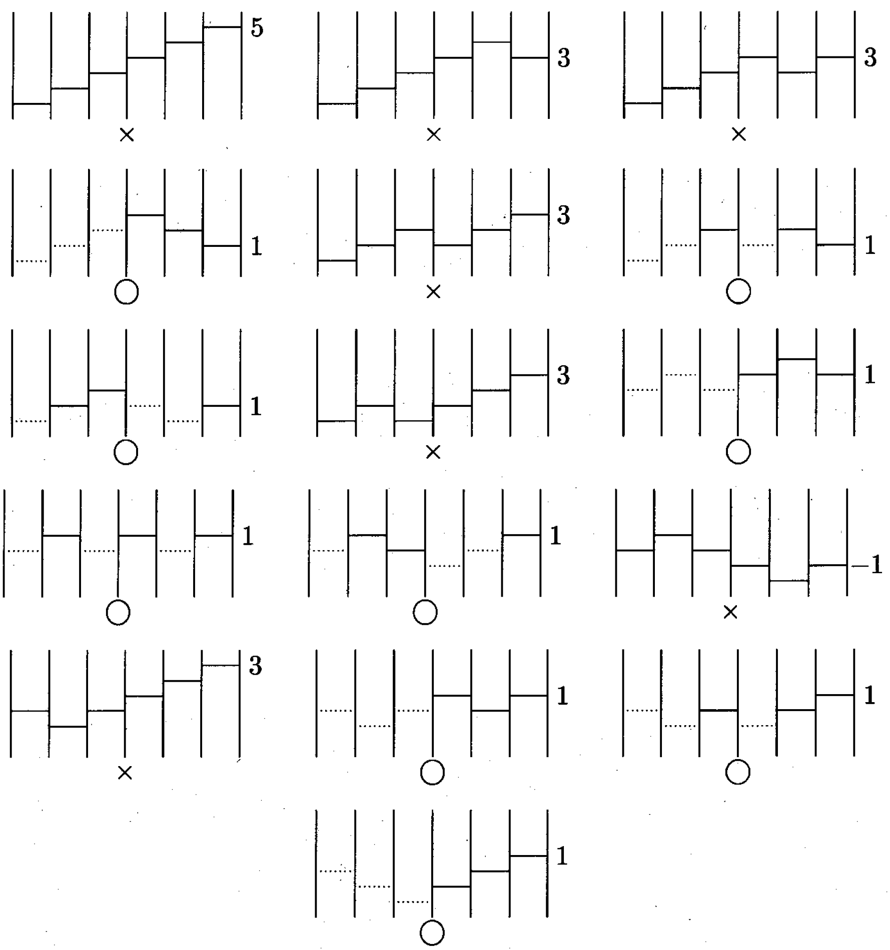
We label each segment of a Coxeter element with its height. Define the height of the left most segment to be 0. To draw an Amida diagram of a Coxeter element from left to right, we attach a segment to the lower or higher position to the previous segment. Assume that the current segment has its height s. If the next segment is placed on higher position than the current segment, then we label it with $s + 1$. If the new segment is placed on lower position than the previous one, then we label it with $s - 1$.

We define the height of C to be the height of the right most segment and denote it by $h(C)$.

Definition 5.

$$h(C) = \text{height of } C = \text{height of the right most segment.}$$

The following are Coxeter elements in $\mathfrak{S}_7$. We put their heights to the right of the diagrams. Also we put ○ signs if they are good ones, while we put $\times$ signs if they are bad ones. (Furter dashed segments are properly chosed lower halves.)



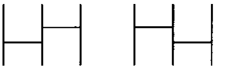
From the above diagrams we find that each good Coxeter element has its heights 1 or -1. In fact the heights provide a necessary condition for good Coxeter elements.

If C is a good Coxeter element, then $h(C) = \pm 1$.

However this is not a sufficient condition. (See the right most diagram in the 3-rd row.) Some extra conditions will be needed. To describe the extra conditions, we observe how a Coxeter element is made from a small Coxeter element.

3.3 The basic operations

In case $n = 2$, there are two Coxeter elements.

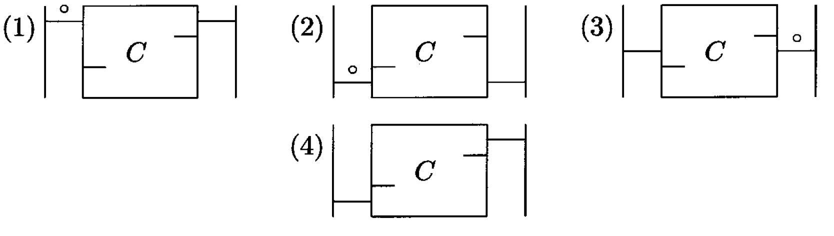


(6)

Both are good ones. In order to obtain a Coxeter element for other even numbers, the following observation will do the trick.

A Coxeter element $\tilde{C} \in \mathfrak{S}_{n+3}$ is obtained from a Coxeter element $C \in \mathfrak{S}_{n+1}$ by adding two segments to the BOTH sides of C one by one.

There are 4 possibilities. We call them the basic operations.



Basic operations

(1) Add the two segments to the upper positions of C.

(2) Add the two segments to the lower positions of C.

(3) Add one segment to the upper and the other segment to the lower so that $|h(\tilde{C})| \leq |h(C)|$

(4) Add one segment to the upper and the other segment to the lower so that $|h(\tilde{C})| > |h(C)|$

Operations (1) and (2) do not change their heights. If C is a good Coxeter element of height 1, Operation (3) changes the heights from $+1$ to -1 and operation (4) raises the heights from +1 to +3. This operation necessarily makes a bad Coxeter element. Hence we call the operations (1)-(3) good operations and the operation (4) bad operation.

Now we state the conclusion.

3.4 Conclusion

Theorm 2. *A good Coxeter element in $\mathfrak{S}_{n+1}$ is obtained from either of the Coxeter element of $\mathfrak{S}_3$ in (6) by iterating one of the good operations only.*

On the other hand we have

Theorm 3. *If $C \in \mathfrak{S}_{n+1}$ is a bad Coxeter element, then $\tilde{C} \in \mathfrak{S}_{n+3}$ obtained from C by any one of the basic operations is a bad Coxeter element.*

In other words, once a Coxeter element is extended by the operation (4), it has no chance to get back to a good one.

3.5 How to choose the lower halves

The good operations also provide us how we divide C into w_1 and w_2. To make the word w_1, we have only to choose the lower segment as a letter of w_1 in each good operation. These segments are marked in the figures which illustrate the Basic operations. Some suspicious people might think that other choice of w_1 can also make a good Coxeter element. But this choice is the only way to divide w_1 and w_2 properly. In fact the following remark holds.

REMARK 3. Let $C = w_2 w_1 \in \mathfrak{S}_{2m+1}$ be a Coxeter element in $\mathfrak{S}_{2m+1}$ such that $\ell(w_1) = \ell(w_2) = m$. If $C_{w_1}^{H/2}$ in the expression (5) affords the longest word in $\mathfrak{S}_{2m+1}$, then such w_1 is uniquely determined.

PROOF. Assume that C is a good Coxeter element and that $w_2 w_1$ and $w_2' w_1'$ are both expressions of C. Then both $C_{w_1}^{H/2} = w_1 C^m$ and $C_{w_1'}^{H/2} = w_1' C^m$ are the longest word in $\mathfrak{S}_{2m+1}$. Since the longest word is unique, they coincide. Hence we have $w_1 = w_1'$. □

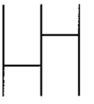
Hence if C is a good Coxeter element, we can drop the subscript w_1 in $C_{w_1}^{H/2}$

4 Example

Finally we see a bad operation makes a bad Coxeter element and good operations make a good Coxeter element.

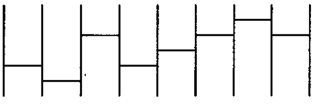
4.1 Example of bad Coxeter element

We begin with this diagram.



First apply the good operation (1). Then apply the bad operations (4) and (3).

We obtain the following diagram.



This diagram corresponds to the following single $(n+1)$-cycle[*1] :

$$C = (124897653).$$

The 4.5-th power of this element is

$$w_1 C^4 = w_1(124897653)^4 = w_1(193854627).$$

We express C^4 in two line form:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 7 & 8 & 6 & 4 & 2 & 1 & 5 & 3 \end{pmatrix}.$$

There are many candidate for w_1 which satisfies these equations:

$$C = w_1 w_2 \quad \text{and} \quad \ell(w_1) = 4.$$

Note that w_1 consists of 4 segments. We may check that 4.5-th power of C does not make the longest word for each possible choice of w_1, one by one. But here we take

*1 Not only in this case, all Coxeter elements are expressed by a single $(n+1)$-cycle. In particular, all Coxeter elements are conjugate and their orders are $n+1$. This proves Remark 1 and 2.

another strategy. We measure how far from the longest word this element is. To do this, here we count the inversion number.

Imagine there is two line form of a permutation. If each number in the second row can see only smaller numbers in his left, it is the identity permutation. If a larger number is sitting in his left, the number of those members is the contribution to the inversion number. In this example, 7 in the second row sees larger number 9 in his left. So the contribution to the inversion number is 1. 8 also sees larger number 9 in his left. So the contribution is also 1. 6 sees larger numbers 9 7 8 in his left. So the contribution is 3. Similarly the remaining numbers have the contributions, 4,5,6,4,6. The sum of these numbers is $1 + 1 + 3 + 4 + 5 + 6 + 4 + 6 = 30$. Hence the total inversion number of this element is equal to 30:

$$\text{inversion number of } C^4 = 30.$$

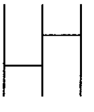
On the other hand the length of the longest word w_0 in $\mathfrak{S}_9$ is $(9 \times 8)/2$, which is equal to 36:

$$\ell(w_0) = (9 \times 8)/2 = 36.$$

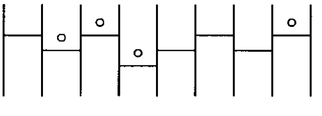
This implies addition of 4 extra segments does not realize the longest word. Hence this Coxeter element is bad one.

We again begin with the following diagram.

4.2 Example of good Coxeter element

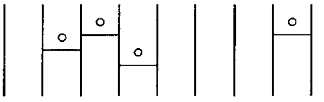


Applying the operations (1)(2) and (3), we have the following diagram.



In operation (1), the segment added to the left has height $+1$, while the segment added to the right has height $+2$. So the left one becomes a letter of w_1, while the right one becomes a letter of w_2. Similarly, in operation (2), the left one has the height -1 and the right one has the height 0. So the left one becomes a letter of w_1.

In operation (3), the left one has the height +1, while the right one has the height 0. Hence the right one becomes a letter of w_1. The product of these letters certainly makes the lower half of C.



In order to check that this Coxeter element is good one, we calculate $C^{H/2}$. Since $n = 8$ accordingly $H = 9$ and $H/2 = 4.5$, $C^{H/2} = C^{4.5} = w_1 C^4$. We can obtain the cyclic forms of C and w_1 from their Amida diagrams.

$$C = (124789653), \quad w_1 = (2453)(89).$$

Since $C^4 = (124789653)^4 = (183754629)$, we have

$$\begin{aligned} C^{H/2} = C^{4.5} = w_1 C^4 &= (2453)(89)(183754629) \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix} \end{aligned}$$

So C is good Coxeter element.

参考文献

[1] N. Bourbaki, Groupes et algèbres de Lie IV, V, VI, Hermann, Paris, 1968.

[2] C. Ceballos, J. -P. Labbé and C. Stump, "Subword complexes, cluster complexes, and generalized multiassociahedra", J. Algebraic Combin. **39** (2014), 17–51.

[3] J. E. Humphreys, Reflection Groups and Coxeter Groups Cambridge University Press, 1990

[4] M. Kosuda "Coxeter elements of the symmetric groups whose powers afford the longest elements" to appear in Tokyo J. Math (2017).