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Generalized Hardy spaces based on Banach function spaces

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Abstract

In this note, we recall a result in [11] together with some examples.

1 Example of function spaces

The following function spaces are fundamental in harmonic analysis and we want to understarnd them in a unified manner. Here are some example of function spaces that we envisage.

Lebesgue spaces for $0 < p \leq \infty$ One of our staring points is the Banach space $L^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$. Although it is not a Banach space, we can define $L^p(\mathbb{R}^n)$ for $0 < p < 1$.

Weighted Lebesgue spaces By a weight we mean a measurable function which satisfies $0 < w(x) < \infty$ for almost all $x \in \mathbb{R}^n$. Let $0 < p < \infty$ and w be a weight. One defines

$$\|f\|_{L^p(w)} \equiv \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx\right)^{\frac{1}{p}}.$$

Morrey spaces Let $0 < q \leq p < \infty$. Define the *Morrey norm* $\| \star \|_{\mathcal{M}^p_q}$ by

$$\|f\|_{\mathcal{M}^p_q} \equiv \sup\left\{ |B|^{\frac{1}{p}-\frac{1}{q}} \left(\int_B |f(x)|^q\, dx\right)^{\frac{1}{q}} \,:\, B \text{ is a ball in } \mathbb{R}^n \right\} \tag{1.1}$$

for a measurable function f. The *Morrey space* $\mathcal{M}^p_q(\mathbb{R}^n)$ is the set of all measurable functions f for which $\|f\|_{\mathcal{M}^p_q}$ is finite. Among of them, the author would like to understand the property of Morrey spaces. Morrey spaces grasp more than $L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ in general; see [10].

If we start Morrey spaces, we are led to Hardy-Morrey spaces. See [19]

Homogeneous and non-homogeneous Herz spaces Write $Q_0 \equiv [-1,1]^n$ and $C_j \equiv [-2^j, 2^j]^n \setminus [-2^{j-1}, 2^{j-1}]^n$ for $j \in \mathbb{Z}$.

Let $0 < p, q \le \infty$ and $\alpha \in \mathbb{R}$. The *non-homogeneous Herz space* $K^\alpha_{pq}(\mathbb{R}^n)$ is the set of all measurable functions f for which the norm

$$\|f\|_{K^\alpha_{pq}} \equiv \|\chi_{Q_0} \cdot f\|_p + \left(\sum_{j=1}^{\infty} (2^{j\alpha}\|\chi_{C_j} \cdot f\|_p)^q\right)^{\frac{1}{q}}$$

is finite. The *homogeneous Herz space* $\dot{K}^\alpha_{pq}(\mathbb{R}^n)$ is the set of all measurable functions f for which the norm $\|f\|_{\dot{K}^\alpha_{pq}} \equiv \left(\sum_{j=-\infty}^{\infty} (2^{j\alpha}\|\chi_{C_j} \cdot f\|_p)^q\right)^{\frac{1}{q}}$ is finite.

Hardy-Herz spaces, made from Herz spaces, are studied in [13, 17].

Orlicz spaces, Musielak-Orlicz spaces Although we do not define these spaces, we remark that

2 General function spaces

Let $L^0(\mathbb{R}^n)$ be the space of all measurable functions defined on $\mathbb{R}^n$.

Definition 2.1. A linear space $X = X(\mathbb{R}^n) \subset L^0(\mathbb{R}^n)$ is said to be a quasi-Banach function space if X is equipped with a functional $\|\cdot\|_X : L^0(\mathbb{R}^n) \to [0,\infty]$ enjoying the following properties:
Let f, g, $f_j \in L^0(\mathbb{R}^n)$ $(j = 1, 2, \ldots)$ and $\lambda \in \mathbb{C}$.

(1) $f \in X$ holds if and only if $\|f\|_X < \infty$.

(2) (Norm property):

- (A1) (Positivity): $\|f\|_X \ge 0$.
- (A2) (Strict positivity) $\|f\|_X = 0$ if and only if $f = 0$ a.e..
- (B) (Homogeneity): $\|\lambda f\|_X = |\lambda| \cdot \|f\|_X$.
- (C) (Triangle inequality): For some $\alpha \ge 1$, $\|f + g\|_X \le \alpha(\|f\|_X + \|g\|_X)$.

(3) (Symmetry): $\|f\|_X = \| \, |f| \, \|_X$.

(4) (Lattice property): If $0 \le g \le f$ a.e., then $\|g\|_X \le \|f\|_X$.

(5) (Fatou property): If $0 \le f_1 \le f_2 \le \cdots$ and $\lim_{j\to\infty} f_j = f$, then $\lim_{j\to\infty} \|f_j\|_X = \|f\|_X$.

(6) For all measurable sets E with $|E| < \infty$, we have $\|\chi_E\|_X < \infty$.

This framework is not enough in view of the following facts:

Remark 2.2. If $\alpha = 1$ and the following condition:

For all measurable sets E with $|E| < \infty$ and $f \in X$

holds, we have $f \cdot \chi_E \in L^1(\mathbb{R}^n)$; holds, then X is said to be a Banach function space; see [2]. Note that, we do not postulate this condition in the definition of quasi-Banach function space. As we have seen in [15], the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ with $1 \le q < p < \infty$ violates this additional condition.

For this reason, we need to introduce the following notion:

Definition 2.3. A linear space $X \subset L^0(\mathbb{R}^n)$ is said to be a ball quasi-Banach function space if X is equipped with a functional $\| \cdot \|_X : L^0(\mathbb{R}^n) \to [0,\infty]$ enjoying (1)–(5) as well as the following properties (6) and (7):
Let f, g, $f_j \in L^0(\mathbb{R}^n)$ $(j = 1, 2, \ldots)$ and $\lambda \in \mathbb{C}$.

(6) For all balls B, we have $\|\chi_B\|_X < \infty$.

(7) For all balls B and $f \in X$, we have $f \cdot \chi_B \in L^1(\mathbb{R}^n)$.

If $\alpha = 1$, then X is said to be a ball Banach function space.

We recall the notion of the *Köthe dual* of a ball Banach function space X. If $\| \cdot \|_X$ is a ball function norm, its *associate norm* $\| \cdot \|_{X'}$ is defined on $L^0(\mathbb{R}^n)$ by

$$\|g\|_{X'} \equiv \sup\left\{\|f \cdot g\|_{L^1} \, : \, f \in L^0(\mathbb{R}^n), \|f\|_X \le 1\right\}, \quad (g \in L^0(\mathbb{R}^n)). \tag{2.1}$$

The space X' collects all measurable functions $f \in L^0(\mathbb{R}^n)$ for which the quantity $\|f\|_{X'}$ is finite. The space X' is called the *Köthe dual* of X or the *associated space* of X.

3 Hardy spaces

3.1 Classical definitions

Suppose that $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfies the non-degenerate condition

$$\int_{\mathbb{R}^n} \psi(x)\,dx \neq 0.$$

Using this function, the Hardy norm is defined by;

$$\|f\|_{H^p}^{\psi} \equiv \left\| \sup_{j \in \mathbb{Z}} |\psi^j * f| \right\|_p, \; 0 < p < \infty, \quad f \in \mathcal{S}'(\mathbb{R}^n) \tag{3.1}$$

for $f \in \mathcal{S}'(\mathbb{R}^n)$. Here

$$\psi^j \equiv 2^{jn}\psi(2^j \cdot)$$

for $j \in \mathbb{Z}$. The space $H^p(\mathbb{R}^n)$ is defined uniquely despite the ambiguity of the choice of ψ. This fact justifies that we can omit ψ in the notation $\| \cdot \|_{H^p}^{\psi}$.

We have a couple of motivations of investigating Hardy spaces.

- The singular integral operators, which are represented by the j-th Riesz transform given by

  $$R_j f(x) \equiv \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\,dy, \tag{3.2}$$

  are integral operators with singularity (mainly at the origin). The boundedness of such operators can be characterized by Hardy spaces. For example, let $f \in L^1(\mathbb{R}^n)$. Then the estimate

  $$\|f\|_{L^1} + \sum_{j=1}^{n} \|R_j f\|_{L^1} < \infty \tag{3.3}$$

  holds if and only if $f \in H^1(\mathbb{R}^n)$. The Hardy space $H^p(\mathbb{R}^n)$ with $0 < p < 1$ also characterizes $L^p(\mathbb{R}^n)$. But the matters are subtler. So we do not go into the details.

- Let $1 < p < \infty$. The Hardy space $H^p(\mathbb{R}^n)$ is isomorphic to $L^p(\mathbb{R}^n)$, so that we have a different expression of $L^p(\mathbb{R}^n)$, which in turm yields the decomposition results for the Lebesgue space $L^p(\mathbb{R}^n)$, for example.

- A spirit similar to above is that the Hardy space $H^p(\mathbb{R}^n)$ and the Triebel-Lizorkin space $\dot{F}^0_{p2}(\mathbb{R}^n)$ are isomorphic for $0 < p < \infty$. So the Hardy space $H^p(\mathbb{R}^n)$ can play the model role of Triebel-Lizorkin spaces.

- An experience shows that many other operators can be bounded from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ but are not bounded on $L^p(\mathbb{R}^n)$.

3.2 Our main results

Let us go back to our fundamental setting: Let X be a quasi-ball Banach function space. So we are going to define $HX(\mathbb{R}^n)$. Let ψ be a function non-degenerate in the above sense. We want to define

$$\|f\|_{HX}^{\psi} \equiv \left\| \sup_{j\in\mathbb{Z}} |\psi^j * f| \right\|_X, \; 0 < p < \infty, \quad f \in \mathcal{S}'(\mathbb{R}^n) \tag{3.4}$$

for $f \in \mathcal{S}'(\mathbb{R}^n)$. Once we can show that different choices of admissible ψ yield equivalent norms, we can define $HX(\mathbb{R}^n)$ to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $\|f\|_{HX}^{\psi}$ is defined.

The powered Hardy-Littlewood maximal operator $M^{(\eta)}$ is defined by:

$$M^{(\eta)} f \equiv [M[\,|f|^{\eta}\,]]^{\frac{1}{\eta}}, \quad \eta > 0. \tag{3.5}$$

We write $M = M^{(1)}$. In [11], we proposed that the following condition to develop the theory of $HX(\mathbb{R}^n)$:

$$\left\| \left(\sum_{j=1}^{\infty} M^{(\eta)} f_j{}^Q \right)^{\frac{1}{Q}} \right\|_X \lesssim \left\| \left(\sum_{j=1}^{\infty} |f_j|^Q \right)^{\frac{1}{Q}} \right\|_X . \tag{3.6}$$

We formulate the atomic decomposition, the main result in this paper after giving the definition.

Definition 3.1. Let X be a ball quasi-Banach function space and $q \in [1, \infty]$. Assume that $d \in \mathbb{Z}_+$ satisfies $d \ge d_X$. Then the function a is called an (X, q, d)-*atom* if there exists $Q \in \mathcal{Q}$ such that $\mathrm{supp}(a) \subset Q$,

$$\|a\|_{L^q} \le \frac{|Q|^{\frac{1}{q}}}{\|\chi_Q\|_X}, \int_{\mathbb{R}^n} x^{\alpha} a(x)\, dx = 0 \tag{3.7}$$

as long as $|\alpha| \le d$.

We also let $d_X = \left[\frac{n}{\eta} - n \right]$.

For $a > 0$, we define $\|f\|_{X^a} = [\| |f|^a \|_X]^{1/a}$ for a measurable function f, so that X^a is a ball quasi-Banach space.

Theorem 3.2 (Reconstruction, [11]). *Let $s \in (0, 1]$, $q \in (1, \infty]$ and d_X be as above. Assume that X is a ball quasi-Banach function space such that the Köthe dual of $X^{1/s}$ is isomorphic to a Banach function space Y such that*

$$\|M^{((q/Q)')} f\|_Y \lesssim \|f\|_Y, \tag{3.8}$$

and, for any $f \in L^0(\mathbb{R}^n)$. *Let* $\{a_j\}_{j=1}^{\infty}$ *be a sequence of* (X, q, d_X)*-atoms, supported on the cubes* $\{Q_j\}_{j=1}^{\infty} \subset \mathcal{Q}$, *and* $\{\lambda_j\}_{j=1}^{\infty} \subset [0, \infty)$ *satisfy that*

$$\left\| \left\{ \sum_{j=1}^{\infty} \left(\frac{\lambda_j}{\|\chi_{Q_j}\|_X} \right)^s \chi_{Q_j} \right\}^{\frac{1}{s}} \right\|_X < \infty. \tag{3.9}$$

Then the series $f := \sum_{j=1}^{\infty} \lambda_j a_j$ *converges in* $\mathcal{S}'(\mathbb{R}^n)$, $f \in HX(\mathbb{R}^n)$ *and*

$$\|f\|_{HX} \lesssim \left\| \left\{ \sum_{j=1}^{\infty} \left(\frac{\lambda_j}{\|\chi_{Q_j}\|_X} \right)^s \chi_{Q_j} \right\}^{\frac{1}{s}} \right\|_X,$$

where the implicit positive constant is independent of f.

Theorem 3.3 (Decomposition, [11]). *Let* X *be a ball quasi-Banach function space satisfying* $d \geq d_X$ *be a fixed integer and* $f \in HX(\mathbb{R}^n)$. *Then there exist a sequence* $\{a_j\}_{j=1}^{\infty}$ *of* (X, ∞, d)*-atoms, supported on the cubes* $\{Q_j\}_{j=1}^{\infty} \subset \mathcal{Q}$, *and a sequence* $\{\lambda_j\}_{j=1}^{\infty} \subset [0, \infty)$ *such that* $f = \sum_{j=1}^{\infty} \lambda_j a_j$ *in* $\mathcal{S}'(\mathbb{R}^n)$ *and*

$$\left\| \left\{ \sum_{j=1}^{\infty} \left(\frac{\lambda_j}{\|\chi_{Q_j}\|_X} \right)^s \chi_{Q_j} \right\}^{\frac{1}{s}} \right\|_X \lesssim_s \|f\|_{HX},$$

where the implicit positive constant is independent of f, *but depends on* s.

3.3 A reduction from HX to X

By mimicking the proof of $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ [16], we can prove the following proposition.

Proposition 3.4. *If* X *is a ball Banach function space that admits a predual and that* $\|Mf\|_X \lesssim \|f\|_X$ *for all* $f \in X(\mathbb{R}^n)$, *then* $HX(\mathbb{R}^n) = X(\mathbb{R}^n)$ *with coincidence of norms.*

Proof. We generalized Proposition 3.4 in [11]. Here for the sake of convenience for readers we supply a proof. Let $f \in X$. Since

$$\int_{B(x,1)} |f(z)|\, dz \lesssim (1 + |x|)^n Mf(y)$$

for all $y \in B(1) = \{|z| < 1\}$, we obtain

$$\int_{B(x,1)} |f(z)|\, dz \lesssim (1 + |x|)^n \|f\|_X.$$

Thus, $f \in \mathcal{S}'(\mathbb{R}^n)$. For each $j \in \mathbb{Z}$, we have $|\psi^j * f| \lesssim Mf$; see [7, Proposition 2.7]. Thus,

$$\|f\|_{HX} = \left\| \sup_{j \in \mathbb{Z}} |\psi^j * f| \right\|_X \lesssim \|Mf\|_X \lesssim \|f\|_X.$$

Let $f \in HX(\mathbb{R}^n)$. Then $\{\psi^j * f\}_{j=1}^{\infty}$ forms a bounded set in X since $f \in HX(\mathbb{R}^n)$. Thus, if we pass to a subsequence, then $\{\psi^j * f\}_{j=1}^{\infty}$ converges to $g \in X(\mathbb{R}^n)$ in the weak-* topology of $X(\mathbb{R}^n)$ thanks to the Banach-Alaoglu theorem. Meanwhile, $\{\psi^j * f\}_{j=1}^{\infty}$ converges to f in $\mathcal{S}'(\mathbb{R}^n)$. Since $HX(\mathbb{R}^n)$ is embedded into $\mathcal{S}'(\mathbb{R}^n)$, $f = g$. Thus, $f \in X(\mathbb{R}^n)$. □

Putting together this proposition and our main result, we can obtain decomposition results for many function spaces; see [1, 9] for some recent works.

4 Main idea of the proof of the main result

4.1 Some problems

One of the attractive ways to prove the main result is to reexamine the book [16]. We need to show $HX(\mathbb{R}^n) \cap L^1_{\rm loc}(\mathbb{R}^n)$ is dense in $HX(\mathbb{R}^n)$. However, as the example of the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ with $0 < q < 1 \le p < \infty$ shows, this does not seem to be true. One sufficient condition that makes this argument possible is the notion of absolute continuity of the norms. A quasi-Banach function space X is said to have an *absolutely continuous quasi-norm*, if $\|\chi_{E_j}\|_X \downarrow 0$ whenever $\{E_j\}_{j=1}^{\infty}$ is a sequence decreasing to the empty set. In this case, we can also show that $HX(\mathbb{R}^n) \cap L^\infty_{\rm c}(\mathbb{R}^n)$ is dense in $HX(\mathbb{R}^n)$.

4.2 What does assumption (3.6) implies ?

Assume that X is a ball quasi-Banach function space satisfying (3.6). To consider the meaning of (3.6), we consider non-homogeneous Herz spaces. Let Q_0 and C_j as before. Then as the inequality

$$M^{(\eta)}[f\chi_{C_j}](x) \gtrsim \left(\frac{1}{|C_j|} \int_{C_j} |f(y)|^\eta \, dy \right)^{\frac{1}{\eta}} \quad (x \in Q_0)$$

implies, $X \hookrightarrow K^{-n/\eta}_{\eta q}(\mathbb{R}^n)$. We use this fact to prove the main theorem in [11]. Note that $1 \notin K^{-n/\eta}_{\eta q}(\mathbb{R}^n)$.

5 Some problems

5.1 Improvement of our key assumption

We postulated assumption (3.6) because it appeared many times in the proof.

In general, it is demanding that we verify assumption (3.6). Here and below we write $\langle a\rangle = \sqrt{1+|a|^2}$ for $a \in \mathbb{R}^n$. So we propose here replace the operator $M^{(\eta)}$ by

$$\varphi \mapsto \sup_{z\in\mathbb{R}^n} \langle z\rangle^{-\frac{n}{\eta}}|\varphi(\cdot - z)|$$

motivated by the well-known Plancherel-Polya-Nikolski'i inequality:

$$\sup_{z\in\mathbb{R}^n} \langle z\rangle^{-\frac{n}{\eta}}|\varphi(x - z)| \lesssim M^{(\eta)}\varphi(x) \tag{5.1}$$

when $\mathrm{supp}(\mathcal{F}\varphi)$ is contained in a fixed compact set. The operator

$$f \mapsto \sup_{z\in\mathbb{R}^n} \langle z\rangle^{-\frac{n}{\eta}}|f(\cdot - z)|$$

is called the *Peetre maximal operator.* For Besov spaces and Triebel-Lizorkin spaces, we succeeded in this attempt; see [12]. A direct consequence of the definition of the Peetre maximal operator is that

$$\sup_{z\in\mathbb{R}^n} \langle z\rangle^{-\frac{n}{\eta}}|f(\cdot + a - z)| \le 2^{\frac{a}{2}}\langle a\rangle^{\frac{n}{\eta}} \sup_{z\in\mathbb{R}^n} \langle z\rangle^{-\frac{n}{\eta}}|f(\cdot - z)|$$

for all $a \in \mathbb{R}^n$.

We can control $\sup\limits_{z\in\mathbb{R}^n} \langle z\rangle^{-\frac{n}{\eta}}|f(\cdot + a - z)|$ by $\sup\limits_{z\in\mathbb{R}^n} \langle z\rangle^{-\frac{n}{\eta}}|f(\cdot - z)|$ at the cost of the factor of $\langle a\rangle^{\frac{n}{\eta}}$.

Although this attempt does not work, we are also interested in the following parametrized space: $\|f\|_{HX} \equiv \left\|\sup_{z\in\mathbb{R}^n} \langle z\rangle^{-\frac{n}{\eta}}|f(\cdot - z)|\right\|_X$. For Besov spaces, the following condition is sufficient: $\|f(\cdot + x)\|_X \lesssim \langle x\rangle^N\|f\|_X$ for some $N \in \mathbb{N}$. See [14] for more details.

5.2 The boundedness of the fractional integral operators

Let I_α be the fractional integral operator of order α given by

$$I_\alpha f(x) \equiv \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy.$$

Here we igore the problem of the convergence of the integral which will be justified later. In fact, in many cases "$f \in X$" is a sufficient condition of the integral to converge for almost all $x \in \mathbb{R}^n$.

Problem 5.1. *Let X, Y be ball Banach function spaces. When is $I_\alpha : X \to Y$ bounded ?*

5.3 The condition on X for (3.6) to hold

It may be interesting to look for the condition for (3.6) to hold. If X is a Banach function space, then a beautiful result is known; see [6].

5.4 The characterization of $HX(\mathbb{R}^n)$ in terms of the Riesz transform

It is not known whether $HX(\mathbb{R}^n)$ can be characterized in terms of the Riesz transform. Maybe, the method in [18] can be used.

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