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LOCAL REGULARITY FOR THE EVOLUTIONARY P-LAPLACE OPERATOR AND ITS APPLICATION TO THE P-HARMONIC FLOWS

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1 Introduction

In this note we report on a local regularity for the evolution of $p-$harmonic maps, the $p-$harmonic flow, in the super- and sub- quadratic cases, which has been recently obtained by the authour.

Let $\mathcal{N}$ be a $n-$dimensional smooth compact Riemannian manifold without boundary and isometrically embedded in $\mathbb{R}^{l}$ $(l>n)$. For a smooth map u from time-space region $\mathbb{R}_{\infty}^{m}:=(0, \infty)\times \mathbb{R}^{m}$ $(m\geq 2)$ to $\mathbb{R}^{l}$ we consider the quasilinear parabolic type system of 2nd-ordered partial differential equations

$$(1.1)\qquad \begin{cases} \partial_{t}u-\operatorname{div}\left(|Du|^{p-2}Du\right)=|Du|^{p-2}A(u)(Du, Du)\\ u\in \mathcal{N}\subset \mathbb{R}^{l}. \end{cases}$$

In this note we study a local regularity of solutions to the $p-$harmonic flow (1.1). Here $p>1$, and $u=(u^{i})$, $i=1, \ldots, l$, is a $\mathbb{R}^{l}$-valued function, $Du=(D_{\alpha}u^{i})$ is the gradient of a map u with partial derivatives $D_{\alpha}=\partial/\partial x_{\alpha}$, $\alpha=1, \ldots, m$, and $|Du|^{2}=\sum_{\alpha=1}^{m}\sum_{i=1}^{l}(D_{\alpha}u^{i})^{2}$, and $A(u)(Du, Du)$ is the second fundamental form of $\mathcal{N}\subset \mathbb{R}^{l}$ (provided that, if necessary, the manifold $\mathcal{N}$ is assumed to be orientable). The solution of (1.1) is the trajectory of negative direction gradient flow of the $p-$energy

$$(1.2)\qquad E(u)=\int_{\mathbb{R}^{m}}\frac{1}{p}\ |Du|^{p}\ dx$$

defined for maps u from $\mathbb{R}^{m}$ to $\mathcal{N}\subset \mathbb{R}^{l}$. A critical point of the $p-$energy is prescribed as a solution of the Euler-Lagrange equation

$$(1.3)\qquad \begin{cases} -\operatorname{div}\left(|Du|^{p-2}Du\right)=|Du|^{p-2}A(u)(Du, Du)\\ u\in \mathcal{N}\subset \mathbb{R}^{l}. \end{cases}$$

and is named the $p-$harmonic map.

Here our interest is to have the restriction that the image of maps is imposed on the manifold $\mathcal{N}$, yielding the second fundamental form of $\mathcal{N}$ in the corresponding equations. Now we explicitly look at the second fundamental form of $\mathcal{N}$ in $\mathbb{R}^{l}$.

First we simply derive the Euler-Lagrange equation of (1.2) and the gradient flow (1.1). Let u be a smooth map from $\mathbb{R}^{m}$ to $\mathcal{N}$ and ϕ a smooth $\mathbb{R}^{l}-$vector valued function on $\mathbb{R}^{m}$ with compact support. Let $\Pi:\mathbb{R}^{l}\supset \mathcal{O}(\mathcal{N})\rightarrow \mathcal{N}\subset \mathbb{R}^{l}$ be the nearest point projection from a tubular neighborhood $\mathcal{O}(\mathcal{N})\subset \mathbb{R}^{l}$ of $\mathcal{N}$, to $\mathcal{N}$. For any sufficient small number τ, $|\tau|\ll|\phi|_{\infty}$, the map $u+\tau\phi$ has its value in $\mathcal{O}(\mathcal{N})$ and so, $\Pi(u+\tau\phi)\in \mathcal{N}$ is a admissible

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comparison map. The first variation (Gâteaux derivative) is computed by integration by parts as

(1.4)
$$\frac{d}{d\tau}E\left(\Pi(u+\tau\phi)\right)\Big|_{\tau=0} \;=\; \int\limits_{\mathbb{R}^m}\left(-\operatorname{div}\left(|Du|^{p-2}Du\right)+|Du|^{p-2}\frac{d^2\Pi}{du^2}(u)(Du,Du)\right)\cdot\phi\,dx.$$

Thus, the Euler-Lagrange equation (1.3) is the first variational formula, (1.4)= 0. For smooth maps $u \in C^\infty(\mathbb{R}^m, \mathcal{N})$ † , its gradient-like vector field $\nabla E(u)$ of the p−energy is formally defined as

$$\langle \nabla E(u), \phi\rangle^{\ddagger} \;=\; \frac{d}{d\tau}E\left(\Pi(u+\tau\phi)\right)\Big|_{\tau=0}$$

and thus, by (1.4)

$$\nabla E(u) = -\operatorname{div}\left(|Du|^{p-2}Du\right)+|Du|^{p-2}\frac{d^2\Pi}{du^2}(u)(Du,Du)$$

and so, the solution-curve $\{u(t)\} \subset C^\infty(\mathbb{R}^m, \mathcal{N})$, $0 \le t < \infty$, of its negative direction gradient vector field is the solution to the differential equation (1.1).

Next let $\mathbb{R}^l = \mathcal{T}_u\mathcal{N} \oplus (\mathcal{T}_u\mathcal{N})^\perp$ be the orthogonal decomposition of $\mathbb{R}^l$ with respect to the tangent space $\mathcal{T}_u\mathcal{N}$ at each $u \in \mathcal{N}$. The corresponding orthonormal basis is $(e_1(u), \ldots, e_n(u))$ of the tangent space $\mathcal{T}_u\mathcal{N}$ and $(e_{n+1}(u), \ldots, e_l(u))$ of its orthogonal complement $(\mathcal{T}_u\mathcal{N})^\perp$. Then the sencond fundamental form can be written as

$$A(u)(Du,Du) = \sum_{j=n+1}^{l}\sum_{i=1}^{l}\left(Du\cdot Du^i\frac{\partial e_j}{\partial u^i}(u)\right)e_j(u)$$

and thus, $A(u)(Du, Du) \in (\mathcal{T}_u\mathcal{N})^\perp$ for each $u \in \mathcal{N}$. On the other hand, $\partial_t u \in \mathcal{T}_u\mathcal{N}$ and $D_\alpha u \in \mathcal{T}_u\mathcal{N}$, $\alpha = 1, \ldots, m$, because the image of maps $u = u(t, x)$ restricted on the manifold $\mathcal{N}$. Thus, making the Euclidean inner product in $\mathbb{R}^l$ with the equation (1.1) gives

$$|\partial_t u|^2 - \Delta_p u\cdot\partial_t u = 0, \qquad \partial_t u\cdot D_\alpha u - \Delta_p u\cdot D_\alpha u = 0,$$

and the crucial formulas for local energy estimates, respectively,

$$|\partial_t u|^2 - \operatorname{div}(|Du|^{p-2}Du\cdot\partial_t u) + \partial_t\frac{1}{p}|Du|^p = 0,$$
$$\partial_t u\cdot D_\alpha u - \operatorname{div}(|Du|^{p-2}Du\cdot D_\alpha u) + D_\alpha\frac{1}{p}|Du|^p = 0, \quad \alpha = 1, \ldots, m.$$

In particular, the first formula is integrated on space and yields, through integration by parts,

$$\frac{d}{dt}E(u(t)) = -\|\partial_t u(t)\|_2^2$$

and thus, the p−energy $E(u(t))$ is decreasing along the solution $u(t)$ of the p−harmonic flow. A global in time solution to (1.1) for any initial data may converge to the critical

† $C^\infty(\Omega, \mathcal{N})$ is a Banach manifold

‡ $\langle\nabla E(u), \cdot\rangle$ is a bounded linear functional on a tangent space $\bigcup_{u\in\mathcal{X}} C^\infty(\Omega, T_u(\mathcal{N}))$ of a Banach manifold $\mathcal{X} := C^\infty(\Omega, \mathcal{N})$.

points of the p–energy, the p–harmonic maps, as time tending to ∞. This *heat flow method* is originally realized by J. Eells and J. H. Sampson in the harmonic flow case $p = 2$ ([7]). Their fundamental result also holds similarly for the p–harmonic flow under the condition on target that the sectional curvature of $\mathcal{N}$ is non-positive (see [15, 8]).

Without any curvature restriction on the target manifold, there is a blowing up solution at a finite time (see [2] in the case $p = m = 3$). Thus, a weak solution is naturally considered. A weak solution which is locally continuous on time-space together with its gradient is called a regular solution.

Theorem 1 *[12] Let $p = m \geq 2$ and let the initial data be in the set of Sobolev maps $W^{1,p}(\mathcal{M},\mathcal{N})$ between two smooth, compact Riemannian manifolds $\mathcal{M}$ and $\mathcal{N}$ without boundaries. Then, there exists a global in time weak solution of Cauchy problem for the m–harmonic flow. The solution is regular, except for at most finitely many time slices.*

In the two-dimensional harmonic flow case $p = m = 2$, the solution is smooth except for at most finitely many points [20]. In the case $p = m$, a nice Sobolev type inequality on time-space, referred as Ladyzhenskaya or Nash inequality in $p = m = 2$, plays an important role in regularity estimate.

The global in time existence of partial regular weak solution to the harmonic flow in the case $p = 2$ has been established by M. Struwe et al. in [21, 4] The crucial ingredient for the result is the so-called small energy regularity estimate as follows : Let $T > 0$ and $X \in \mathbb{R}^m$, and let the *backward* in time heat kernel with pole at (T, X) be

$$G(t,x) = \frac{1}{(4\pi\,(T-t))^{m/2}} \exp\left(\frac{|x-X|^2}{4\,(T-t)}\right), \quad t < T.$$

The scaled energy is defined as

$$I(T,\, X\,;\, r) = r^2 \int_{\{t=T-r^2\}} \frac{1}{2}|Du(t,x)|^2\, G(t,x)\, dx, \quad 0 < r \leq T^{1/2}.$$

The following monotonicity estimate holds true (see [21, Lemma 3.2, pp. 489-490]).

Lemma 2 (monotonicity formula) *Let $p = 2$ and let u be a smooth solution of the harmonic flow (1.1) on $\mathbb{R}^m_T = (0,T) \times \mathbb{R}^m$ for $T > 0$. For any positive $r < \rho \leq T^{1/2}$ it holds that*

$$I(T,\, X\,;\, r) \leq I(T,\, X\,;\, \rho).$$

From the monotonicity estimate of scaled enegy and the gradient L^∞–estimate on *small region* for harmonic flow, the following regularity estimate is obtained (see [21, Proposition 4.1, p. 490 ; Theorem 5.1, its proof, pp. 491-493 ; Theorem 5.3, p. 494] and also [22, Proof of Theorem, pp. 171-172]).

Theorem 3 (small energy regularity) *Let $p = 2$ and let u be a smooth solution of the harmonic flow (1.1) on $\mathbb{R}^m_T = (0,T) \times \mathbb{R}^m$. Then there exist positive constants ϵ_0 and C depending only on m and $\mathcal{N}$ such that the following holds true : If $I(T,\, X\,;\, R) \leq \epsilon_0$ for some $X \in \mathbb{R}^m$ and some positive $R \leq T^{1/2}$, then it holds that*

$$\sup_{(T-(R/4)^2,\,T)\times B(R/4,\,X)} |Du| \leq C\, R^{-1}.$$

There also exist blowing up solutions at a finite time (see [1, 3, 5, 10]).

Based on the a-priori estimate for smooth solutions in Theorem 3 with an appropriate approximation method, it is shown in [4] that, for the Cauchy problem for harmonic flow in the case $p = 2$, there exists a global in time weak solution which is *partial regular* in the sense of regularity outside exceptional closed set. The local regularity estimate has recently been established for the $p-$harmonic flow in the superqudratic case $p > 2$ (see [16, 17]), which corresponds to the small energy regularity result as in Theorem 3 for the $p-$harmonic flow.

Now we will present our main result, the *small energy regularity* estmate for the $p-$harmonic flow.

Theorem 4 *Let λ_0, B_0 and a_0 be positive numbers satisfying the conditions : In the superquadratic case $p > 2$,*

$$\frac{4(p-1)}{p} < \lambda_0 = B_0 < p \quad ; \quad \frac{\lambda_0 - 2}{p-2} < a_0 \leq 1 \tag{1.5}$$

and, in the subquadratic case $\frac{2m}{m+2} < p < 2$,

$$p < \lambda_0 = B_0 < \min\left\{\frac{4}{4-p}, 3 - \frac{2}{p}\right\} \quad ; \quad \frac{2-\lambda_0}{2-p} < a_0 \leq 1. \tag{1.6}$$

Let u be a regular solution of (1.1) on $\mathbb{R}^m_T$ for a positive $T < \infty$, satisfying the energy bound

$$p\,\|\partial_t u\|^2_{L^2(\mathbb{R}^m_T)} + \sup_{0<t<T} \|Du(t)\|^p_{L^p(\mathbb{R}^m)} \leq C$$

for a positive number C depending only on m, p and $\mathcal{N}$. Then, there exists a small positive numeber $R_0 < 1$, depending only on m, p, B_0 and a_0, and the following holds true : If, for some small positive $R < \min\{R_0, T^{1/\lambda_0}\}$ and some $X \in \mathbb{R}^m$,

$$\limsup_{r\searrow 0} r^{\gamma_0 - m} \int_{\{t=T-R^{\lambda_0}\}\times B(r,X)} |Du(t,x)|^p\,dx \leq 1, \quad \gamma_0 = \frac{p(B_0-2)}{p-2}, \tag{1.7}$$

then, the inequality holds

$$\sup_{\left(T-(R/4)^{\lambda_0},T\right)\times B(R/4,X)} |Du| \leq C\,R^{-a_0}, \tag{1.8}$$

where the positive constant C depends only on λ_0, B_0, a_0, m,p and $\mathcal{N}$.

The condition (1.7) is the local regularity criterion for regular solutions with energy boundedness of the $p-$harmonic flow to be the uniformly locally bounded of gradients as in (1.8) and thus, uniformly locally continuously differentiable (see [6, 13, 14]). The scale order in condition in (1.7) is *almost optimal*, comparing with the corresponding uniform regularity criterion for regular solutions of stationary $p-$harmonic maps because the exponent γ_0 can be chosen as close to p as possible, by the condtion of B_0 in (1.5) or (1.6).

The main ingredients of Theorem 4 are the monotonicity estimates of scaled energy and the $L^\infty-$estimate of gradients (see [6, 13, 14]), well-combined under a time-space scaling. The technical novelty here is a new *monotonicity* type estimate of a *localized* scaled $p-$energy, which may be of its own interest. Let us define our localized scaled

$p-$energy in the following way: Let $T \geq 0$ and $X \in \mathbb{R}^m$ be given, and (t_0, x_0) in the parabolic like envelope

$$\left\{(t,\, x) \in (0, \infty) \times \mathbb{R}^m \,:\, t - T \geq |x - X|^{\lambda_0}\right\} \quad ; \quad \lambda_0 > 2.$$

The localized scaled energy is defined as

$$E_{\pm}(r) = \frac{1}{\Lambda^p} \int_{\{t=t_0 \pm \Lambda^{2-p} r^2\} \times \mathbb{R}^m} \frac{1}{p}\, |Du(t,x)|^p\, \mathcal{B}_{\pm}(t_0, x_0; t, x)\, \mathcal{C}^q(t,x)\, dx, \tag{1.9}$$

where $\Lambda = \Lambda(r)$ is a function of a scale radius r, defined as

$$\Lambda = \Lambda(r) = r^{\frac{B_0-2}{2-p}} \quad ; \quad \lambda_0 = B_0 \text{ is as in (1.5) or (1.6).} \tag{1.10}$$

The *forward* or *backward* in time Barenblatt like function denoted by $\mathcal{B}_+$ and $\mathcal{B}_-$, respectively, are defined as

$$\mathcal{B}_{\pm}(t_0,\, x_0\,;\, t,\, x) = \frac{1}{(\mp t_0 \pm t)^{\frac{m}{B_0}}} \left(1 - \left(\frac{|x - x_0|}{(\mp t_0 \pm t)^{\frac{1}{B_0}}}\right)^a\right)_+^b, \quad \mp t < \mp t_0 \quad ; \tag{1.11}$$

$a, b > 1\,;$ determined later.

The localized function $\mathcal{C}$ is defined and used as

$$\mathcal{C}(t,\, x) := \left((t-T)^{1/\lambda_0} - |x - X|\right)_+ \quad ; \quad q > 2. \tag{1.12}$$

We call $E_+(r)$ and $E_-(r)$ the forward and backward localized scaled $p-$energy, respectively.

The followings are our monotonicity type estimate of scaled energy.

Lemma 5 (backward monotonicity estimate) *Suppose that $t_0 - T \leq 2$. For any regular solution to the $p-$harmonic flow the following estimate holds for all positive numbers r, ρ, $r^{B_0} = \Lambda(r)^{2-p} r^2 < \rho^{B_0} = \Lambda(\rho)^{2-p}\rho^2 \leq \min\{1,\, (t_0 - T)/2\}$*

$$\begin{aligned} E_-(r) \;&\leq\; E_-(\rho) + C\,(\rho^\mu - r^\mu) \\ &\quad + C \int_{t_0-\rho^{B_0}}^{t_0-r^{B_0}} \|\mathcal{C}^{\tilde q}(t)\, |Du(t)|^{\hat p}\|_{L^\infty(B((t_0-t)^{1/B_0}, x_0))}\, dt, \end{aligned} \tag{1.13}$$

where $\hat p = \max\{2(p-1),\, 2\}$ and, $\tilde q = \min\{q-2,\, q(p-1)/p\}$, B_0 as in (1.10), and the positive exponent μ depends only on $\mathcal{N}$, m, p and B_0, and the positive constant C depends only on the same ones as μ and q.

Lemma 6 (forward monotonicity estimate) *Suppose that $t_0 - T \leq 1$. For any regular solution to the $p-$harmonic flow the following estimate holds for all positive numbers r, ρ, $r^{B_0} = \Lambda(r)^{2-p} r^2 < \rho^{B_0} = \Lambda(\rho)^{2-p}\rho^2 \leq 1$*

$$\begin{aligned} E_+(\rho) \;&\leq\; E_+(r) + C\,(\rho^\mu - r^\mu) \\ &\quad + C \int_{t_0+r^{B_0}}^{t_0+\rho^{B_0}} \|\mathcal{C}^{\tilde q}(t)\, |Du(t)|^{\hat p}\|_{L^\infty(B((t-t_0)^{1/B_0}, x_0))}\, dt, \end{aligned} \tag{1.14}$$

where $\hat p = \max\{2(p-1),\, 2\}$ and, $\tilde q = \min\{q-2,\, q(p-1)/p\}$, B_0 as in (1.10), and the positive constants μ and C have the same dependence as those in Lemma 5.

From Theorem 4, a compactness for regular $p-$harmonic flows with uniform boundedness of $p-$energy is obtained (see [21, Theorem 6.1 ; its proof, pp. 494-497] for the harmonic flow). The compactness result will be the key ingredient for the global in time existence of $p-$harmonic flow, which will be studied in the near future work (refer to [4] for the harmonic flow case).

Theorem 7 (compactness of regular $p-$harmonic flows) *Suppose that a family $\{u_k\}$ of regular $p-$harmonic flows on $\mathbb{R}^m_\infty = (0, \infty) \times \mathbb{R}^m$ satisfies the $p-$energy boundedness with uniform positive constant C*

$$p\,\|\partial_t u_k\|^2_{L^2(\mathbb{R}^m_\infty)} + \sup_{0<t<\infty} \|Du_k(t)\|^p_{L^p(\mathbb{R}^m)} \le C \tag{1.15}$$

and converges to a limit map u in the sense

$$u_k \longrightarrow u \quad \text{weakly} * \text{ in } L^\infty\left(0, T\,;\, W^{1,p}(\mathbb{R}^m_\infty, \mathbb{R}^l)\right), \tag{1.16}$$

$$Du_k \longrightarrow Du \quad \text{weakly in } L^p\left(\mathbb{R}^m_\infty, \mathbb{R}^{ml}\right), \tag{1.17}$$

$$\partial_t u_k \longrightarrow \partial_t u \quad \text{weakly in } L^2\left(\mathbb{R}^m_\infty, \mathbb{R}^l\right). \tag{1.18}$$

Then, the limit map u is a global weak solution on $\mathbb{R}^m_\infty$ of the $p-$harmonic map heat flow such that $u \in \mathcal{N}$ almost everywhere in $\mathbb{R}^m_\infty$, and the $p-$energy boundedness is valid, replacing u_k by u in (1.15). Moreover, the limit map u is partial regular in the sense : Let $R_0 < 1$ be a positive number, defined in Theorem 4 and a subset $\mathcal{S} \subset \mathbb{R}^m_\infty$ be defined as

$$\mathcal{S} := \Big\{(\tau, x_0) \in \mathbb{R}^m_\infty \;:\; \text{for all positive } R < \min\{R_0, \tau^{1/\lambda_0}\},$$

$$\limsup_{k\to\infty}\left(\limsup_{r\searrow 0} r^{\gamma_0 - m} \int_{\{t=\tau-R^{\lambda_0}\}\times B(r, x_0)} |Du_k(t,x)|^p\,dx\right) \ge 1\Big\}. \tag{1.19}$$

Then, $\mathcal{S}$ is closed in $\mathbb{R}^m_\infty$ and u and its gradient Du are locally in time-space continuous in the complement $\mathbb{R}^m_\infty \setminus \mathcal{S}$. The size of $\mathcal{S}$ is also estimated by the Hausdorff measure : Let γ_0, λ_0, B_0 and a_0 be the same positive numbers as in (1.5) and (1.7) in Theorem 4. The set $\mathcal{S}$ is of at most locally zero $m-$dimensional Hausdorff measure with respect to the time-space metric $|t|^{1/\gamma_0} + |x|$, $\mathcal{H}^m(\mathcal{S} \cap K) = 0$ for any open subset K compactly contained in $\mathbb{R}^m_\infty$, and, furthermore, for any positive time $\tau < \infty$, the $(m - \gamma_0)-$dimensional Hausdorff measure of $\{\tau\} \times \mathcal{S}$ with respect to the usual Euclidean metric is locally zero, $\mathcal{H}^{m-\gamma_0}(\{\tau\} \times \mathcal{S} \cap K) = 0$ for any open subset K compactly contained in $\mathbb{R}^m$.

2 Monotonicity estimate

We demonstrate the monotonicity estimates in the superquadratic case $p > 2$. For brevity, we reset as $\mathcal{C} \equiv 1$.

Let $z_0 = (t_0, x_0) \in (0, T] \times \mathbb{R}^m$. As before, we put

$$\Lambda = r^{\frac{B_0-2}{2-p}}, \quad B_0 > \frac{4(p-1)}{p}$$

and let r any positive number in the range $0 < r \le \min\left\{1, (t_0)^{1/B_0}\right\}$. We make a scaling transformation intrinsic to the evolutionary $p-$Laplace operator (refer to [6, 13, 14])

$$t = t_0 + \Lambda^{2-p} r^2\, s\,; \quad x = x_0 + r\,y\,; \quad v(s, y) = \frac{u(t_0 + \Lambda^{2-p} r^2\, s,\, x_0 + r\,y)}{\Lambda\, r} \tag{2.1}$$

and, under the scaling transformation

$$t = t_0 - \Lambda^{2-p} r^2 \iff s = -1.$$

Then the scaled solution v is a solution of the scaled equation on $\{s = -1\} \times \mathbb{R}^m$

$$\partial_s v - \operatorname{div}\left(|Dv|^{p-2}\, Dv\right) = -\Lambda\, r\, |Dv|^{p-2}\, A(r\Lambda v)(Dv,\, Dv) \tag{2.2}$$

and the scaled $p-$energy is rewritten as

$$E(r) = \int_{\{s=-1\}\times\mathbb{R}^m} \frac{1}{p} |Dv(s,\, y)|^p\, \mathcal{B}(s,\, y)\, dy \quad ; \quad \mathcal{B}(s,\, y) = (1 - \, |y|^a)_+^b \tag{2.3}$$

by simply computing as

$$Dv(s,\, y) = \frac{1}{\Lambda} D_x u(t,\, x) \quad ; \quad \Lambda = r^{\frac{B_0-2}{2-p}} \iff \Lambda^{\frac{p-2}{B_0}}\, r^{\frac{B_0-2}{B_0}} = 1 \quad ;$$
$$\mathcal{B}(s,\, y)\, dy = \mathcal{B}\,(t_0,\, x_0\,;t,\, x)\; dx.$$

Our main task in monotonicity estimate is to derive appropriate values of parameters

$$B_0 > \frac{4(p-1)}{p} \quad ; \tag{2.4}$$

$$0 < \delta \le \frac{B_0(p-2)}{B_0 - 2}\left(-1 + \frac{2(p-1)(B_0-2)}{B_0(p-2)}\right). \tag{2.5}$$

Step 1: differentiation of $E(r)$ on r. Now we compute differentiation of $E(r)$ on r. By the equation (2.2) and integration by parts,

$$\begin{aligned}
\frac{d}{dr} E(r) &= \int_{\{s=-1\}\times\mathbb{R}^m} |Dv|^{p-2} Dv \cdot \frac{d}{dr} Dv\, \mathcal{B}(s,\, y)\, dy \\
&= \int_{\{s=-1\}\times\mathbb{R}^m} \frac{dv}{dr} \cdot \left(-\Delta_p v\, \mathcal{B}(s,\, y) - |Dv|^{p-2} Dv \cdot D\mathcal{B}(s,\, y)\right)\, dy \\
&= r^{-1} \int_{\{s=-1\}\times\mathbb{R}^m} (-s)\, \left((2-p)\, r\, \Lambda^{-1}\, \Lambda' + 2\right)\, |\partial_s v|^2\, \mathcal{B}(s,\, y)\, dy \\
&\quad - r^{-1} \int_{\{s=-1\}\times\mathbb{R}^m} (y \cdot Dv) \cdot \partial_s v\, \mathcal{B}(s,\, y)\, dy \\
&\quad + r^{-1} \int_{\{s=-1\}\times\mathbb{R}^m} \left(1 + r\, \Lambda^{-1}\, \Lambda'\right)\, v \cdot \partial_s v\, \mathcal{B}(s,\, y)\, dy \\
&\quad + \int_{\{s=-1\}\times\mathbb{R}^m} \Lambda\, \left(1 + r\, \Lambda^{-1}\, \Lambda'\right)\, |Dv|^{p-2} v \cdot A(r\, \Lambda\, v)(Dv,\, Dv)\, \mathcal{B}(s,\, y)\, dy \\
&\quad + ab\, r^{-1} \int_{\{s=-1\}\times\mathbb{R}^m} \Big\{ |Dv|^{p-2}\, |y \cdot Dv|^2 \\
&\qquad\qquad + \left((2-p)\, r\, \Lambda^{-1}\, \Lambda' + 2\right)\, |Dv|^{p-2}\, (y \cdot Dv) \cdot (s\, \partial_s v) \\
&\qquad\qquad - \left(1 + r\, \Lambda^{-1}\, \Lambda'\right)\, |Dv|^{p-2}\, (y \cdot Dv) \cdot v \Big\} \times \\
&\qquad \times |y|^{a-2}\, (1 - \, |y|^a)_+^{b-1}\, dy,
\end{aligned} \tag{2.6}$$

where, noting that $\Lambda = r^{(B_0-2)/(2-p)}$, the generator of dilation is computed as

$$\frac{dv}{dr} = r^{-1}\left(-\left(1+r\,\Lambda^{-1}\,\Lambda'\right)\,v+\left((2-p)\,r\,\Lambda^{-1}\,\Lambda'+2\right)\,s\,\partial_s v+y\cdot Dv\right)$$

Now each term in (2.6) is separately estimated.

1st term of (2.6). In the first term of (2.6) the coefficient is positive, because, by definition of Λ and $s=-1$

$$(-s)\,\left((2-p)\,r\,\Lambda^{-1}\,\Lambda'+2\right) = B_0 > 0 \Longleftrightarrow \Lambda = r^{(B_0-2)/(2-p)}.$$

2nd term of (2.6). By Cauchy's inequality with small $c>0$ the second term of (2.6) is estimated below by

$$-\frac{c}{2}\,r^{-1}\int\limits_{\{s=-1\}\times\mathbb{R}^m}|\partial_s v|^2\,\mathcal{B}\,dy-\frac{1}{2c}\,r^{-1}\int\limits_{\{s=-1\}\times\mathbb{R}^m}|y|^2\,|Dv|^2\,\mathcal{B}\,dy.$$

The time-derivative term is absorbed into the first term and, by the support of the Barenblatt like weight $\mathcal{B}$, $\{y\in\mathbb{R}^m \mid |y|\le 1\}$, and by Young's inequality with $\delta>0$, the spatial gradient term is estimated below by

$$-\,C\,r^{-1}\,\Lambda^{\delta}\int\limits_{\{s=-1\}\times\mathbb{R}^m}|Dv|^{2(p-1)}\,\mathcal{B}\,dy-\,C\,r^{-1}\,\Lambda^{-\frac{\delta}{p-2}}\int\limits_{\{s=-1\}\times\mathbb{R}^m}\mathcal{B}\,dy. \tag{2.7}$$

3rd term of (2.6). For estimation of the third term of (2.6) we use the Poincaré type inequality with weight of Barenblatt like function [§] [18, Theorem 5.3.4, p. 134].

Lemma 8

$$\int\limits_{\{s=-1\}\times\mathbb{R}^m}|v|^2\,\mathcal{B}\,dy \le C\int\limits_{\{s=-1\}\times\mathbb{R}^m}|Dv|^2\,\mathcal{B}\,dy. \tag{2.8}$$

By Cauchy's inequality with small $c>0$ the third term is estimated below by

$$-\frac{c}{2}\,r^{-1}\int\limits_{\{s=-1\}\times\mathbb{R}^m}|\partial_s v|^2\,\mathcal{B}\,dy-\frac{1}{2c}\,r^{-1}\int\limits_{\{s=-1\}\times\mathbb{R}^m}|v|^2\,\mathcal{B}\,dy, \tag{2.9}$$

where, by definition of Λ, $1+r\,\Lambda^{-1}\,\Lambda' = (p-B_0)/(p-2)$. The first time-derivative term is absorbed into that of (2.6). By the support of $\mathcal{B}$, again, the Poincaré inequality, Lemma 8, and Young's inequality, the second term is bounded below by (2.7).

4th term of (2.6). By the definition of Λ, the fourth term of (2.6) is estimated below as

$$-\frac{|p-B_0|}{p-2}\,C'(\mathcal{N})\,\Lambda\int\limits_{\{s=-1\}\times\mathbb{R}^m}|v|\,|Dv|^p\,\mathcal{B}\,dy \ge -\,C\,r^{-1}\int\limits_{\{s=-1\}\times\mathbb{R}^m}|Dv|^p\,\mathcal{B}\,dy, \tag{2.10}$$

where the second fundamental form A is bounded by the compactness of target $\mathcal{N}$ and, by scaling back and the compactness of target $\mathcal{N}$,

$$\Lambda\,|v| = \Lambda\,\frac{1}{\Lambda\,r}\,|u| \le \frac{1}{r}\,\|u\|_{L^\infty(\mathbb{R}^m)} \le r^{-1}\,C(\mathcal{N}).$$

§ Our estimations here remained unchanged, even if v is replaced by $v-\bar{v}$ with weighted integral mean $\bar{v} = \int_{\{s=-1\}\times\mathbb{R}^m} v\,\mathcal{B}\,dy / \int_{\{s=-1\}\times\mathbb{R}^m}\mathcal{B}\,dy$.

By Young's inequality, (2.10) is bounded below for $\delta > 0$ by

$$- C r^{-1} \Lambda^{\delta} \int_{\{s=-1\}\times\mathbb{R}^m} |Dv|^{2(p-1)} \mathcal{B}\, dy - C r^{-1} \Lambda^{-\frac{\delta p}{p-2}} \int_{\{s=-1\}\times\mathbb{R}^m} \mathcal{B}\, dy. \tag{2.11}$$

5th term of (2.6). The fifth term of (2.6) is clearly nonnegative.

6th term of (2.6). The sixth term of (2.6) appears from the nonhomogeneity of evolutionary p−Laplace operator and is estimated below by Cauchy's inequality as

$$\begin{aligned} &-\frac{c}{2} r^{-1} \int_{\{s=-1\}\times\mathbb{R}^m} |\partial_s v|^2 \mathcal{B}\, dy \\ &-\frac{C}{2c} r^{-1} \int_{\{s=-1\}\times\mathbb{R}^m} |Dv|^{2(p-1)} |y|^{2(a-1)} (1 - |y|^a)_+^{b-2}\, dy, \end{aligned} \tag{2.12}$$

where, by definition of Λ, $(2-p)\, r\, \Lambda^{-1} \Lambda' + 2 = B_0$, as before. The first term of (2.12) is absorbed into that of (2.6). The second term of (2.12) is estimated below by

$$-\frac{C}{2\delta} r^{-1} \frac{1}{\Lambda^{2(p-1)}} \|Du(\tau)\|^{2(p-1)}_{L^\infty(\mathrm{supp}\mathcal{B}(\tau))}\Big|_{\tau=t_0-\Lambda^{2-p} r^2}, \tag{2.13}$$

where, by a scaling back,

$$\int_{\mathbb{R}^m} |y|^{2(a-1)} (1 - |y|^a)_+^{b-2}\, dy < \infty,$$

$$1 + \frac{m-2}{a} > -1 \Longleftarrow a > 0 \quad ; \quad b - 2 > -1 \Longleftrightarrow b > 1.$$

7th term of (2.6). As in (2.12), the seventh term of (2.6) is bounded below by

$$- C r^{-1} \int_{\{s=-1\}\times\mathbb{R}^m} |v|^2 \mathcal{B}\, dy - C r^{-1} \int_{\{s=-1\}\times\mathbb{R}^m} |Dv|^{2(p-1)} |y|^{2(a-1)} (1 - |y|^a)_+^{b-2}\, dy,$$

where the first one is the same as the second term in (2.9) and bounded below for $\delta > 0$ by (2.7) and, the second one is the same as in (2.12), together with the first ones of (2.7) and (2.11), estimated below by

$$-C r^{-1} (\Lambda^{\delta} + 1) \frac{1}{\Lambda^{2(p-1)}} \|Du(\tau)\|^{2(p-1)}_{L^\infty(\mathrm{supp}\mathcal{B}(\tau))}\Big|_{\tau=t_0-\Lambda^{2-p} r^2}. \tag{2.14}$$

Resulting estimation of (2.6). Combining all of the estimations above we have

$$\begin{aligned} \frac{d}{dr} E(r) \ \geq \ & I - C r^{-1} \left(\Lambda^{-\frac{\delta p}{p-2}} + \Lambda^{-\frac{\delta}{p-2}} \right) \\ & - C r^{-1} \frac{1}{\Lambda^{2(p-1)}} (\Lambda^{\delta} + 1) \|Du(\tau)\|^{2(p-1)}_{L^\infty(\mathrm{supp}\mathcal{B}(\tau))}\Big|_{\tau=t_0-\Lambda^{2-p} r^2}, \end{aligned} \tag{2.15}$$

where $\Lambda = r^{(B_0-2)/(2-p)}$, and we put

$$\begin{aligned} I \ = \ & \frac{1}{2} B_0 r^{-1} \int_{\{s=-1\}\times\mathbb{R}^m} (-s) |\partial_s v|^2 \mathcal{B}(s, y)\, dy \\ & + ab\, r^{-1} \int_{\{s=-1\}\times\mathbb{R}^m} |Dv|^{p-2} |y \cdot Dv|^2 |y|^{a-2} (1 - |y|^a)_+^{b-1}\, dy. \end{aligned}$$

The terms I is clearly nonnegative. From (2.15) integrated on (r, ρ)

$$
\begin{aligned}
& E(\rho) - E(r) \\
& \geq -C \int_r^\rho r^{-1} \left(\Lambda^{-\frac{\delta p}{p-2}} + \Lambda^{-\frac{\delta}{p-2}} \right) dr \\
& \qquad - C \int_r^\rho r^{-1} \frac{1}{\Lambda^{2(p-1)}} \left(\Lambda^\delta + 1 \right) \|Du(\tau)\|^{2(p-1)}_{L^\infty(\mathrm{supp}\mathcal{B}(\tau))} \Big|_{\tau = t_0 - \Lambda^{2-p} r^2} dr. \qquad (2.16)
\end{aligned}
$$

Step 2 : a uniform bound. We will make a bound of each term in the right hand side of (2.16).

2nd line of (2.16). The first term in the second line of (2.16) is computed as

$$
\begin{aligned}
\int_r^\rho r^{-1} \Lambda^{-\frac{\delta p}{p-2}} dr &= \int_r^\rho r^{-1-\frac{p\delta(B_0'-2)}{(p-2)^2}} dr \\
&= \frac{(p-2)^2}{p\delta(B_0-2)} \left(\rho^{\frac{p\delta(B_0-2)}{(p-2)^2}} - r^{\frac{p\delta(B_0-2)}{(p-2)^2}} \right),
\end{aligned}
$$

where

$$
\Lambda = r^{\frac{B_0-2}{2-p}}, \quad \frac{p\delta(B_0-2)}{(p-2)^2} > 0 \iff \delta > 0 \quad ; \quad B_0 > 2.
$$

Similarily as above, another term in the second line of (2.16) is

$$
\int_r^\rho r^{-1} \Lambda^{-\frac{\delta}{p-2}} dr = \frac{(p-2)^2}{\delta(B_0-2)} \left(\rho^{\frac{\delta(B_0-2)}{(p-2)^2}} - r^{\frac{\delta(B_0-2)}{(p-2)^2}} \right).
$$

3rd line of (2.16). The term in the third line of (2.16) is bounded by

$$
\begin{aligned}
& \Lambda = r^{(B_0-2)/(2-p)} \quad ; \\
& \int_r^\rho r^{-1} \left(-B_0 \Lambda^{2-p} r \right)^{-1} \frac{1}{\Lambda^{2(p-1)}} \times \Lambda^\delta \times \|Du(\tau)\|^{2(p-1)}_{L^\infty(\mathrm{supp}\,\mathcal{B}(\tau))} \left(-B_0 \Lambda^{2-p} r \right) dr \\
& = \frac{1}{B_0} \int_{t_0-(\Lambda(\rho))^{2-p}\rho^2}^{t_0-(\Lambda(r))^{2-p} r^2} (t_0-\tau)^{-1+\frac{2(p-1)(B_0-2)}{B_0(p-2)} - \frac{\delta(B_0-2)}{B_0(p-2)}} \|Du(\tau)\|^{2(p-1)}_{L^\infty(\mathrm{supp}\,\mathcal{B}(\tau))} d\tau, \qquad (2.17)
\end{aligned}
$$

where by definition of Λ

$$
\Lambda = r^{(B_0-2)/(2-p)} \iff (\Lambda(r))^{2-p} r^2 = r^{B_0}
$$

and, in the last term a changing of variable is performed

$$
\begin{aligned}
& \tau = t_0 - \Lambda^{2-p} r^2 \iff t_0 - \tau = \Lambda^{2-p} r^2 = r^{B_0} \quad ; \\
& \frac{d\tau}{dr} = -B_0 \Lambda^{2-p} r \iff d\tau = -B_0 \Lambda^{2-p} r \, dr.
\end{aligned}
$$

Here the exponent of power of $(t_0 - \tau)$ in (2.17) is estimated as

$$
\begin{aligned}
& -1 + \frac{2(p-1)(B_0-2)}{B_0(p-2)} > 0 \iff B_0 > \frac{4(p-1)}{p} \quad ; \\
& -1 + \frac{2(p-1)(B_0-2)}{B_0(p-2)} - \frac{\delta(B_0-2)}{B_0(p-2)} \geq 0 \\
& \iff 0 < \delta \leq \frac{B_0(p-2)}{B_0-2} \left(-1 + \frac{2(p-1)(B_0-2)}{B_0(p-2)} \right)
\end{aligned}
$$

and then,

$$t_0 - (\Lambda(\rho))^{2-p}\,\rho^2 \le \tau \le t_0 - (\Lambda(r))^{2-p}\,r^2 \iff r^{B_0} \le t_0 - \tau \le \rho^{B_0},$$
$$(t_0-\tau)^{-1+\frac{2(p-1)(B_0-2)}{B_0(p-2)}-\frac{\delta(B_0-2)}{B_0(p-2)}} \le \rho^{B_0\left(-1+\frac{2(p-1)(B_0-2)}{B_0(p-2)}-\frac{\delta(B_0-2)}{B_0(p-2)}\right)} \le 1$$

and thus, the right hand side of (2.17) is bounded above by

$$\frac{1}{B_0}\int_{t_0-(\Lambda(\rho))^{2-p}\,\rho^2}^{t_0-(\Lambda(r))^{2-p}\,r^2} \|Du(\tau)\|^{2(p-1)}_{L^\infty(\operatorname{supp}\mathcal{B}(\tau))}\,d\tau.$$

□

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