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REFINEMENTS OF HÖLDER-MCCARTHY INEQUALITY

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1. Introduction

Throughout this note, a capital letter means a (bounded linear) operator acting on a Hilbert space $\mathcal{H}$. An operator A is said to be positive, denoted by $A \geq 0$, if $(Ax, x) \geq 0$ for all $x \in \mathcal{H}$.

McCarthy [6] proved the following inequalities: Let A be positive operator acting on a Hilbert space $\mathcal{H}$. Then

(i) $(A^{\mu}x, x) \leq (Ax, x)^{\mu}\|x\|^{2(1-\mu)}$ for $\mu \in [0, 1]$ and $x \in \mathcal{H}$.

(ii) $(A^{\mu}x, x) \geq (Ax, x)^{\mu}\|x\|^{2(1-\mu)}$ for $\mu > 1$ and $x \in \mathcal{H}$.

Moreover (i) and (ii) are simplified to the following (iii) and (iv), respectively:

(iii) $(A^{\mu}x, x) \leq (Ax, x)^{\mu}$ for $\mu \in [0, 1]$ and $\|x\| = 1$.

(iv) $(A^{\mu}x, x) \geq (Ax, x)^{\mu}$ for $\mu > 1$ and $\|x\| = 1$.

The inequalities (i) and (ii) are proved by using the integral representation of A and the Hölder inequality. Hence they are called the Hölder-McCarthy inequality. For readers' convenience, we cite a proof of (i) for the case where A is a positive definite diagonal matrix with diagonal entries $a_1, \cdots, a_n$. For $r \in (0, 1)$,

$$\begin{aligned}(A^r x, x) = \sum a_i^r |x_i|^2 &= \sum a_i^r |x_i|^{2r} |x_i|^{2(1-r)} \\ &\leq (\sum a_i |x_i|^2)^r (\sum |x_i|^2)^{1-r} \\ &= (Ax, x)^r \|x\|^{2(1-r)}\end{aligned}$$

On the other hand, the following inequality is named as the Young inequality, cf. [2] and [3]: For $A, B \geq 0$,

$$\mu A + (1 - \mu)B \geq B \#_{\mu} A \quad \text{for } 0 \leq \mu \leq 1,$$

where $B \#_{\mu} A = B^{\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{\mu}B^{\frac{1}{2}}$ is the μ-operator geometric mean. Its simplified form is as follows: For $A \geq 0$,

$$\mu A + 1 - \mu \geq A^{\mu} \quad \text{for } 0 \leq \mu \leq 1.$$

It is known that the Hölder-McCarthy inequality (iii) and the Young inequality are equivalent [3] and [2; §3.1.3].

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As a refinement of the Young inequality, Kittaneh and Manasrah [4] proposed that

$$(1-\mu)a+\mu b \geq a^{1-\mu}b^{\mu}+\min\{\mu, 1-\mu\}(\sqrt{a}-\sqrt{b})^2$$

for all positive numbers a, b and $\mu \in [0,1]$. It is simplified as follows:

$$\mu a+1-\mu-a^{\mu} \geq \min\{\mu, 1-\mu\}(1+a-2\sqrt{a})$$

for all positive numbers a and $\mu \in [0,1]$. We now understand it as the inequality

$$\mu A+1-\mu-A^{\mu} \geq \min\{\frac{1-\mu}{1-\nu}, \frac{\mu}{\nu}\}(\nu A+1-\nu-A^{\nu}).$$

As a matter of fact, if we take $\nu=\frac{1}{2}$ and $A=aI$, where I is the identity operator, then we easily obtain the simplified inequality mentioned above. In succession, Manasrah and Kittaneh generalized refined Young inequalities in [5].

Based on recent results on refinements of Young inequality, Alzer et al. proposed the following estimation [1: Theorem 2.1]: If $0<\mu<\nu<1$, $\lambda \geq 1$ and $a, b>0$, then

$$\left(\frac{1-\nu}{1-\mu}\right)^{\lambda} < \frac{A_{\nu}^{\lambda}-G_{\nu}^{\lambda}}{A_{\mu}^{\lambda}-G_{\mu}^{\lambda}} < \left(\frac{\nu}{\mu}\right)^{\lambda}$$

holds, where $A_{\tau}=(1-\tau)a+\tau b$ and $G_{\tau}=a^{1-\tau}b^{\tau}$.

In this paper, we improve the Hölder-McCarthy inequality, whose point is the convexity of the function $f(\mu)=\frac{(A^{\mu}x,x)}{(Ax,x)^{\mu}}$. Moreover we point out that the improved Hölder-McCarthy inequality is equivalent to an improved Young inequality in the sense of Kittaneh and Manasrah.

2. Hölder-McCarthy inequality

As an approach to the Hölder-McCarthy inequality, we consider the function defined by the ratio; $f(\mu)=\frac{(A^{\mu}x,x)}{(Ax,x)^{\mu}}$. We first show the convexity of the function.

Theorem 2.1. *Let A be a positive operator on $\mathcal{H}$ and $x \in \mathcal{H}$ with $Ax \neq 0$. If $f(\mu)=\frac{(A^{\mu}x,x)}{(Ax,x)^{\mu}}$, then $f(\mu)$ is a convex function on $[0,\infty)$. Moreover if A is invertible, then $f(\mu)$ is a convex function on $(-\infty,\infty)$.*

Proof. First of all, we note that $(A^{\mu}x,x)$ is log-convex, i.e.,

$$(A^{\frac{\mu+\nu}{2}}x,x) \leq (A^{\mu}x,x)^{\frac{1}{2}}(A^{\nu}x,x)^{\frac{1}{2}}.$$

It is easily checked as follows:

$$(A^{\frac{\mu+\nu}{2}}x,x) \leq \|A^{\frac{\mu}{2}}x\|\|A^{\frac{\nu}{2}}x\| = (A^{\mu}x,x)^{\frac{1}{2}}(A^{\nu}x,x)^{\frac{1}{2}}.$$

By this and the arithmetic-geometric mean inequality, we have

$$\frac{1}{2}\left(\frac{(A^{\mu}x,x)}{(Ax,x)^{\mu}}+\frac{(A^{\nu}x,x)}{(Ax,x)^{\nu}}\right) \geq \frac{(A^{\mu}x,x)^{\frac{1}{2}}(A^{\nu}x,x)^{\frac{1}{2}}}{(Ax,x)^{\frac{\mu+\nu}{2}}} \geq \frac{(A^{\frac{\mu+\nu}{2}}x,x)}{(Ax,x)^{\frac{\mu+\nu}{2}}},$$

that is, $f(\frac{\mu+\nu}{2}) \leq \frac{1}{2}(f(\mu)+f(\nu))$. □

Remark 2.2. It is remarkable that the convexity of $f(\mu)$ implies the Hölder-McCarthy inequality. As a matter of fact, if $x \in \mathcal{H}$ is unit vector, then $f(\mu)$ defined in above satisfies $f(0) = f(1) = 1$. Hence the convexity of it implies the Hölder-McCarthy inequality (iii) and (iv).

Next we propose a refinement of the Hölder-McCarthy inequality:

Theorem 2.3. *Let $A \geq 0$, $\|x\| = 1$ and $\lambda \geq 1$. Then*

$$m(\mu,\nu)\left(1-\left(\frac{(A^\nu x,x)}{(Ax,x)^\nu}\right)^\lambda\right) \leq 1-\left(\frac{(A^\mu x,x)}{(Ax,x)^\mu}\right)^\lambda \leq M(\mu,\nu)\left(1-\left(\frac{(A^\nu x,x)}{(Ax,x)^\nu}\right)^\lambda\right)$$

hold for $\mu,\nu \in (0,1)$, where $m(\mu,\nu) = \min\{\frac{1-\mu}{1-\nu},\frac{\mu}{\nu}\}$ and $M(\mu,\nu) = \max\{\frac{1-\mu}{1-\nu},\frac{\mu}{\nu}\}$. Moreover two inequalities in above are equivalent.

Proof. It follows from the preceding theorem that $f^\lambda(\mu)$ is a convex function by $\lambda \geq 1$.

If $\nu \geq \mu$, then we have

$$\frac{f^\lambda(\mu)-f^\lambda(0)}{\mu-0} \leq \frac{f^\lambda(\nu)-f^\lambda(0)}{\nu-0},$$

that is,

$$1-f^\lambda(\mu) \geq \frac{\mu}{\nu}(1-f^\lambda(\nu)).$$

Next, if $\mu \geq \nu$, then we have

$$\frac{f^\lambda(1)-f^\lambda(\mu)}{1-\mu} \geq \frac{f^\lambda(1)-f^\lambda(\nu)}{1-\nu},$$

that is,

$$1-f^\lambda(\mu) \geq \frac{1-\mu}{1-\nu}(1-f^\lambda(\nu)).$$

Hence the first inequality is proved. Finally, the equivalence between two inequalities is ensured by permuting μ and ν. Actually, if we do in the first inequality, then we have the second one by $\max\{a,b\} = [\min\{\frac{1}{a},\frac{1}{b}\}]^{-1}$ for $a,b > 0$; the converse is shown by the same way. □

We here discuss the previous result under the case $\lambda \in (0,1]$.

Theorem 2.4. *Let $A \geq 0$, $\|x\| = 1$ and $0 < \lambda \leq 1$. If $1 \geq \nu \geq \mu > 0$, then*

$$1-\left(\frac{(A^\mu x,x)}{(Ax,x)^\mu}\right)^\lambda \geq \frac{\mu}{\nu}\left(1-\left(\frac{(A^\nu x,x)}{(Ax,x)^\nu}\right)^\lambda\right).$$

Proof. It follows from the arithmetic-geometric mean inequality that

$$1-\frac{\mu}{\nu}+\frac{\mu}{\nu}\left(\frac{(A^\nu x,x)}{(Ax,x)^\nu}\right)^\lambda \geq \left(\frac{(A^\nu x,x)}{(Ax,x)^\nu}\right)^{\lambda\cdot\frac{\mu}{\nu}} = \left(\frac{(A^\nu x,x)^{\frac{\mu}{\nu}}}{(Ax,x)^{\nu\frac{\mu}{\nu}}}\right)^\lambda \geq \left(\frac{(A^\mu x,x)}{(Ax,x)^\mu}\right)^\lambda$$

by $\frac{\mu}{\nu} \in (0,1)$. □

3. Hölder-McCarthy inequality and Young inequality

We first give an elementary proof to the following known refinement of the Young inequality

Theorem 3.1. *Let $A \geq 0$ and $0 \leq \mu, \nu \leq 1$, and $m(\mu,\nu)$ and $M(\mu,\nu)$ be as in Theorem 2.3. Then*

$$m(\mu,\nu)(\nu A + 1 - \nu - A^{\nu}) \leq \mu A + 1 - \mu - A^{\mu} \leq M(\mu,\nu)(\nu A + 1 - \nu - A^{\nu}).$$

Moreover, two inequalities in above are equivalent.

Proof. It is sufficient to prove the numerical case for the left hand side, i.e.,

$$\mu a + 1 - \mu - a^{\mu} \geq m(\mu,\nu)(\nu a + 1 - \nu - a^{\nu}) \quad \text{for } a > 0.$$

If $\mu \geq \nu$, then $\frac{1-\mu}{1-\nu} \leq 1$ and $\frac{\mu-\nu}{1-\nu} + \frac{1-\mu}{1-\nu} = 1$ and so

$$\begin{aligned}\mu a + 1 - \mu - &\frac{1-\mu}{1-\nu}(\nu a + 1 - \nu - a^{\nu}) \\ &= \mu a - \frac{\nu(1-\mu)}{1-\nu}a + \frac{1-\mu}{1-\nu}a^{\nu} \\ &= \frac{\mu-\nu}{1-\nu}a + \frac{1-\mu}{1-\nu}a^{\nu} \\ &\geq a^{\frac{\mu-\nu}{1-\nu}} a^{\frac{\nu(1-\mu)}{1-\nu}} = a^{\mu}.\end{aligned}$$

If $\nu \geq \mu$, then

$$\mu a + 1 - \mu - \frac{\mu}{\nu}(\nu a + 1 - \nu - a^{\nu}) = 1 - \frac{\mu}{\nu} + \frac{\mu}{\nu}a^{\nu} \geq a^{\mu}.$$

Hence we have the first inequality.

The second inequality and the equivalence between two inequalities are obtained by $\max\{a,b\} = [\min\{\frac{1}{a},\frac{1}{b}\}]^{-1}$ for $a, b > 0$, as in the proof of Theorem 2.3. □

Finally, we discuss the equivalence between refined Hölder-McCarthy inequality and refined Young inequality, which is analogous to the result in [3].

Theorem 3.2. *Refined Hölder-McCarthy inequality and refined Young inequality are equivalent, i.e.,*

$$(1) \qquad 1 - \frac{(A^{\mu}x,x)}{(Ax,x)^{\mu}} \geq m(\mu,\nu)\left(1 - \frac{(A^{\nu}x,x)}{(Ax,x)^{\nu}}\right) \quad \textit{for unit vectors } x,$$

$$(2) \qquad \mu A + 1 - \mu - A^{\mu} \geq m(\mu,\nu)(\nu A + 1 - \nu - A^{\nu})$$

are equivalent for given $\mu, \nu \in (0,1)$, where $m(\mu,\nu)$ is as in Theorem 2.3.

Proof. Assume that (1) holds and x is a unit vector. If $\nu \geq \mu$, then we have

$$\begin{aligned}\mu(Ax,x) + 1 - \mu &- \frac{\mu}{\nu}(\nu(Ax,x) + 1 - \nu - (A^{\nu}x,x)) \\ &= \frac{\nu-\mu}{\nu} + \frac{\mu}{\nu}(A^{\nu}x,x) \geq (A^{\nu}x,x)^{\frac{\mu}{\nu}} \geq (A^{\mu}x,x)\end{aligned}$$

by the (classical) Young inequality and Hölder-McCarthy inequality.

If $\mu \geq \nu$, then

$$\mu(Ax,x)+1-\mu-\frac{1-\mu}{1-\nu}(\nu(Ax,x)+1-\nu-(A^\nu x,x))$$
$$=\left(\left(\frac{\mu-\nu}{1-\nu}A+\frac{1-\mu}{1-\nu}A^\nu\right)x,x\right)\geq (A^{\frac{\mu-\nu}{1-\nu}}A^{\frac{\nu(1-\mu)}{1-\nu}}x,x)=(A^\mu x,x).$$

For the reverse implication (2) $\Rightarrow$ (1), we replace A by kA in (2) where $k = (Ax,x)^{-1}$. Thus we have

$$\mu(Ax,x)^{-1}(Ax,x)+1-\mu-(Ax,x)^{-\mu}(A^\mu x,x)$$
$$\geq m(\mu,\nu)(\nu(Ax,x)^{-1}(Ax,x)+1-\nu-(Ax,x)^{-\nu}(A^\nu x,x)),$$

which is just arranged as (1), i.e.,

$$1-\frac{(A^\mu x,x)}{(Ax,x)^\mu}\geq m(\mu,\nu)\left(1-\frac{(A^\nu x,x)}{(Ax,x)^\nu}\right).$$

□

Note. This paper is based on our recent work [7].

References

[1] H. Alzer, C. M. da Fonseca and A.Kovačec, *Young-type inequalities and their matrix analogues*, Linear Multilinear Algebra, **63** (2015), 622-635.

[2] T. Furuta, *Invitation to Linear Operators*, Taylor & Francis, 2001.

[3] T. Furuta, *The Hölder-McCarthy and Young inequalities are equivalent for Hilbert space operators*, Amer. Math. Monthly, **108** (2001), 68-69.

[4] F. Kittaneh and Y. Manasrah, *Improved Young and Heinz inequalities for matrices*, J. Math. Anal. Appl., **361** (2010), 262-269.

[5] Y. Manasrah and F. Kittaneh, *A generalization of two refined Young inequalities*, Positivity, **19** (2015), 757-768.

[6] C. A. McCarthy, C_p, Israel J. Math., **5** (1967), 249-271.

[7] M. Fujii and R. Nakamoto, *Refinements of Hölder-McCarthy inequalities and Young inequality*, Adv. Oper. Theory, **1** (2016), 184-188.

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