

# Quasi-metrics on the function spaces $L_p(0 \leq p \leq \infty)$ for a fuzzy measure

Aoi Honda, Kyushu Institute of Technology  
and  
Yoshiaki Okazaki, Fuzzy Logic Systems Institute

## Abstract

We introduce the function spaces  $L_p(0 \leq p \leq \infty)$  and the natural quasi-metrics on them based on the Sugeno integral for a fuzzy measure. It is proved that for every  $0 < p, q < \infty$ ,  $L_p = L_q$  holds.

## 1 Introduction

We study the metric and quasi-metric structures of the function spaces  $L_p$  for a fuzzy measure. The definition of  $L_p$  for  $0 < p < \infty$  depends on the integration for a fuzzy measure. In fact, many definitions of the fuzzy integrals are proposed such as the Choquet integral[1], Sugeno integral[12, 18], Shilkret integral[17], Imaoka integral[9], pan-integral[14], Lehrer integral(concave-integral)[11], convex-integral[10], and so on. In this note, we shall consider  $L_p(0 < p < \infty)$  spaces for the Sugeno integral. In the case where  $p = 1, \infty$ , the function spaces  $L_0, L_\infty$  are defined without utilizing the fuzzy integral.

**Definition 1** [3, 4, 5, 7, 8, 16, 19] Let  $T$  be a set. A function  $\rho(s, t) : T \times T \rightarrow [0, +\infty)$  is called a quasi-metric

$\iff$

1.  $\rho(s, t) \geq 0, \quad \rho(s, t) = 0 \iff s = t, \quad s, t \in T$
2.  $\rho(s, t) = \rho(t, s), \quad s, t \in T, \quad \text{and}$
3.  $\exists K \geq 1 ; \rho(s, t) \leq K(\rho(s, u) + \rho(u, t)), \quad s, t, u \in T.$

In the case where  $K = 1$ , then  $\rho$  is called a metric.

**Definition 2** [1, 2, 12, 15, 18] Let  $(X, \mathcal{B}(X))$  be a measure space on a set  $X$ , that is  $\mathcal{B}(X)$  is a  $\sigma$ -algebra on  $X$ . A set function  $\mu : \mathcal{B}(X) \rightarrow [0, +\infty]$  is called a fuzzy measure

$\iff$

1.  $\mu(\emptyset) = 0,$
2.  $A \subset B, A, B \in \mathcal{B}(X) \Rightarrow \mu(A) \leq \mu(B).$

A fuzzy measure  $\mu$  is called subadditive(or  $\mu$  is a submeasure)

$\iff$

$$\mu(A \cup B) \leq \mu(A) + \mu(B) \text{ for every } A, B \in \mathcal{B}(X).$$

A fuzzy measure  $\mu$  is called weakly subadditive

$\iff$

$\exists k \geq 1 ; \mu(A \cup B) \leq \mu(A) + k\mu(B)$  for every  $A, B \in \mathcal{B}(X)$ .

A fuzzy measure  $\mu$  is said to be continuous from below

$\iff$

$\mu(A_n) \uparrow \mu(A)$  for any  $A_n, A \in \mathcal{B}(X)$  such that  $A_n \uparrow A$ .

A fuzzy measure  $\mu$  is said to be continuous from above

$\iff$

$\mu(B_n) \downarrow \mu(B)$  for any  $B_n, B \in \mathcal{B}(X)$  such that  $B_n \downarrow B$  with  $\mu(B_1) < +\infty$ .

**Definition 3** [12, 18] A set  $N \in \mathcal{B}(X)$  is called the strongly null set

$\iff$

$\mu(A \cup N) = \mu(A)$  for every  $A \in \mathcal{B}(X)$ .

**Lemma 4** [12, 18] Assume  $\mu$  is subadditive or weakly subadditive. Then  $N$  is a strongly null set if and only if  $\mu(N) = 0$ .

A function  $f : (X, \mathcal{B}(X)) \rightarrow (-\infty, +\infty)$  is called measurable if for every real number  $r$ , it holds that  $\{f > r\} := \{x \in X \mid f(x) > r\} \in \mathcal{B}(X)$ .

Let  $\mu : (X, \mathcal{B}(X)) \rightarrow [0, +\infty]$  be a fuzzy measure and  $f : (X, \mathcal{B}(X)) \rightarrow [0, +\infty)$  be a non-negative measurable function. Then the Sugeno integrals[1, 18] of  $f$  with respect to  $\mu$  are defined by

$$(Su) \int_X f d\mu := \sup_{r \geq 0} r \wedge \mu(\{f > r\}).$$

## 2 The space $L_0$ of all real measurable functions

Let  $(X, \mathcal{B}(X))$  be a measure space and  $\mu$  is a fuzzy measure on  $(X, \mathcal{B}(X))$ . Denote by  $\mathcal{L}_0$  the set of all real valued measurable functions  $f : (X, \mathcal{B}(X)) \rightarrow (-\infty, +\infty)$ . For  $f, g \in \mathcal{L}_0$ , we set

$$d_0(f, g) = \inf_{r > 0} \arctan \{r + \mu(\{x \in X \mid |f(x) - g(x)| > r\})\}.$$

**Lemma 5** Assume  $\mu$  is weakly subadditive. Then  $d_0(f, g)$  is a translation invariant quasi-metric on  $\mathcal{L}_0$ . Furthermore if  $\mu$  is subadditive, then  $d_0(f, g)$  is a metric.

Remark that for  $h \in \mathcal{L}_0$ ,  $d_0(h, 0) = 0$  if and only if  $\mu(|f| > r) = 0$  for every  $r > 0$ . So that we have the following lemma.

**Lemma 6** Assume  $\mu$  is weakly subadditive. If  $d_0(h, 0) = 0$ , then  $d_0(f \pm h, g) = d_0(f, g)$  for every  $f, g \in \mathcal{L}_0$ .

We set

$$\begin{aligned}\mathcal{N}_0 &= \{h \in \mathcal{L}_0 \mid d_0(h, 0) = 0\}, \text{ and} \\ \mathcal{L}_0 &= \mathcal{L}_0/\mathcal{N}_0.\end{aligned}$$

By the above lemma,  $d_0$  induces naturally the quasi-metric  $\tilde{d}_0$  on  $L_0$  by

$$\tilde{d}_0(f + \mathcal{N}_0, g + \mathcal{N}_0) = d_0(f, g).$$

In the sequel we shall identify the equivalence class  $f + \mathcal{N}_0$  with  $f$ , and also the quasi-metric  $\tilde{d}_0$  with  $d_0$ .

### 3 Sugeno $L_p$ space $L_p(Su)$ ( $0 < p < \infty$ )

In this section we introduce the  $L_p$  space  $L_p(Su)$  with respect to the Sugeno integral for  $0 < p < \infty$ . We shall show that if  $\mu$  is weakly subadditive, then  $L_p(Su)$  is a quasi-metric space. Furthermore for every  $0 < p, q < \infty$ , we have  $L_p(Su) = L_q(Su)$  as a set and the quasi-metrics on these two spaces define the same topology.

Let  $p$  be  $0 < p < \infty$ . For a measurable function  $f : (X, \mathcal{B}) \rightarrow (-\infty, +\infty)$ , we set

$$\begin{aligned}|f|_p &= \left[ \sup_{r \geq 0} r \wedge \mu(|f|^p > r) \right]^{\frac{1}{p}}, \\ \mathcal{L}_p &= \{f \mid |f|_p < +\infty\}, \text{ and} \\ \mathcal{O}_p &= \{f \in \mathcal{L}_p \mid |f|_p = 0\}.\end{aligned}$$

**Lemma 7** *Assume that  $\mu$  is a weakly subadditive fuzzy measure. Then we have for  $p \geq 1$*

$$|f + g|_p \leq 2k^{\frac{1}{p}} (|f|_p + |g|_p), \quad f, g \in \mathcal{L}_p,$$

and for  $0 < p \leq 1$

$$|f + g|_p \leq (2k)^{\frac{1}{p}} (|f|_p + |g|_p), \quad f, g \in \mathcal{L}_p.$$

**Proof**

$$\begin{aligned}
|f + g|_p^p &= \sup_{r \geq 0} r \wedge \mu(|f + g|^p > r) = \sup_{r \geq 0} r \wedge \mu\left(|f + g| > r^{\frac{1}{p}}\right) \\
&\leq \sup_{r \geq 0} r \wedge \mu\left(\left\{|f| > \frac{1}{2}r^{\frac{1}{p}}\right\} \cup \left\{|g| > \frac{1}{2}r^{\frac{1}{p}}\right\}\right) \\
&\leq \sup_{r \geq 0} r \wedge \left[\mu\left(|f| > \frac{1}{2}r^{\frac{1}{p}}\right) + k\mu\left(|g| > \frac{1}{2}r^{\frac{1}{p}}\right)\right] \\
&\leq \left[\sup_{r \geq 0} r \wedge \mu\left(|f| > \frac{1}{2}r^{\frac{1}{p}}\right) + \sup_{r \geq 0} r \wedge k\mu\left(|g| > \frac{1}{2}r^{\frac{1}{p}}\right)\right] \\
&= \left[\sup_{r \geq 0} \left(2^p \frac{r}{2^p}\right) \wedge \mu\left(|f|^p > \frac{r}{2^p}\right) + \sup_{r \geq 0} \left(2^p \frac{r}{2^p}\right) \wedge k\mu\left(|g|^p > \frac{r}{2^p}\right)\right] \\
&\leq 2^p \left[\sup_{r \geq 0} \left(\frac{r}{2^p}\right) \wedge \mu\left(|f|^p > \frac{r}{2^p}\right) + \sup_{r \geq 0} \left(\frac{r}{2^p}\right) \wedge k\mu\left(|g|^p > \frac{r}{2^p}\right)\right] \\
&\leq 2^p [|f|_p^p + k|g|_p^p] \leq 2^p k [|f|_p^p + |g|_p^p],
\end{aligned}$$

where we have used the inequality  $a \wedge (b + c) \leq a \wedge b + a \wedge c$ . So that we have for  $p \geq 1$ ,

$$|f + g|_p \leq 2k^{\frac{1}{p}} (|f|_p + |g|_p),$$

and for  $0 < p \leq 1$ ,

$$|f + g|_p \leq 2k^{\frac{1}{p}} 2^{\frac{1}{p}-1} (|f|_p + |g|_p) = (2k)^{\frac{1}{p}} (|f|_p + |g|_p).$$

**Lemma 8** We have

$$|cf|_p \leq \text{Max}\{|c|, 1\} |f|_p \text{ for real number } c \text{ and } f \in \mathcal{L}_p.$$

**Proof**

$$\begin{aligned}
|cf|_p^p &= \sup_{r \geq 0} r \wedge \mu(|cf|^p > r) \\
&= \sup_{r \geq 0} r \wedge \mu\left(|f|^p > \frac{r}{|c|^p}\right) \\
&= \sup_{r \geq 0} \left(|c|^p \frac{r}{|c|^p}\right) \wedge \mu\left(|f|^p > \frac{r}{|c|^p}\right).
\end{aligned}$$

If  $|c| > 1$ , then we have

$$\sup_{r \geq 0} \left(|c|^p \frac{r}{|c|^p}\right) \wedge \mu\left(|f|^p > \frac{r}{|c|^p}\right) \leq |c|^p \sup_{r \geq 0} \left(\frac{r}{|c|^p}\right) \wedge \mu\left(|f|^p > \frac{r}{|c|^p}\right) = |c|^p |f|_p^p.$$

If  $|c| \leq 1$ , then we have

$$\sup_{r \geq 0} \left( |c|^p \frac{r}{|c|^p} \right) \wedge \mu \left( |f|^p > \frac{r}{|c|^p} \right) \leq \sup_{r \geq 0} \left( \frac{r}{|c|^p} \right) \wedge \mu \left( |f|^p > \frac{r}{|c|^p} \right) = |f|_p^p.$$

So that we have the assertion.

**Lemma 9** Assume that  $h \in \mathcal{O}_p$ , that is,  $|h|_p = 0$ . Then we have

$$\mu(|h| > r) = 0 \quad \text{for every } r > 0.$$

In particular, if  $\mu$  is continuous from below then  $h = 0$   $\mu$ -almost everywhere, that is

$$\mu(|h| > 0) = 0.$$

**Proof** By the definition of  $|\cdot|_p$  we have the assertion.

**Lemma 10** Let  $\mu$  be a weakly subadditive fuzzy measure. Then we have

$$|f \pm h|_p = |f|_p$$

for every  $f \in \mathcal{L}_p$  and  $h \in \mathcal{O}_p$ .

**Proof** By Lemma 4 and Lemma 9, for every  $r \geq 0$  it follows that the subset  $N(r) := \{|h| > r\}$  is a null set. Let  $0 < \varepsilon < 1$  be arbitrarily fixed. Then we have

$$\begin{aligned} \mu(|f \pm h|^p > r) &= \mu(|f \pm h| > r^{\frac{1}{p}}) \\ &\leq \mu\left(\left(\{|f \pm h| > r^{\frac{1}{p}}\} \cap N(\varepsilon r^{\frac{1}{p}})^c\right) \cup N(\varepsilon r^{\frac{1}{p}})\right) \\ &= \mu\left(\{|f \pm h| > r^{\frac{1}{p}}\} \cap N(\varepsilon r^{\frac{1}{p}})^c\right) \\ &= \mu(|f \pm h| > r^{\frac{1}{p}}, |h| \leq \varepsilon r^{\frac{1}{p}}) \\ &\leq \mu(|f| > (1 - \varepsilon)r^{\frac{1}{p}}) \\ &= \mu(|f|^p > (1 - \varepsilon)^p r). \end{aligned}$$

So that we have

$$\begin{aligned} r \wedge \mu(|f \pm h|^p > r) &\leq r \wedge \mu(|f|^p > (1 - \varepsilon)^p r) \\ &\leq \frac{1}{(1 - \varepsilon)^p} [(1 - \varepsilon)^p r \wedge \mu(|f|^p > (1 - \varepsilon)^p r)] \\ &\leq \frac{1}{(1 - \varepsilon)^p} |f|_p^p. \end{aligned}$$

Taking  $\sup_{r \geq 0}$  in the left hand side, we have

$$|f \pm h|_p^p \leq \frac{1}{(1 - \varepsilon)^p} |f|_p^p.$$

Letting  $\varepsilon \downarrow 0$ , we have the assertion.

**Definition 11** Let  $\mu$  be a weakly subadditive fuzzy measure. We set

$$L_p := \mathcal{L}_p / \mathcal{O}_p$$

$$\|f + \mathcal{O}_p\|_p := |f|_p \text{ for } f + \mathcal{O}_p \in L_p.$$

By Lemma 10, the value  $\|f + \mathcal{O}_p\|_p$  does not depend on the choice of the representative  $f$  of the equivalence class  $f + \mathcal{O}_p$ . In the sequel we identify the equivalence class  $f + \mathcal{O}_p$  with  $f$  and write

$$\|f\|_p = |f + \mathcal{O}_p|_p \text{ for } f \in L_p.$$

Then  $\|f\|_p$  determines a translation invariant quasi-metric on  $L_p$  as follows.

**Theorem 12** Let  $\mu$  be a weakly subadditive fuzzy measure. Then the space  $(L_p, \|f\|_p)$  is a linear space. The function  $\gamma_p(f, g) := \|f - g\|_p$  is a quasi-metric satisfying :

1.  $\gamma_p(cf, 0) \leq \text{Max}\{|c|, 1\} \gamma_p(f, 0)$  for a real number  $c$  and  $f \in L_p$ ,

2. in the case where  $p \geq 1$ ,

$$\gamma_p(f, g) \leq 2k^{\frac{1}{p}} (\gamma_p(f, h) + \gamma_p(h, g)) \text{ for } f, g, h \in L_p,$$

3. in the case where  $0 < p \leq 1$ ,

$$\gamma_p(f, g) \leq (2k)^{\frac{1}{p}} (\gamma_p(f, h) + \gamma_p(h, g)) \text{ for } f, g, h \in L_p,$$

4.  $\gamma_p(f + h, g + h) = \gamma_p(f, g)$  for  $f, g, h \in L_p$  (translation invariance of  $\gamma$ ).

**Proof** The assertions 1 and 2 follow from Lemma 8 and Lemma 7. The translation invariance is clear.

**Definition 13** We say the pair  $(L_p, \|f\|_p)$  the Sugeno  $L_p$  space and denote it by  $L_p(Su)$ .

**Remark 14**  $L_p(Su)$  is a topological additive group but not necessarily a topological linear space. The linear topological structure of  $L_p(Su)$  shall be studied in [7, 13].

**Theorem 15** For every  $0 < p, q < \infty$ , we have  $L_p = L_q$ . Furthermore  $\gamma_p$  and  $\gamma_q$  determine the same topology.

**Proof** We shall prove  $L_1(Su) = L_p(Su)$  set theoretically and topologically. By the definitions, we have

$$\|f\|_p^p = \sup_{r \geq 0} r \wedge \mu(|f|^p > r) = \sup_{r \geq 0} r^p \wedge \mu(|f| > r)$$

and

$$\|f\|_1 = \sup_{r \geq 0} r \wedge \mu(|f| > r).$$

By the inequality  $a^p \wedge b \leq (a \wedge b)^p + a \wedge b$ , we have

$$\|f\|_p^p \leq \|f\|_1^p + \|f\|_1,$$

which shows  $L_1(Su) \subset L_p(Su)$  and the identity map  $i : L_1(Su) \rightarrow L_p(Su)$  is continuous.

Conversely by the inequality  $(a \wedge b)^p \leq (a^p \wedge b)^p + a^p \wedge b$ , we have

$$\|f\|_1^p \leq \|f\|_p^{p^2} + \|f\|_p^p.$$

**Remark 16**  $L_1(Su)$  is also realized as a truncated  $L_\infty$  space  $M_\infty$  and we have  $L_1(Su) \neq L_0$  in general. If  $\mu$  is weakly subadditive, continuous from above and  $\mu(X) < +\infty$ , then we have  $L_1 = L_0$ , see [7, 13].

## 4 $L_\infty$ space

A measurable function  $f$  is called essentially bounded if

$$\exists \alpha \geq 0 ; \mu(|f| > \alpha) = 0.$$

For an essentially bounded function  $f$ , we set

$$|f|_\infty = \inf\{\alpha \geq 0 \mid \mu(|f| > \alpha) = 0\}.$$

**Proposition 17** Assume  $\mu$  is continuous from below. Then the following conditions are equivalent.

$$|f|_\infty = a$$

$$\iff$$

- (1)  $\mu(|f| > a) = 0$  and
- (2) for every  $b < a$ ,  $\mu(|f| > b) > 0$ .

**Proof**  $(\Leftarrow)$  is clear.

$(\Rightarrow)$

Assume  $\mu(|f| > a) > 0$ . Then we have  $\{|f| > a + \frac{1}{n}\} \uparrow \{|f| > a\}$ , so that  $\mu(\{|f| > a + \frac{1}{n}\}) \uparrow \mu(\{|f| > a\}) > 0$  by the continuity from below. Therefore there exists  $n_0$  such that  $\mu(\{|f| > a + \frac{1}{n_0}\}) > 0$ , which implies  $|f|_\infty \geq a + \frac{1}{n_0} > a$ . This contradicts to  $|f|_\infty = a$ .

If there exists  $b < a$  satisfying  $\mu(|f| > b) = 0$ , then it must be  $|f|_{\infty \text{leb}} < a$ , which also contradicts to  $|f|_\infty = a$ .

We set

$$\mathcal{L}_\infty = \{f \mid f \text{ is essentially bounded}\}, \text{ and}$$

$$\mathcal{O}_\infty = \{f \in \mathcal{L}_\infty \mid |f|_\infty = 0\}.$$

**Lemma 18** *If  $h \in \mathcal{O}_\infty$  then for every  $\alpha > 0$  it follows that  $\mu(|h| > \alpha) = 0$ .*

**Lemma 19** *Assume that  $\mu$  is weakly subadditive. Let  $f \in \mathcal{L}_\infty$  and  $h \in \mathcal{O}_\infty$ . Then we have  $|f \pm h|_\infty = |f|_\infty$ .*

**Proof** *Assume  $|f|_\infty = a$ . Let  $\varepsilon > 0$  and  $\alpha > a$  be arbitrarily fixed. We have*

$$\{|f \pm h| > \alpha + \varepsilon\} \subset \{|f| > \alpha\} \cup \{|h| > \varepsilon\}.$$

*Since  $\{|h| > \varepsilon\}$  is a strongly null set, we have*

$$\mu(\{|f \pm h| > \alpha + \varepsilon\}) \leq \mu(\{|f| > \alpha\}) = 0.$$

*So that we have*

$$|f \pm h|_\infty \leq \alpha + \varepsilon$$

*for every  $\varepsilon > 0$  and  $\alpha > a$ , which implies  $|f \pm h|_\infty \leq a = |f|_\infty$ .*

*Conversely, assume that  $|f \pm h|_\infty = b$ . Let  $\varepsilon > 0$  and  $\beta > b$  be arbitrarily fixed. We have*

$$\{|f| > \beta + \varepsilon\} \subset \{|f \pm h| > \beta\} \cup \{|h| > \varepsilon\}.$$

*Since  $\{|h| > \varepsilon\}$  is a strongly null set, we have*

$$\mu(\{|f| > \beta + \varepsilon\}) \leq \mu(\{|f \pm h| > \beta\}) = 0.$$

*So that we have*

$$|f|_\infty \leq \beta + \varepsilon$$

*for every  $\varepsilon > 0$  and  $\beta > b$ , which implies  $|f|_\infty \leq b = |f \pm h|_\infty$ .*

**Definition 20** *Let  $\mu$  be a weakly subadditive fuzzy measure. We set*

$$L_\infty := \mathcal{L}_\infty / \mathcal{O}_\infty$$

$$\|f + \mathcal{O}_\infty\|_\infty := |f|_\infty \text{ for } f + \mathcal{O}_\infty \in L_\infty.$$

By the preceding Lemma, the value  $\|f + \mathcal{O}_\infty\|_\infty$  does not depend on the choice of the representative  $f$  of the equivalence class  $f + \mathcal{O}_\infty$ . In the sequel we identify the equivalence class  $f + \mathcal{O}_\infty$  with  $f$  and write

$$\|f\|_\infty = |f + \mathcal{O}_\infty|_\infty \text{ for } f \in L_\infty.$$

Then  $\|f\|_\infty$  determines a norm on  $L_\infty$ .

**Theorem 21**  *$\|f\|_\infty$  is a norm on  $L_\infty$ .*



**Proof** For every  $\varepsilon > 0$ , we have

$$\mu(|f| > \|f\|_\infty + \varepsilon) = 0, \quad \mu(|g| > \|g\|_\infty + \varepsilon) = 0.$$

Since  $\{|f+g| > \|f\|_\infty + \|g\|_\infty + 2\varepsilon\} \subset \{|f| > \|f\|_\infty + \varepsilon\} \cup \{|g| > \|g\|_\infty + \varepsilon\}$ , by the weak subadditivity of  $\mu$ , we have

$$\mu(|f+g| > \|f\|_\infty + \|g\|_\infty + 2\varepsilon) \leq \mu(|f| > \|f\|_\infty + \varepsilon) + k\mu(|g| > \|g\|_\infty + \varepsilon) = 0,$$

which implies

$$\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty + 2\varepsilon.$$

Letting  $\varepsilon \downarrow 0$  we have the triangle inequality.

We show  $\|cf\|_\infty = |c| \cdot \|f\|_\infty$ . Remark that  $\mu(|f| > \alpha) = 0$  if and only if  $\mu(|cf| > |c|\alpha) = 0$ . This implies  $\{|c|\alpha \mid \mu(|f| > \alpha) = 0\} = \{\beta \mid \mu(|cf| > \beta) = 0\}$ . Consequently it follows that

$$|c| \cdot \|f\|_\infty = \inf\{|c|\alpha \mid \mu(|f| > \alpha) = 0\} = \inf\{\beta \mid \mu(|cf| > \beta) = 0\} = \|cf\|_\infty.$$

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**Acknowledgments.** This work was supported by JSPS KAKENHI Grant number 15K05003 and 26400155.

Adresses:

Aoi Honda

Department of Computer Science and Systems Engineering, Kyushu Institute of Technology, Kawazu 680-4, Iizuka, 820 JAPAN

E-mail: aoi@ces.kyutech.ac.jp

Yoshiaki Okazaki

Fuzzy Logic Systems Institute, Kawazu 680-41, Iizuka, 820 JAPAN

E-mail: okazaki@flsi.or.jp