数理解析研究所講究録
第2041巻 2017年 220-226

Fractional integrals on martingale spaces

茨城大学　理学部　中井 英一 (Eiichi Nakai)
Department of Mathematics,
Ibaraki University
大阪教育大学　教育学部　貞末 岳 (Gaku Sadasue)
Department of Mathematics,
Osaka Kyoiku University

1 Introduction

In this paper, we review known results on fractional integrals of martingales and state some new results. We introduce commutator of fractional integral of martingales, and state a characterization of Lipschitz martingales by boundedness of these commutators on martingale Morrey spaces. We also state a property of sharp functions on martingale Morrey spaces. This paper is an announcement of the authors' recent results [11].

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{\mathcal{F}_n\}_{n\geq 0}$ be a nondecreasing sequence of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma(\bigcup_n \mathcal{F}_n)$. We suppose that every σ-algebra $\mathcal{F}_n$ is generated by countable atoms, where $B \in \mathcal{F}_n$ is called an atom (more precisely a $(\mathcal{F}_n, P)$-atom), if any $A \subset B$ with $A \in \mathcal{F}_n$ satisfies $P(A) = P(B)$ or $P(A) = 0$. Denote by $A(\mathcal{F}_n)$ the set of all atoms in $\mathcal{F}_n$. We also suppose that $(\Omega, \mathcal{F}, P)$ is non-atomic.

The expectation operator is denoted by E. Let $L_{p,\mathrm{loc}}$ be the set of all measurable functions such that $|f|^p\chi_B$ is integrable for all $B \in A(\mathcal{F}_0)$. If $\mathcal{F}_0 = \{\Omega, \emptyset\}$,

2000 *Mathematics Subject Classification.* Primary 46E30, 60G46; Secondary 42B35, 26A33.
Key words and phrases. martingale, fractional integral, commutator, Morrey-Campanato space.
The first author was supported by Grant-in-Aid for Scientific Research (B), No. 15H03621, Japan Society for the Promotion of Science. The second author was supported by Grant-in-Aid for Scientific Research (C), No. 16K05203, Japan Society for the Promotion of Science.

then $L_{p,\mathrm{loc}} = L_p$. An $\mathcal{F}_n$-measurable function $g \in L_{1,\mathrm{loc}}$ is called the conditional expectation of $f \in L_{1,\mathrm{loc}}$ relative to $\mathcal{F}_n$ if

$$E[g\chi_B\chi_G] = E[f\chi_B\chi_G] \quad \text{for all} \quad B \in A(\mathcal{F}_0) \quad \text{and} \quad G \in \mathcal{F}_n.$$

We denote by $E_n f$ the conditional expectation of f relative to $\mathcal{F}_n$. We say a sequence $(f_n)_{n\ge 0}$ in $L_{1,\mathrm{loc}}$ is a martingale relative to $\{\mathcal{F}_n\}_{n\ge 0}$ if it is adapted to $\{\mathcal{F}_n\}_{n\ge 0}$ and satisfies $E_n[f_m] = f_n$ for every $n \le m$.

2 Definitions, notation and known results

In this section, we give definitions and recall known results.

We first recall the definition of martingale Morrey spaces $L_{p,\lambda}$ and martingale Campanato spaces $\mathcal{L}_{p,\lambda}$.

Definition 2.1. Let $p \in [1,\infty)$ and $\lambda \in (-\infty,\infty)$. For $f \in L_{1,\mathrm{loc}}$, let

$$\|f\|_{L_{p,\lambda}} = \sup_{n\ge 0} \sup_{B\in A(\mathcal{F}_n)} \frac{1}{P(B)^\lambda} \left(\frac{1}{P(B)} \int_B |f|^p \, dP \right)^{1/p},$$

$$\|f\|_{\mathcal{L}_{p,\lambda}} = \sup_{n\ge 0} \sup_{B\in A(\mathcal{F}_n)} \frac{1}{P(B)^\lambda} \left(\frac{1}{P(B)} \int_B |f - E_n f|^p \, dP \right)^{1/p},$$

and define

$$L_{p,\lambda} = \{f \in L_{p,\mathrm{loc}} : \|f\|_{L_{p,\lambda}} < \infty\}, \quad \mathcal{L}_{p,\lambda} = \{f \in L_{p,\mathrm{loc}} : \|f\|_{\mathcal{L}_{p,\lambda}} < \infty\}.$$

Then functionals $\|f\|_{L_{p,\lambda}}$ and $\|f\|_{\mathcal{L}_{p,\lambda}}$ are norms.

We regard martingale BMO spaces and martingale Lipschitz spaces as special classes of martingale Campanato spaces.

Definition 2.2. Let $\mathrm{BMO} = \mathcal{L}_{1,0}$ and $\mathrm{Lip}(\delta) = \mathcal{L}_{1,\delta}$ for $\delta > 0$.

The filtration $\{\mathcal{F}_n\}_{n\ge 0}$ is said to be regular, if there exists a constant $R \ge 2$ such that

$$f_n \le R f_{n-1} \tag{2.1}$$

holds for all nonnegative martingales $(f_n)_{n\ge 0}$.

The following theorem is well-known. See [6], [16] and [7].

Theorem 2.1. *Assume that* $\{\mathcal{F}_n\}_{n\geq 0}$ *is regular. Let* $1 < p < \infty$*. Then,*

$$\|f\|_{\mathrm{BMO}} \sim \|f\|_{\mathcal{L}_{p,0}} \quad \textit{and} \quad \|f\|_{\mathrm{Lip}(\delta)} \sim \|f\|_{\mathcal{L}_{p,\delta}}.$$

Fractional integrals for martingales was first introduced by Chao and Ombe [3] as follows.

Definition 2.3 ([3])**.** Let $\alpha > 0$. For a dyadic martingale $f = (f_n)_{n\geq 0}$, its fractional integral $I_\alpha f = ((I_\alpha f)_n)_{n\geq 0}$ is defined by

$$(I_\alpha f)_n = \sum_{k=0}^{n} 2^{-k\alpha}(f_k - f_{k-1}).$$

Later, I_α is defined for more general martingales. Recall our assumption that every σ-algebra $\mathcal{F}_n$ is generated by countable atoms. In [9], I_α is defined for this case.

Let

$$\beta_n = \sum_{B\in A(\mathcal{F}_n)} P(B)\chi_B, \quad n = 0, 1, 2, \cdots . \tag{2.2}$$

Definition 2.4 ([9])**.** Let $\alpha > 0$. For a martingale $f = (f_n)_{n\geq 0}$, its fractional integral $I_\alpha f = ((I_\alpha f)_n)_{n\geq 0}$ is defined by

$$(I_\alpha f)_n = \sum_{k=0}^{n} \beta_{k-1}^{\alpha}(f_k - f_{k-1}).$$

with convention $\beta_{-1} = \beta_0$ and $f_{-1} = 0$.

In above two definitions, I_α is defined on martingale spaces. In this paper, we define I_α on function spaces.

Definition 2.5. Let $\alpha > 0$. For $f \in L_{1,\mathrm{loc}}$, its fractional integral $I_\alpha f$ with respect to $\{\mathcal{F}_n\}_{n\geq 0}$ is defined by

$$I_\alpha f = \sum_{k=0}^{\infty} \beta_{k-1}^{\alpha}(E_k f - E_{k-1} f) \tag{2.3}$$

with convention $\beta_{-1} = \beta_0$ and $E_{-1} f = 0$.

Remark 2.1. As is shown in [9], the series $\chi_B \sum_{k=0}^{\infty}(\beta_{k-1})^{\alpha}(E_k f - E_{k-1}f)$ converges in L_1 for every $B \in A(\mathcal{F}_0)$ and $f \in L_{1,\mathrm{loc}}$. Moreover,

$$E_n[I_\alpha f] = \sum_{k=0}^{n} \beta_{k-1}^{\alpha}(E_k f - E_{k-1}f).$$

We recall the following result on the boundedness of I_α.

Theorem 2.2. *Assume that $\{\mathcal{F}_n\}_{n\geq 0}$ is regular. Let $1 < p < q < \infty$, $\alpha > 0$ and $-1/p \leq \lambda < 0$. If $\alpha+\lambda < 0$ and $\alpha = 1/p - 1/q$, then there exists a positive constant C depending only on R and α such that*

$$\|I_\alpha f\|_{L_{q,\alpha+\lambda}} \leq C\|f\|_{L_{p,\lambda}}.$$

Remark 2.2. Theorem 2.2 extends [3, Theorem 1] in several ways: from dyadic martingales to more general martingales, from L_p spaces to Morrey spaces.

Further, we recall the definition of generalized fractional integrals of martingales, and the definition of generalized Morrey spaces.

Definition 2.6. Let $(\gamma_n)_{n\geq 0}$ be a non-increasing sequence of non-negative bounded functions adapted to $\{\mathcal{F}_n\}_{n\geq 0}$. For a martingale $(f_n)_{n\geq 0}$, its generalized fractional integral $I_\gamma f = ((I_\gamma f)_n)_{n\geq 0}$ is defined as a martingale by

$$(I_\gamma f)_n = \sum_{k=0}^{n} \gamma_{k-1}(f_k - f_{k-1})$$

with convention $\gamma_{-1} = \gamma_0$ and $f_{-1} = 0$.

Definition 2.7. For $p \in [1,\infty)$ and $\phi : (0,1] \to (0,\infty)$, let

$$L_{p,\phi} = \{f \in L_{p,\mathrm{loc}} : \|f\|_{L_{p,\phi}} < \infty\},$$

where

$$\|f\|_{L_{p,\phi}} = \sup_{n\geq 0} \sup_{B\in A(\mathcal{F}_n)} \frac{1}{\phi(P(B))} \left(\frac{1}{P(B)} \int_B |f|^p \, dP\right)^{1/p}.$$

Boundedness of generalized fractional integrals is studied extensively in [10].

Theorem 2.3 ([10]). *Let $1 < p < q < \infty$ and $\phi : (0,1] \to (0,\infty)$. Assume that ϕ is almost decreasing. If there exists a positive constant C such that*

$$\sum_{k=0}^{n}(\gamma_{k-1} - \gamma_k)\phi(b_k) + \gamma_n\phi(b_n) \leq C\phi(b_n)^{p/q} \quad \textit{for all} \quad n \geq 0 \tag{2.4}$$

with convention $\gamma_{-1} = \gamma_0$, then I_γ is bounded from $L_{p,\phi}$ to $L_{q,\phi^{p/q}}$.

3 Some new properties: fractional maximal functions, sharp functions and commutators.

In this section, we state new properties of fractional maximal functions, sharp functions and commutators. The proofs of these properties will be given in [11].

For $f \in L_1$ its fractional maximal function $M_\alpha f$ is defined by

$$M_\alpha f = \sup_{n\geq 0}(\beta_n)^\alpha |E_n f|. \tag{3.1}$$

Using Theorem 2.2 and the positivity of I_α, we have the following theorem.

Theorem 3.1. *Assume that $\{\mathcal{F}_n\}_{n\geq 0}$ is regular. Let $1 < p < q < \infty$ and $-1/p+\alpha = -1/q$. Then M_α is bounded from $L_{p,\lambda}$ to $L_{q,\alpha+\lambda}$.*

We next recall the definition of sharp functions.

$$M^\sharp f = \sup_{n\geq 0} E_n[|f - E_{n-1}f|]. \tag{3.2}$$

The following theorem is well-known. See [16] and [6].

Theorem 3.2. *Let $1 < p < \infty$. Then, there exists a positive constant C depending only on p such that*

$$\|f\|_{L_p} \leq C\|M^\sharp f\|_{L_p}.$$

Our result on sharp functions is to give an extension of Theorem 3.2 to martingale Morrey spaces.

Theorem 3.3. *Assume that $\{\mathcal{F}_n\}_{n\geq 0}$ is regular. Let $1 < p_0 < p < \infty$ and $-1/p \leq \lambda < 0$. If $Mf \in L_{p_0,\lambda}$, then*

$$\|f\|_{L_{p,\lambda}} \leq C_{p,\lambda,R}\|M^\sharp f\|_{L_{p,\lambda}}, \tag{3.3}$$

where $C_{p,\lambda,R}$ is a positive constant depending only on p, λ and R in (2.1).

To show Theorem 3.3, we use a good λ-inequality which is a martingale version of Komori-Furuya's result in [5].

We now introduce commutators. Let $p \geq 1$ and let p' be the conjugate exponent of p. If $f \in L_{p,\text{loc}}$ and $b \in L_{p',\text{loc}}$, then the commutator

$$[b, I_\alpha]f := bI_\alpha f - I_\alpha(bf)$$

is well-defined.

In [3], Chao and Ombe showed the following characterization theorem for dyadic BMO-martingales.

Theorem 3.4 ([3]). *Let* $1 < p < q < \infty$ *and* $\alpha = 1/p - 1/q$. *Let* I_α *be the fractional integral defined in Defintition 2.3. Then, b belongs to dyadic* BMO *space if and only if the commutator* $[b, I_\alpha]$ *is bounded from* L_p *to* L_q.

We extend Theorem 3.4 to the following theorem.

Theorem 3.5. *Assume that* $\{\mathcal{F}_n\}_{n\geq 0}$ *is regular. Let* $1 < p < q < \infty$, $\alpha = 1/p - 1/q$, $\delta \geq 0$ *and* $\lambda < 0$. *Suppose that* $\delta + \alpha + \lambda < 0$. *Then,* $b \in \mathrm{Lip}(\delta)$, $b \in \mathrm{BMO}$ *when* $\delta = 0$, *if and only if the commutator* $[b, I_\alpha]$ *is bounded from* $L_{p,\lambda}$ *to* $L_{q,\delta+\alpha+\lambda}$.

Remark 3.1. In Theorem 3.5, we extend Theorem 3.4 in several ways. We extend Theorem 3.4 from dyadic martingales to more general martingales, from BMO-martingales to Lipschitz martingales and from L_p spaces to Morrey spaces.

The proof of Theorem 3.5 consists of the use of Theorem 3.3 and some computations. The detailed proof will be given in [11].

References

[1] D. R. Adams, *A note on Riesz potentials*, Duke Math. J. 42 (1975), no. 4, 765–778.

[2] D. L. Burkholder, Martingale transforms. Ann. Math. Stat., 37 (1966), 1494–1504.

[3] J.-A. Chao and H. Ombe, Commutators on Dyadic Martingales, Proc. Japan Acad., 61, Ser. A (1985), 35–38.

[4] F. Chiarenza and M. Frasca, Morrey spaces and Hardy-Littlewood maximal function, Rend. Mat. Appl. (7) 7 (1987), no. 3-4, 273–279.

[5] Y. Komori-Furuya, Local good-λ estimate for the sharp maximal function and weighted Morrey space. J. Funct. Spaces 2015, Art. ID 651825, 4 pp.

[6] R. L. Long, Martingale spaces and inequalities, Peking University Press, Beijing, 1993. ISBN: 7-301-02069-4

[7] T. Miyamoto, E. Nakai and G. Sadasue, Martingale Orlicz-Hardy spaces, Math. Nachr. 285 (2012), no. 5-6, 670-686.

[8] E. Nakai, G. Sadasue and Y. Sawano, Martingale Morrey-Hardy and Campanato-Hardy Spaces, J. Funct. Spaces Appl. 2013 (2013), Article ID 690258, 14 pages. DOI:10.1155/2013/690258

[9] E. Nakai and G. Sadasue, Martingale Morrey-Campanato spaces and fractional integrals, J. Funct. Spaces Appl. 2012 (2012), Article ID 673929, 29 pages. DOI:10.1155/2012/673929

[10] E. Nakai and G. Sadasue, Characterizations of boundedness for generalized fractional integrals on martingale Morrey spaces, to appear in Math. Inequal. Appl.

[11] E. Nakai and G. Sadasue, Commutators of fractional integrals on martingale Morrey spaces, in preparation.

[12] J. Neveu, Discrete-parameter martingales, North-Holland, Amsterdam, 1975. ISBN 0720428106

[13] J. Peetre, *On the theory of $\mathcal{L}_{p,\lambda}$ spaces*, J. Funct. Anal. 4 (1969), 71–87.

[14] C. Watari, Multipliers for Walsh Fourier series. Tohoku Math. J., 16 (1964), 239–251.

[15] F. Weisz, Martingale Hardy spaces for $0 < p \leq 1$. Probab. Theory Related Fields 84 (1990), no. 3, 361–376.

[16] F. Weisz, Martingale Hardy spaces and their applications in Fourier analysis, Lecture Notes in Mathematics, 1568, Springer-Verlag, Berlin, 1994. ISBN: 3-540-57623-1