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Asymptotic stability in a two-species chemotaxis-competition system

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1. Introduction

We consider the two-species chemotaxis system

$$(1.1) \quad \begin{cases} u_t = d_1 \Delta u - \nabla \cdot (u \chi_1(w) \nabla w) + \mu_1 u(1 - u - a_1 v), & x \in \Omega, t > 0, \\ v_t = d_2 \Delta v - \nabla \cdot (v \chi_2(w) \nabla w) + \mu_2 v(1 - a_2 u - v), & x \in \Omega, t > 0, \\ w_t = d_3 \Delta w + h(u, v, w), & x \in \Omega, t > 0, \\ \nabla u \cdot \nu = \nabla v \cdot \nu = \nabla w \cdot \nu = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), w(x, 0) = w_0(x), & x \in \Omega, \end{cases}$$

where Ω is a bounded domain in \mathbb{R}^n ($n \in \mathbb{N}$) with smooth boundary $\partial\Omega$ and ν is the outward normal vector to $\partial\Omega$. The initial data u_0, v_0 and w_0 are assumed to be nonnegative functions. The unknown functions $u(x, t)$ and $v(x, t)$ represent the population densities of two species and $w(x, t)$ shows the concentration of the substance at place x and time t .

The problem (1.1) consists of the influence of chemotaxis, diffusion, and the Lotka-Volterra kinetics. In mathematical view, global existence and behavior of solutions are fundamental theme. In the case $\chi_i(w) = \chi_i$ and $h(u, v, w) = \alpha u + \beta v - \gamma w$, Bai-Winkler [1] considered asymptotic behavior of solutions to (1.1). When $a_1, a_2 \in (0, 1)$, they proved that the solution (u, v, w) satisfies $u(t) \rightarrow u^*, v(t) \rightarrow v^*, w(t) \rightarrow \frac{\alpha u^* + \beta v^*}{\gamma}$ in $L^\infty(\Omega)$ as $t \rightarrow \infty$, where $u^* = \frac{1-a_1}{1-a_1 a_2}, v^* = \frac{1-a_2}{1-a_1 a_2}$, under the conditions

$$(1.2) \quad \mu_1 > \frac{d_2 \chi_1^2 u^*}{\frac{4a_1 \gamma (1-a_1 a_2) d_1 d_2 d_3}{(a_1 \alpha^2 + a_2 \beta^2 - 2a_1 a_2 \alpha \beta)} - \frac{d_1 a_1 \chi_2^2 v^*}{4\mu_2 a_2}}, \quad \mu_2 > \frac{\chi_2^2 v^* (a_1 \alpha^2 + a_2 \beta^2 - 2a_1 a_2 \alpha \beta)}{16d_2 d_3 a_2 \gamma (1 - a_1 a_2)}.$$

These conditions are not natural because they are not symmetric.

The purpose of the present report is to improve the method in [1] for obtaining asymptotic stability of solutions to (1.1) under a more general and sharp condition for the sensitivity function $\chi_i(w)$. We shall suppose throughout this report that h, χ_i ($i = 1, 2$) satisfy the following conditions:

$$(1.3) \quad \chi_i \in C^{1+\theta}([0, \infty)) \cap L^1(0, \infty) \quad (0 < \exists \theta < 1), \quad \chi_i > 0 \quad (i = 1, 2),$$

$$(1.4) \quad h \in C^1([0, \infty) \times [0, \infty) \times [0, \infty)), \quad h(0, 0, 0) \geq 0,$$

$$(1.5) \quad \exists \gamma > 0; \quad \frac{\partial h}{\partial u}(u, v, w) \geq 0, \quad \frac{\partial h}{\partial v}(u, v, w) \geq 0, \quad \frac{\partial h}{\partial w}(u, v, w) \leq -\gamma,$$

$$(1.6) \quad \exists \delta > 0, \exists M > 0; \quad |h(u, v, w) + \delta w| \leq M(u + v + 1),$$

$$(1.7) \quad \exists k_i > 0; \quad -\chi_i(w)h(0, 0, w) \leq k_i \quad (i = 1, 2).$$

We also assume that

(1.8)

$$\exists p > n; 2d_i d_3 \chi_i'(w) + \left((d_3 - d_i)p + \sqrt{(d_3 - d_i)^2 p^2 + 4d_i d_3 p} \right) [\chi_i(w)]^2 \leq 0 \quad (i = 1, 2).$$

The above conditions cover the prototypical example $\chi_i(w) = \frac{K_i}{(1+w)^{\sigma_i}}$ ($K_i > 0$, $\sigma_i > 1$), $h(u, v, w) = u + v - w$. We assume that the initial data u_0, v_0, w_0 satisfy

$$(1.9) \quad 0 \leq u_0 \in C(\bar{\Omega}) \setminus \{0\}, 0 \leq v_0 \in C(\bar{\Omega}) \setminus \{0\}, 0 \leq w_0 \in W^{1,q}(\Omega) \quad (\exists q > n).$$

The following result which is concerned with global existence and boundedness in (1.1) was established in [2].

Theorem 1.1. *Let $d_1, d_2, d_3 > 0$, $\mu_1, \mu_2 > 0$, $a_1, a_2 \geq 0$. Assume that h, χ_1, χ_2 satisfy (1.3)–(1.8). Then for any u_0, v_0, w_0 satisfying (1.9) for some $q > n$, there exists an exactly one pair (u, v, w) of nonnegative functions*

$$u, v, w \in C(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)),$$

which satisfy (1.1). Moreover, the solutions u, v, w are uniformly bounded, i.e., there exists a constant $C_1 > 0$ such that

$$\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)} + \|w(t)\|_{W^{1,\infty}(\Omega)} \leq C_1 \quad \text{for all } t \geq 0,$$

and the solutions u, v, w are the Hölder continuous functions, i.e., there exist $\alpha \in (0, 1)$ and $C_2 > 0$ such that

$$\|u\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{\Omega} \times [1, t])} + \|v\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{\Omega} \times [1, t])} + \|w\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{\Omega} \times [1, t])} \leq C_2 \quad \text{for all } t \geq 1.$$

Since Theorem 1.1 guarantees that u, v and w exist globally and are bounded and nonnegative, it is possible to define nonnegative numbers $\alpha_1, \alpha_2, \beta_1, \beta_2$ by

$$(1.10) \quad \begin{aligned} \alpha_1 &:= \min_{(u,v,w) \in I} h_u(u, v, w), & \alpha_2 &:= \max_{(u,v,w) \in I} h_u(u, v, w), \\ \beta_1 &:= \min_{(u,v,w) \in I} h_v(u, v, w), & \beta_2 &:= \max_{(u,v,w) \in I} h_v(u, v, w), \end{aligned}$$

where $I = (0, C_1)^3$ and C_1 is defined in Theorem 1.1.

In the case $a_1, a_2 \in (0, 1)$ asymptotic behavior of solutions to (1.1) will be discussed under the following additional conditions: there exists $\delta_1 > 0$ such that

$$(1.11) \quad 4\delta_1 - a_1 a_2 (1 + \delta_1)^2 > 0$$

and

$$(1.12) \quad \mu_1 > \frac{\chi_1(0)^2 u^* (1 + \delta_1) (\alpha_2^2 a_1 \delta_1 + \beta_2^2 a_2 - \alpha_1 \beta_1 a_1 a_2 (1 + \delta_1))}{4a_1 d_1 d_3 \gamma (4\delta_1 - a_1 a_2 (1 + \delta_1)^2)},$$

$$(1.13) \quad \mu_2 > \frac{\chi_2(0)^2 v^* (1 + \delta_1) (\alpha_2^2 a_1 \delta_1 + \beta_2^2 a_2 - \alpha_1 \beta_1 a_1 a_2 (1 + \delta_1))}{4a_2 d_2 d_3 \gamma (4\delta_1 - a_1 a_2 (1 + \delta_1)^2)}.$$

Now the main result reads as follows. The main theorem is concerned with asymptotic stability in (1.1) in the case $a_1, a_2 \in (0, 1)$.

Theorem 1.2. Let $d_1, d_2, d_3 > 0$, $\mu_1, \mu_2 > 0$ and $a_1, a_2 \in (0, 1)$. Under the conditions (1.3)–(1.9) and (1.11)–(1.13), the unique global solution (u, v, w) of (1.1) has the following asymptotic behavior:

$$\|u(t) - u^*\|_{L^\infty(\Omega)} \rightarrow 0, \quad \|v(t) - v^*\|_{L^\infty(\Omega)} \rightarrow 0, \quad \|w(t) - w^*\|_{L^\infty(\Omega)} \rightarrow 0 \quad (t \rightarrow \infty).$$

where

$$u^* := \frac{1 - a_1}{1 - a_1 a_2}, \quad v^* := \frac{1 - a_2}{1 - a_1 a_2}$$

and $w^* \geq 0$ such that $h(u^*, v^*, w^*) = 0$.

Remark 1.1. Theorem 1.2 can be applied to the case $\chi_i(w) = \chi_i$ and $h(u, v, w) = \alpha u + \beta v - \gamma w$. Then the conditions (1.11)–(1.13) have symmetry and relax the condition (1.2) assumed in [1]. Indeed, the conditions (1.2) are stronger than (1.11)–(1.13) when $\delta_1 = 1$. Moreover, in view of considering the function

$$f(x) = \frac{a_1(\alpha^2 - \alpha\beta a_2)x^2 + (\beta^2 a_2 - \alpha^2 a_1)x}{-a_1 a_2 x^2 + 4x - 4}$$

(we put $x = 1 + \delta_1$), $x = 2$ ($\delta_1 = 1$) is not a minimizer of the right-hand sides of (1.12) and (1.13) except the case $\beta^2 a_2 = \alpha^2 a_1$. Thus the conditions (1.11)–(1.13) relax (1.2).

Remark 1.2. In Theorem 1.2 we can find $w^* \geq 0$ satisfying $h(u^*, v^*, w^*) = 0$. Indeed, from (1.4)–(1.6) for every $a, b \geq 0$ there exists \bar{w} such that $h(a, b, \bar{w}) = 0$. Indeed, if we choose $w_1 \geq \frac{M(a+b+1)}{\delta}$, then (1.6) yields $h(a, b, w_1) \leq M(a+b+1) - \delta w_1 \leq 0$. On the other hand, (1.4) and (1.5) imply that $h(a, b, 0) \geq h(0, 0, 0) \geq 0$. Hence, by the intermediate value theorem there exists $\bar{w} \geq 0$ such that $h(a, b, \bar{w}) = 0$.

The strategy for the proof of Theorem 1.2 is to modify an argument in [1]. The key for this strategy is to construct the following energy estimate:

$$\frac{d}{dt} E(t) \leq -\varepsilon \left(\int_{\Omega} (u - \bar{u})^2 + \int_{\Omega} (v - \bar{v})^2 + \int_{\Omega} (w - \bar{w})^2 + \int_{\Omega} |\nabla w|^2 \right)$$

with some function $E(t) \geq 0$ and some $\varepsilon > 0$, where $(\bar{u}, \bar{v}, \bar{w}) \in \mathbb{R}^3$ is a solution of (1.1). For finding the above inequality we apply more “suitable” estimates for

$$\int_{\Omega} \frac{\chi_1(w)}{u} \nabla u \cdot \nabla w \quad \text{and} \quad \int_{\Omega} \frac{\chi_1(w)}{v} \nabla v \cdot \nabla w.$$

These enable us to improve the condition (1.2).

2. Proof of the main result

In this section we will establish asymptotic stability of solutions to (1.1) in the case $a_1, a_2 \in (0, 1)$. For the proof of Theorem 1.2, we shall prepare some elementary results.

Lemma 2.1 (see [1, Lemma 3.1]). Suppose $f : (1, \infty) \rightarrow \mathbb{R}$ is a uniformly continuous nonnegative function satisfying $\int_1^\infty f(t) dt < \infty$. Then $f(t) \rightarrow 0$ as $t \rightarrow \infty$.

Lemma 2.2. Let $a, b, c, d, e, f \in \mathbb{R}$. Suppose that

$$(2.1) \quad a > 0, \quad d - \frac{b^2}{4a} > 0, \quad f - \frac{c^2}{4a} - \frac{(2ae - bc)^2}{4a(4ad - b^2)} > 0.$$

Then

$$(2.2) \quad ax^2 + bxy + cxz + dy^2 + eyz + fz^2 \geq 0$$

holds for all $x, y, z \in \mathbb{R}$.

Proof. From straightforward calculations we obtain

$$\begin{aligned} & ax^2 + bxy + cxz + dy^2 + eyz + fz^2 \\ &= a \left(x + \frac{by + cz}{2a} \right)^2 + \left(d - \frac{b^2}{4a} \right) \left(y + \frac{2ae - bc}{4ad - b^2} \right)^2 + \left(f - \frac{c^2}{4a} - \frac{(2ae - bc)^2}{4a(4ad - b^2)} \right) z^2. \end{aligned}$$

In view of the above equation, (2.1) leads to (2.2). \square

Now we will prove the key estimate for the proof of Theorem 1.2.

Lemma 2.3. Let $a_1, a_2 \in (0, 1)$ and (u, v, w) a solution to (1.1). Under the conditions (1.3)–(1.9) and (1.11)–(1.13), there exist $\delta_1, \delta_2 > 0$ and $\varepsilon > 0$ such that the nonnegative functions E_1 and F_1 defined by

$$E_1(t) := \int_{\Omega} \left(u - u^* - u^* \log \frac{u}{u^*} \right) + \delta_1 \frac{a_1 \mu_1}{a_2 \mu_2} \int_{\Omega} \left(v - v^* - v^* \log \frac{v}{v^*} \right) + \frac{\delta_2}{2} \int_{\Omega} (w - w^*)^2$$

and

$$F_1(t) := \int_{\Omega} (u - u^*)^2 + \int_{\Omega} (v - v^*)^2 + \int_{\Omega} (w - w^*)^2 + \int_{\Omega} |\nabla w|^2$$

satisfy

$$(2.3) \quad \frac{d}{dt} E_1(t) \leq -\varepsilon F_1(t) \quad (t > 0).$$

Proof. Thanks to (1.11)–(1.13) we can choose $\delta_1 > 0$ defined in (1.11)–(1.13) and $\delta_2 > 0$ satisfying

$$\frac{\chi_1(0)^2 u^*(1 + \delta_1)}{4d_1 d_3} < \delta_2 < \frac{a_1 \mu_1 \gamma (4\delta_1 - a_1 a_2 (1 + \delta_1)^2)}{\alpha_2^2 a_1 \delta_1 + \beta_2^2 a_2 - \alpha_1 \beta_1 a_1 a_2 (1 + \delta_1)}$$

and

$$\frac{a_1 \mu_1 \chi_2(0)^2 v^*(1 + \delta_1)}{4a_2 \mu_2 d_2 d_3} < \delta_2 < \frac{a_1 \mu_1 \gamma (4\delta_1 - a_1 a_2 (1 + \delta_1)^2)}{\alpha_2^2 a_1 \delta_1 + \beta_2^2 a_2 - \alpha_1 \beta_1 a_1 a_2 (1 + \delta_1)}.$$

We denote by $A_1(t)$, $B_1(t)$, $C_1(t)$ the functions defined as

$$\begin{aligned} A_1(t) &:= \int_{\Omega} \left(u - u^* - u^* \log \frac{u}{u^*} \right), & B_1(t) &= \int_{\Omega} \left(v - v^* - v^* \log \frac{v}{v^*} \right), \\ C_1(t) &:= \frac{1}{2} \int_{\Omega} (w - w^*)^2, \end{aligned}$$

and we write as

$$E_1(t) = A_1(t) + \delta_1 \frac{a_1 \mu_1}{a_2 \mu_2} B_1(t) + \delta_2 C_1(t).$$

The Taylor formula applied to $H(s) = s - u^* \log s$ ($s \geq 0$) yields $A_1(t) = \int_{\Omega} (H(u) - H(u^*))$ is a nonnegative function for $t > 0$ (more detail, see [1, Lemma 3.2]). Similarly, we have that $B_1(t)$ is a positive function. By the straightforward calculations we infer

$$\begin{aligned} \frac{d}{dt} A_1(t) &= -\mu_1 \int_{\Omega} (u - u^*)^2 - a_1 \mu_1 \int_{\Omega} (u - u^*)(v - v^*) - d_1 u^* \int_{\Omega} \frac{|\nabla u|^2}{u^2} \\ &\quad + u^* \int_{\Omega} \frac{\chi_1(w)}{u} \nabla u \cdot \nabla w, \\ \frac{d}{dt} B_1(t) &= -\mu_2 \int_{\Omega} (v - v^*)^2 - a_2 \mu_2 \int_{\Omega} (u - u^*)(v - v^*) - d_2 v^* \int_{\Omega} \frac{|\nabla v|^2}{v^2} \\ &\quad + v^* \int_{\Omega} \frac{\chi_2(w)}{v} \nabla v \cdot \nabla w, \\ \frac{d}{dt} C_1(t) &= \int_{\Omega} h_u (u - u^*)(w - w^*) + \int_{\Omega} h_v (v - v^*)(w - w^*) + \int_{\Omega} h_w (w - w^*)^2 \\ &\quad - d_3 \int_{\Omega} |\nabla w|^2 \end{aligned}$$

with some derivatives h_u , h_v and h_w . Hence we have

$$(2.4) \quad \frac{d}{dt} E_1(t) = I_3(t) + I_4(t),$$

where

$$\begin{aligned} I_3(t) &:= -\mu_1 \int_{\Omega} (u - u^*)^2 - a_1 \mu_1 (1 + \delta_1) \int_{\Omega} (u - u^*)(v - v^*) - \delta_1 \frac{a_1 \mu_1}{a_2} \int_{\Omega} (v - v^*)^2 \\ &\quad + \delta_2 \int_{\Omega} h_u (u - u^*)(w - w^*) + \delta_2 \int_{\Omega} h_v (v - v^*)(w - w^*) + \delta_2 \int_{\Omega} h_w (w - w^*)^2 \end{aligned}$$

and

$$(2.5) \quad \begin{aligned} I_4(t) &:= -d_1 u^* \int_{\Omega} \frac{|\nabla u|^2}{u^2} + u^* \int_{\Omega} \frac{\chi_1(w)}{u} \nabla u \cdot \nabla w - d_2 v^* \delta_1 \frac{a_1 \mu_1}{a_2 \mu_2} \int_{\Omega} \frac{|\nabla v|^2}{v^2} \\ &\quad + v^* \delta_1 \frac{a_1 \mu_1}{a_2 \mu_2} \int_{\Omega} \frac{\chi_2(w)}{v} \nabla v \cdot \nabla w - d_3 \delta_2 \int_{\Omega} |\nabla w|^2. \end{aligned}$$

At first, we shall show from Lemma 2.2 that there exists $\varepsilon_1 > 0$ such that

$$(2.6) \quad I_3(t) \leq -\varepsilon_1 \left(\int_{\Omega} (u - u^*)^2 + \int_{\Omega} (v - v^*)^2 + \int_{\Omega} (w - w^*)^2 \right).$$

To see this, we put

$$\begin{aligned} g_1(\varepsilon) &:= \mu_1 - \varepsilon, \\ g_2(\varepsilon) &:= \left(\frac{a_1}{a_2} \mu_1 \delta_1 - \varepsilon \right) - \frac{a_1^2 \mu_1^2 (1 + \delta_1)^2}{4(\mu_1 - \varepsilon)}, \\ g_3(\varepsilon) &:= (-\delta_2 h_w - \varepsilon) - \frac{h_u^2}{4(\mu_1 - \varepsilon)} \delta_2^2 - \frac{(2h_v(\mu_1 - \varepsilon) - h_u a_1 \mu_1 (1 + \delta_1))^2}{4(\mu_1 - \varepsilon)(4\frac{a_1}{a_2} \mu_1 \delta_1 (\mu_1 - \varepsilon) - a_1^2 \mu_1^2 (1 + \delta_1)^2)} \delta_2^2. \end{aligned}$$

Since $\mu_1 > 0$, we have $g_1(0) = \mu_1 > 0$. Due to (1.11), we infer

$$g_2(0) = \frac{a_1 \mu_1}{4a_2} (4\delta_1 - a_1 a_2 (1 + \delta_1)^2) > 0.$$

In light of (1.5) and the definitions of $\delta_2 > 0$, $\alpha_i, \beta_i \geq 0$ (defined in (1.10)) we obtain

$$\begin{aligned} g_3(0) &= \delta_2 \left(-h_w - \left(\frac{h_u^2}{4\mu_1} + \frac{a_2(2h_v - h_u a_1 (1 + \delta_1))^2}{4a_1 \mu_1 (4\delta_1 - a_1 a_2 (1 + \delta_1)^2)} \right) \delta_2 \right) \\ &\geq \delta_2 \left(\gamma - \left(\frac{\alpha_2^2 a_1 \delta_1 + \beta_2^2 a_2 - \alpha_1 \beta_1 a_1 a_2 (1 + \delta_1)}{a_1 \mu_1 (4\delta_1 - a_1 a_2 (1 + \delta_1)^2)} \right) \delta_2 \right) > 0. \end{aligned}$$

Combination of the above inequalities and the continuity argument yields that there exists $\varepsilon_1 > 0$ such that $g_i(\varepsilon_1) > 0$ hold for $i = 1, 2, 3$. Thanks to Lemma 2.2 with

$$\begin{aligned} a &= \mu_1 - \varepsilon_1, & b &= a_1 \mu_1 (1 + \delta_1), & c &= -\delta_2 h_u, \\ d &= \delta_1 \frac{a_1 \mu_1}{a_2} - \varepsilon_1, & e &= -\delta_2 h_v, & f &= -\delta_2 h_w - \varepsilon_1, \\ x &= u(t) - u^*, & y &= v(t) - v^*, & z &= w(t) - w^*, \end{aligned}$$

we obtain (2.6) with $\varepsilon_1 > 0$. Lastly we will find $\varepsilon_2 > 0$ satisfying

$$(2.7) \quad I_4(t) \leq -\varepsilon_2 \int_{\Omega} |\nabla w|^2.$$

By virtue of the definition of $\delta_2 > 0$, we can find $\delta_3 \in \left(\frac{\chi_i(0)^2 u^*(1 + \delta_1)}{4d_1 d_3 \delta_2}, 1 \right)$. Noting that $\chi'_i < 0$ (from (1.8)) and then using the Young inequality, we have

$$\begin{aligned} u^* \int_{\Omega} \frac{\chi_1(w)}{u} \nabla u \cdot \nabla w &\leq \chi_1(0) u^* \int_{\Omega} \frac{|\nabla u \cdot \nabla w|}{u} \\ &\leq \frac{\chi_1(0)^2 u^{*2} (1 + \delta_1)}{4d_3 \delta_2 \delta_3} \int_{\Omega} \frac{|\nabla u|^2}{u^2} + \frac{d_3 \delta_2 \delta_3}{1 + \delta_1} \int_{\Omega} |\nabla w|^2 \end{aligned}$$

and

$$\begin{aligned} v^* \delta_1 \frac{a_1 \mu_1}{a_2 \mu_2} \int_{\Omega} \frac{\chi_2(w)}{v} \nabla v \cdot \nabla w &\leq \chi_2(0) v^* \delta_1 \frac{a_1 \mu_1}{a_2 \mu_2} \int_{\Omega} \frac{|\nabla v \cdot \nabla w|}{v} \\ &\leq \frac{\chi_2(0)^2 v^{*2} \delta_1 (1 + \delta_1)}{4d_3 \delta_2} \left(\frac{a_1 \mu_1}{a_2 \mu_2} \right)^2 \int_{\Omega} \frac{|\nabla v|^2}{v^2} + \frac{d_3 \delta_1 \delta_2}{1 + \delta_1} \int_{\Omega} |\nabla w|^2. \end{aligned}$$

Plugging these into (2.5) we infer

$$\begin{aligned} I_4(t) &\leq -u^* \left(d_1 - \frac{\chi_1(0)^2 u^* (1 + \delta_1)}{4d_3 \delta_2 \delta_3} \right) \int_{\Omega} \frac{|\nabla u|^2}{u^2} \\ &\quad - v^* \delta_1 \frac{a_1 \mu_1}{a_2 \mu_2} \left(d_2 - \frac{a_1 \mu_1 \chi_2(0)^2 v^* (1 + \delta_1)}{4d_3 a_2 \mu_2 \delta_2} \right) \int_{\Omega} \frac{|\nabla v|^2}{v^2} \\ &\quad - d_3 \delta_2 \left(1 - \frac{\delta_1 + \delta_3}{1 + \delta_1} \right) \int_{\Omega} |\nabla w|^2. \end{aligned}$$

We note from the definitions of $\delta_2 > 0$ and $\delta_3 > 0$ that

$$\begin{aligned} d_1 - \frac{\chi_1(0)^2 u^* (1 + \delta_1)}{4d_3 \delta_2 \delta_3} &> 0, \\ d_2 - \frac{a_1 \mu_1 \chi_2(0)^2 v^* (1 + \delta_1)}{4d_3 a_2 \mu_2 \delta_2} &> 0 \end{aligned}$$

and

$$1 - \frac{\delta_1 + \delta_3}{1 + \delta_1} = \frac{1 - \delta_3}{1 + \delta_1} > 0.$$

Therefore we obtain that there exists $\varepsilon_2 > 0$ such that (2.7) holds. Combination of (2.4), (2.6) and (2.7) implies the end of the proof. \square

Proof of Theorem 1.2. We let $f_1(t) := \int_{\Omega} (u - u^*)^2 + \int_{\Omega} (v - v^*)^2 + \int_{\Omega} (w - w^*)^2 \geq 0$. We have $f_1(t)$ is a nonnegative function, and thanks to the regularity of u, v, w (see Theorem 1.1) we can see that $f_1(t)$ is uniformly continuous. Moreover, integrating (2.3) over $(1, \infty)$, we infer from the positivity of $E_1(t)$ that

$$\int_1^{\infty} f_1(t) dt \leq \frac{1}{\varepsilon} E_1(1) < \infty.$$

Therefore we obtain from Lemma 2.1 that $f_1(t) \rightarrow 0$. \square

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