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Some mathematical considerations about a small intestine morphology in the human body

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Abstract

A small intestine has a non-brunching tube structure and is packed in an abdominal cavity which is a finite space. Therefore, a small intestine has a finite number of bending. To investigate this number of tube-bending, I introduce some differential geometrical concepts into the arguments and conclude that a small intestine has a given range of the number of a tube-bending.

1 Introduction

The role of a tube structure for a living organism is very important in the view of evolution. There is almost no organism which has not a tube structure. Animals have a gastrointestinal tract, a capillary, a bronchus, and plants have a conducting vessel, for example. So there are many mathematical researches about a tube structure. Some studies are concerned with fractal dimension, some are with fluid dynamics, and some are with vertex model. In this paper, I discuss some mathematical properties of a non-brunching tube in the view of differential geometrical point: especially, the number of a tube bending of a small intestine.
Of course, there are many differential geometrical studies. These studies use theory of elasticity [1], which is superior to investigate the dynamics of a tube structure. However, these method is too complicated to investigate static properties. Therefore, in this paper, I introduce another differential geometrical concepts to simplify the arguments of a tube structure about static properties like following.

$$\begin{aligned} dE_{bind} &= \alpha dS \\ dE_{bend} &= \beta K^2 dS \end{aligned}$$

These energy are used to investigate a carbon nanotube or graphene [3]. In a vertex model which is very common as the model for mathematical biology, a cell has, ideally, 6-vertices ; in another words, a shape of a cell inclines to be hexagonal. This means a cell is equal to be a hexagon which vertex is a particle. In this sense, we can identify a tube which is constituted with cells, with a carbon nano tube.

2 The first and second fundamental form of a cylinder and a tours.

Let the first fundamental form be g and the second be a h.

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

$$h = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$$

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Of course, we can represent these g and h with a notation of partial differentiation.

$$g = \begin{pmatrix} \boldsymbol{p}_u \cdot \boldsymbol{p}_u & \boldsymbol{p}_u \cdot \boldsymbol{p}_v \\ \boldsymbol{p}_v \cdot \boldsymbol{p}_u & \boldsymbol{p}_v \cdot \boldsymbol{p}_v \end{pmatrix}$$

$$h = \begin{pmatrix} \boldsymbol{p}_{uu} \cdot \boldsymbol{e} & \boldsymbol{p}_{uv} \cdot \boldsymbol{e} \\ \boldsymbol{p}_{vu} \cdot \boldsymbol{e} & \boldsymbol{p}_{vv} \cdot \boldsymbol{e} \end{pmatrix}$$

Here, the denote $\boldsymbol{p}(u,v)$ is a position vector which represents a point on a surface and $\boldsymbol{e}$ is a norm vector of that point.
A vector to a cylinder C is of the form

$$\boldsymbol{p}(u,v) = \begin{pmatrix} rcosu \\ rsinu \\ v \end{pmatrix}$$

Therefore, The coefficients of the first and second fundamental form of C are

$$g = \begin{pmatrix} r^2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$h = \begin{pmatrix} -r^2 & 0 \\ 0 & 0 \end{pmatrix}$$

We rotate a circle of radius r, lying in the x_1x_3-plane with the centre at $(x_1, x_3) = (R, 0)$, about the x_3-axis. We denote by u the directed angle from positive direction of the x_1-axis to a point P on the circle. We then represent the tours T in the form

$$\boldsymbol{p}(u,v) = \begin{pmatrix} (R + rcosu)cosv \\ (R + rcosu)sinv \\ rsinu \end{pmatrix}$$

The coefficients of the first and second fundamental form are

$$g = \begin{pmatrix} r^2 & 0 \\ 0 & (R + rcosu)^2 \end{pmatrix}$$

$$h = \begin{pmatrix} r & 0 \\ 0 & (R + rcosu)cosu \end{pmatrix}$$

By using these notations, the area of the surface is represented as

$$Area(S) = \int_D \sqrt{g_{11}g_{22} - g_{12}g_{21}}dS$$

And the Gaussian curvature which is the product of the two principal curvature is

$$K = \frac{h_{11}h_{22} - h_{12}h_{21}}{g_{11}g_{22} - g_{12}g_{21}}$$

Therefore, the area and the Gaussian curvature of C, T are

$$\begin{aligned} Area(C) &= 2\pi rh \\ Area(T) &= 2\pi^2 r^2 \\ K_C &= 0 \\ K_T &= \frac{cosu}{r(R + rcosu)} \end{aligned}$$

For simplicity, we regard a tube of a small intestine as composed with only a C and T. That is, when a tube bends, the part of bending is regarded as a half tours. Therefore, a tube of small intestine has a unit which has a one cylinder part and a half tours part. Furthermore, R of the tours should be equal to r to fulfill the space.

Let the space in which a small intestine be packed is the domain $[0, L] \times [0, H] \times [0, r] \in \boldsymbol{R}^3$. The argument we've have, E_{bind} and E_{bend} of the unit of a tube are represented like following:

$$
\begin{aligned}
E_{bind} &= \alpha(Area(C) + \frac{1}{2}Area(T)) \\
&= 2\pi\alpha r(\pi r + H) \\
E_{bend} &= \frac{\beta}{r^4}A
\end{aligned}
$$

Here, the notaiton A is

$$
\begin{aligned}
\int_D {K_T}^2 dS &= \int_{u=0}^{u=2\pi}\int_{v=0}^{v=\pi}\{\frac{cosu}{r^2(1+cosu)}\}^2 dudv \\
&= \frac{1}{r^4}\iint \frac{cos^2u}{(1+cosu)^2}dudv \\
&\equiv \frac{1}{r^4}A \\
& \quad (A = const.)
\end{aligned}
$$

A tube has the n units in the space. The unit has two r wide, so a number of the unit is $\frac{L}{2r}$. Thus, the total energy $E(r)$ of a tube is

$$
\begin{aligned}
E(r) &= (E_{bind} + E_{bend}) \times n \\
&= L\{\pi\alpha H + \pi(\pi - 2)\alpha r + \frac{\beta}{2r^5}A\}
\end{aligned}
$$

We require that the total energy $E(r)$ takes a minimum in the body.

$$
\frac{dE(r)}{dr} = 0
$$

Left-side hand is $\pi(\pi-2)\alpha - \frac{5\beta}{2r^6}A$. So, when r is equal to $\{\frac{5\beta}{2\pi(\pi-2)\alpha}A\}^{\frac{1}{6}}(\equiv r_o)$, $E(r)$ takes a minimum. A number of tube bending is equal to a number of the unit. Therefore, using a Gaussian symbol, a number of a tube bending n is concluded following;

$$
n = [\frac{L}{2r_o}]
$$

3 Discussions

Using the energies which I introduced, we can investigate static properties of a form very easily. But we should validate this energies theoretically and practically. There are many mathematical researches and principles about a form. For example, principle of least action or foam theory. In this paper, I introduce two energy E_{bind} and E_{bend}. The former is, essentially, equivalence to the action function [2] . However, the latter is not so clear whether the energy has some relation with existing energies or not. So I will seek for this relation after this and extend the energies for not only a curvature but also a torsion in $\boldsymbol{R}^3$ space. At the same time, to decide constant of proportionality which appears in this paper, I will investigate the number of bending of a small intestine in the human body.

References

[1] Guillermo R. Lázaro, Ignacio Pagonabarraga, Aurora Hernández-Machado, *Phase-field theories for mathematical modeling of biological membranes*, Chemistry and Physics of Lipids 185 (2015) 46-60.

[2] V.I.Arnold, *Mathematical Methods of Classical Mechanics, 2th ed.*, Springer (1989).

[3] Amarui Libèrio de Lima, et al., *Soliton instability and fold formation in laterally compressed few-layer graphene*, Nanotechnology, (2015) 26.