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Propagation property and inverse scattering for the fractional power of negative Laplacian

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1 Introduction

The fractional power of the negative Laplacian as the self-adjoint operator acting on $L^2(\mathbb{R}^n)$ is defined by the Fourier multiplier with the symbol

$$\omega_\rho(\xi) = |\xi|^{2\rho}/(2\rho) \tag{1.1}$$

for $1/2 \leqslant \rho \leqslant 1$. We denote this operator by

$$H_{0,\rho} = \omega_\rho(D_x), \tag{1.2}$$

where $D_x = -i\nabla_x = -i(\partial_{x_1}, \ldots, \partial_{x_n})$. More specifically, we can represent $H_{0,\rho}$ by the Fourier integral operator

$$(H_{0,\rho}\phi)(x) = (\mathscr{F}^*\omega_\rho(\xi)\mathscr{F}\phi)(x) = \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}\omega_\rho(\xi)\phi(y)dyd\xi/(2\pi)^n \tag{1.3}$$

for $\phi \in \mathscr{D}(H_{0,\rho}) = H^{2\rho}(\mathbb{R}^n)$, which is the Sobolev space of order 2ρ. In particular, if $\rho = 1$, then $H_{0,1}$ is the free Schrödinger operator $\omega_1(D_x) = -\Delta_x/2 = -\sum_{j=1}^n \partial_{x_j}^2/2$. If $\rho = 1/2$, then $H_{0,1/2}$ is the massless relativistic Schrödinger operator $\omega_{1/2}(D_x) = \sqrt{-\Delta_x}$.

In Ishida [I2], we proved the following Enss-type propagation estimate for $e^{-itH_{0,\rho}}$. We denote the usual characteristic function of the set $\{\cdots\}$ by $F(\cdots)$. We also denote the smooth characteristic function $\chi \in C^\infty(\mathbb{R}^n)$ by

$$\chi(x) = \begin{cases} 1 & |x| \geqslant 2 \\ 0 & |x| \leqslant 1. \end{cases} \tag{1.4}$$

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Theorem 1.1. *Let $f \in C_0^\infty(\mathbb{R}^n)$ with* $\operatorname{supp} f \subset \{\xi \in \mathbb{R}^n \mid |\xi| \leqslant \eta\}$ *for some given $\eta > 0$. Choose $v \in \mathbb{R}^n$ such that $|v| \gg 1$. The following estimate holds for $t \in \mathbb{R}$ and $N \in \mathbb{N}$:*

$$\left\| \chi\left(\frac{x - (\nabla_\xi \omega_\rho)(v)t}{|v|^{2\rho-1}|t|/4}\right) e^{-itH_{0,\rho}} f(D_x - v) F\left(|x| \leqslant \frac{|v|^{2\rho-1}|t|}{16}\right) \right\| \leqslant C_N(|v|^{2\rho-1}|t|)^{-N}, \quad (1.5)$$

where $\|\cdot\|$ stands for the operator norm on $L^2(\mathbb{R}^n)$, and the constant $C_N > 0$ also depends on the dimension n and the shape of f.

Enss [E] proved the following estimate for the free Schrödinger operator:

$$\left\| F\left(|x - vt| \geqslant \frac{|v||t|}{4}\right) e^{-itD_x^2/2} f(D_x - v) F\left(|x| \leqslant \frac{|v||t|}{16}\right) \right\| \leqslant C_N(1 + |v||t|)^{-N}. \quad (1.6)$$

This estimate was proved not only for spheres but more generally for measurable subsets of $\mathbb{R}^n$ (see Proposition 2.10 in Enss [E]). Before considering Theorem 1.1 further, we discuss the meaning of the estimate (1.6). From the perspective of classical mechanics, D_x represents the momentum or, in particular, the velocity of the particle when the mass is equal to 1. On the left-hand side of (1.6), D_x is localized to the neighborhood of v by the cut-off function f. Therefore, along the time evolution of the propagator $e^{-itD_x^2/2}$, the position of the particle behaves according to

$$x \sim D_x t \sim vt. \quad (1.7)$$

Because the behavior of the sphere is the same, the center of the sphere moves toward vt from the origin:

$$\left\{x \in \mathbb{R}^n \;\middle|\; |x| \leqslant \frac{|v||t|}{16}\right\} \sim \left\{x \in \mathbb{R}^n \;\middle|\; |x - vt| \leqslant \frac{|v||t|}{16}\right\}. \quad (1.8)$$

We can understand the meaning of the estimate (1.6) from these observations. The behavior of the sphere (1.8) makes the characteristic functions on both sides of (1.6) disjoint. Thus, this gives rise to the decay associated with time and velocity. Theorem 1.1 is the fractional Laplacian version of (1.6). Noting that $(\nabla_\xi \omega_\rho)(v) = |v|^{2\rho-2}v$, the case where $\rho = 1$ in (1.5) is essentially equivalent to (1.6). Conversely, when $\rho = 1/2$ in (1.5), the decay on the right-hand side does not involve $|v|$. However, this does not conflict with the physical meaning. In the case where $\rho = 1/2$, the system is relativistic. In this system, the particle does not have a mass, and its velocity is the light velocity, which is normalized to 1. Therefore, the decay cannot include the velocity v.

Spectral analysis for the relativistic Schrödinger operator was initiated by Weder [Wed1], following which Umeda [U1, U2] studied the resolvent estimate and mapping properties associated with the Sobolev spaces. Wei [Wei] also studied the generalized eigenfunctions. Weder [Wed2] also analyzed the spectral properties of the fractional Laplacian for the massive case, and Watanabe [Wa] investigated the Kato-smoothness. Giere [G] investigated the scattering theory and proved the asymptotic completeness of the wave operators in the case of short-range perturbations. Recently, Kitada [K1, K2] constructed the long-range theory.

2 Inverse Scattering

In this section, we assume that the space dimension satisfies $n \geqslant 2$. As an application of Theorem 1.1, we consider a multidimensional inverse scattering. The high-velocity limit of the scattering operator uniquely determines the interaction potentials that satisfy the short-range condition below by using the Enss-Weder time-dependent method (Enss-Weder [EW]).

Assumption 2.1. *$V \in C^1(\mathbb{R}^n)$ is real-valued and satisfies, for $\gamma > 1$,*

$$|\partial_x^\beta V(x)| \leqslant C_\beta \langle x\rangle^{-\gamma-|\beta|}, \quad |\beta| \leqslant 1, \tag{2.1}$$

where the bracket of x has the usual definition $\langle x\rangle = \sqrt{1+|x|^2}$.

For the full Hamiltonian $H_\rho = H_{0,\rho}+V$, where V belongs to the class above, the existence of the wave operators

$$W_\rho^\pm = \operatorname*{s-lim}_{t\to\pm\infty} e^{itH_\rho}e^{-itH_{0,\rho}} \tag{2.2}$$

and their asymptotic completeness have already been proved. Thus, we can define the scattering operator $S_\rho = S_\rho(V)$ by

$$S_\rho = (W_\rho^+)^* W_\rho^-. \tag{2.3}$$

Under these situations, we obtained the following uniqueness theorem in Ishida [I2].

Theorem 2.2. *Let V_1 and V_2 be interaction potentials that satisfy Assumption 2.1. If $S_\rho(V_1) = S_\rho(V_2)$, then $V_1 = V_2$ holds for $1/2 < \rho \leqslant 1$.*

We note that $\rho = 1/2$ is excluded in this theorem. As mentioned before, in the case where $\rho = 1/2$, the system is relativistic and the light velocity is always equal to 1, that is, $|v| \equiv 1$. The Enss-Weder time-dependent method is also called the high-velocity method. As indicated by this name, deriving the uniqueness of the interaction potentials requires the limit of $|v|$. Thus, this method does not combine well with relativistic phenomena (see also Jung [J]).

In Enss-Weder [EW], it was demonstrated that the estimate (1.6) was very useful for inverse scattering and the Enss-Weder time-dependent method was developed. Since then, the uniqueness of the interaction potentials for various quantum systems has been studied by many authors (Weder [Wed3], Jung [J], Nicoleau [N1, N2, N3], Adachi-Maehara [AM], Adachi-Kamada-Kazuno-Toratani [AKKT], Valencia-Weder [VW], Adachi-Fujiwara-Ishida [AFI] and Ishida [I1]). This paper is motivated by these results. In particular, Enss-Weder [EW] first proved the uniqueness of the potentials in the case where $\rho = 1$ by applying (1.6). On the other hand, Jung [J] treated the case of $\rho = 1/2$ using a slightly different approach. Of course, we cannot consider the limit of the velocity in this case. However, roughly speaking, Jung [J] translated the high-velocity limit into a high energy-limit and, without using an estimate of the type (1.5), obtained the uniqueness of the potentials. Thus, Theorem 2.2 represents an interpolation between the results of Enss-Weder [EW] and Jung [J].

To apply the Enss-Weder time-dependent method, the following Radon transformation-type reconstruction formula is crucial.

Theorem 2.3. *Let $v \in \mathbb{R}^n$ be given and let $\hat{v} = v/|v|$. Suppose that $\eta > 0$, and that $\Phi_0, \Psi_0 \in L^2(\mathbb{R}^n)$ such that $\mathscr{F}\Phi_0, \mathscr{F}\Psi_0 \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\mathscr{F}\Phi_0, \operatorname{supp}\mathscr{F}\Psi_0 \subset \{\xi \in \mathbb{R}^n \mid |\xi| \leqslant \eta\}$. Let $\Phi_v = e^{iv\cdot x}\Phi_0, \Psi_v = e^{iv\cdot x}\Psi_0$. Then*

$$|v|^{2\rho-1}(i(S_\rho - 1)\Phi_v, \Psi_v) = \int_{-\infty}^{\infty}(V(x+\hat{v}t)\Phi_0, \Psi_0)dt + O(|v|^{\max\{1-2\rho+\epsilon, -1/(2+\gamma)\}}) \qquad (2.4)$$

holds as $|v| \to \infty$ for any $0 < \epsilon < 2\rho - 1$ and any V that satisfies Assumption 2.1, where $(\cdot,\cdot)$ is the scalar product of $L^2(\mathbb{R}^n)$.

We emphasize that the error exponent in (2.4) is $-1/(2+\gamma)$ when $\rho = 1$, because $\epsilon > 0$ can be chosen arbitrarily. The corresponding order obtained by Enss-Weder [EW] is $o(1-\gamma)$ for $1 < \gamma < 2$ (see Theorem 2.4 in Enss-Weder [EW]). Note that $1 - \gamma > -1/(2+\gamma)$ is equivalent to $\gamma < (\sqrt{13} - 1)/2$. Therefore, in the case where $1 < \gamma < (\sqrt{13} - 1)/2$, our exponent $-1/(2+\gamma)$ is better than the correspondence obtained by Enss-Weder [EW].

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